Neutrino Mass and Grand Unification

Lecture XXII

18/1/2022

LMU
Winter 2022

Anomalies (III)

- $SU(3)$

$$\left[\begin{array}{l} S \text{ (symmetric)} \\ A_s \text{ (anti-symmetric)} \\ Adj \text{ (adjoint)} \end{array}\right]$$

$$\underset{(F)}{3} \otimes \underset{(F)}{3} = 6 + 3^*$$

$$3 \otimes 3 \otimes 3 = 1 + \cdots$$

$$\parallel$$

$$(3^* \otimes 3)$$

$$SU(3)$$

Anomaly $(F) = 1$

$$\downarrow$$

$$-\parallel- \quad (\bar{F}) = -1 = 3 - 4$$

$\Rightarrow$ Anomaly $(6) = ?$

$$A(3_1 \otimes 3_2) = 3\,A(3_2) + A(3_1)\,3$$

$$= 6$$

$$= A(S) + A(A_S) =$$

$$= A(6) + A(3^*) = A(6) - 1$$

$$\Downarrow$$

$$\left(3+4 = \right. \boxed{\mathcal{A}(6) = 7} \; \boxed{\mathcal{A}(3^*) = -1} \; \boxed{= 3-4}\Big)$$

- $\mathcal{A}(\mathrm{Adj}) = \,?$ $\qquad \mathrm{Adj}\,(SU(3)) = 8$

$$3 \otimes \bar{3} = 8 + 1$$

$$\mathrm{Adj} = \Sigma \longrightarrow U \Sigma U^{\dagger}, \quad \mathrm{Tr}\,\Sigma = 0,$$

$$(\Sigma = \Sigma^{\dagger})$$

$$\Sigma = \sum_{a=1}^{8} T_a \varphi_a$$

$$\Sigma_L = \begin{pmatrix} \text{real} & \overbrace{\varphi_1 - i\varphi_2}^{\varphi^+} & \cdots \\ \underbrace{\varphi_1 + i\varphi_2}_{\varphi^-} & \text{real} & \\ & & \text{real} \end{pmatrix}_L \text{(fermions)}$$

$$f_L^+ + f_L^-$$

$$\Leftrightarrow f_L^+ + \underbrace{c(\overline{f_L^-})^T}_{f_R^+}$$

$$\boxed{\text{zero anomaly}}$$

$\Downarrow$ <u>generalise</u>

$$\boxed{SU(N)}$$

- $\mathcal{A}(\text{adjoint}) = 0$

- $\mathcal{A}(S)$, $\mathcal{A}(A_s)$??

$$\underset{(F)}{N} \otimes \underset{(F)}{N} = S + A_s$$

$$\boxed{\mathcal{A}(S) + \mathcal{A}(A_s) = N\,\mathcal{A}(F) + \mathcal{A}(F)\,N = 2N}$$

$$\mathcal{A}(A_s)^{SU(3)} = -1 = 3-4$$

$$\mathcal{A}(S)^{SU(3)} = 7 = 3+4$$

Guess $\mathcal{A}(A_s)^{SU(N)} = N-4$

$$\mathcal{A}(S)^{SU(N)} = N+4$$

Proof

Induction

$N=3$ works

$N \leftarrow$ assume $\mathcal{A}(A_s)^{SU(N)} = N-4$

$N+1$? $\quad (N+1) \otimes (N+1) =$

$$= N \otimes N + N + N + 1$$

$$\Downarrow$$

$$S(SU(N+1)) = S(SU(N)) + N + 1$$

$$A_S(SU(N+1)) = A_S(SU(N)) + N$$

$$A(A_S)^{SU(N+1)} = A(A_S)^{SU(N)} +$$

$$+ A(N)$$

$$= N - 4 + 1 = (N+1) - 4$$

Q.E.D.

$$(A(1) = 0)$$

$$\Rightarrow \boxed{A(S)^{SU(N+1)} = N+1+4}$$

Prove! $\Uparrow$

$$\boxed{\bullet \quad SU(5) \quad GUT}$$

15 SM fermions =

$$= \underbrace{10_F + \bar{5}_F}_{\text{Left-handed}}$$

$$\underbrace{\bar{5} = \begin{pmatrix} d^c \\ \hdashline \nu \\ e \end{pmatrix}_L}_{\bar{F}} \qquad \underbrace{10_F = \begin{pmatrix} u^c & u & d \\ & & e^c \end{pmatrix}_L}_{A_S}$$

$$A(\bar{5}) = -1$$

$$A(10) = \{N-4\} = 5-4 = +1$$

$$\Rightarrow \boxed{A(\bar{5}) + A(10) = 0}$$

Q. E. D.

$$S_F(SO(5)) = 15_F \qquad \left(\frac{5 \cdot 6}{2} = 15 \right)$$

$$A(15_F) = \{N+4\} = 5+4 = 9$$

- <u>comment</u>

(i) $SU(5) \subseteq SO(10)$

(ii) $SO(10) \rightarrow$ spinor 16_F

$$16_F = \bar{5}_F + 10_F + 1_F$$

$$\nu_R$$

$\Rightarrow$ seesaw

(iii) $\mathcal{A}(SO(10)) = 0$

$$\Leftrightarrow \quad \mathcal{A}(16_F) = 0$$

$$\Rightarrow \quad \mathcal{A}(\bar{5}_F) + \mathcal{A}(10_F) + \mathcal{A}(1_F) = 0$$

$$\Downarrow \qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$$

$$\boxed{\mathcal{A}(\bar{5}_F) + \mathcal{A}(10_F) = 0}$$

Georgi, Glashow
'74 ('75)
on anomaly paper –
to be posted

- $SU(6)$ GUT

anomaly Fermions anomaly

$(1)\ \bar{6}_F + 15_F\ (6-4=2)$

$(\bar{F})$ $\qquad A_S \left(\frac{6\cdot 5}{2} = 15\right)$

$$A = -1 + 2 = 1 \Rightarrow \text{anomalous}$$

add $\overline{6}_F$ (?)

$\nearrow$

minimal addition :

$$\boxed{A(\overline{6}_F) + A(\overline{6}_F) + A(15) = 0}$$

- $SU(7)$ Anomaly ?

Topological defects

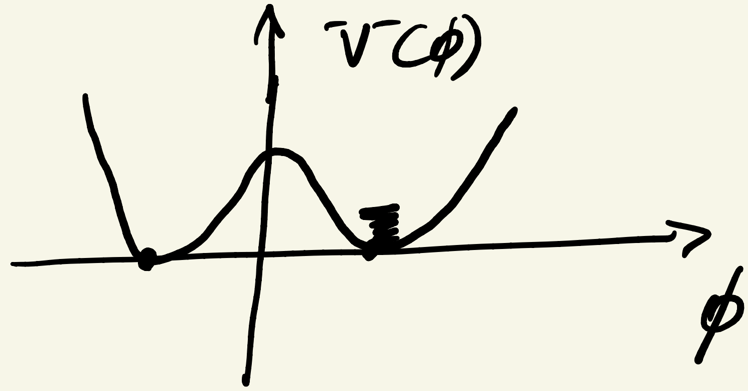
① Domain walls

SSB of D (discrete symm)

$\phi \in R$, D: $\phi \to -\phi$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

$$V(\phi) = \frac{\lambda^2}{2} \left(\phi^2 - v^2 \right)^2$$



$$\boxed{\text{vacuum manifold } \mathcal{M}_0 : \quad \{ \phi_0 : V(\phi_0) = V_{min} \}}$$

$$\mathcal{M}_0 = \{ \phi_0 : V = 0 \Rightarrow \phi_0^2 = v^2 \}$$

$$= \{ \pm v \}$$

- $\phi_0 = v$ = ground state

$$\phi_0 = v + h$$

↑ physical field

$$\Downarrow$$

$$V(h) = \frac{\lambda^2}{2}\left[(v+h)^2 - v^2\right]^2$$

$$= \frac{\lambda^2}{2}\left(2vh + h^2\right)^2$$

$$= \frac{\lambda^2}{2} h^4 + 2\lambda^2 h^3 v + \frac{4\lambda^2}{2} v^2 h^2$$

quartic — cubic — mass term

NO $h \to -h$

$$\Rightarrow \boxed{m_h^2 = 4\lambda^2 v^2}$$

· vacuum $\quad \boxed{\phi = \phi_0 \text{ everywhere}}$

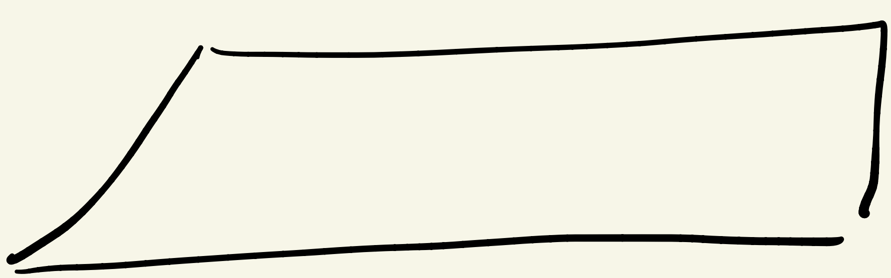
trivial classical solution

$(E = 0)$

• finite energy $(\neq 0)$ energy classical solution ϕ_{cl}!

$\boxed{\text{STATIC}}$

$\phi_{cl}(z) \qquad (z \to -z)$



(x, y)

(infinite in x, y)

$$\boxed{\text{finite energy density} = E/S}$$

$$\frac{E}{S} = \underbrace{\int_{-\infty}^{+\infty} dz \left[\frac{1}{2} \left(\frac{d\phi_{cl}}{dz} \right)^2 + \bar{V}(\phi_{cl}) \right]}_{\text{finite}}$$

$$\Rightarrow \boxed{\begin{array}{ll} \bar{V}(\phi) \xrightarrow[z \to \pm\infty]{} 0 & (1) \\ \\ \dfrac{d\phi_{cl}}{dz} \xrightarrow[z \to \pm\infty]{} 0 & (2) \end{array}}$$

$$\Rightarrow \boxed{\phi_{cl}^2 \rightarrow \phi_0^2 = v^2}$$

$$\Downarrow$$

(a) $\phi_{cl}(z) = \phi_0 = v$

everywhere $\boxed{\text{vacuum}}$

$\therefore \quad \phi_{cl}(+\infty) = \phi_{cl}(-\infty) = v$

(b) $\left.\begin{aligned} \phi_{cl}(+\infty) &= +v \\ \phi_{cl}(-\infty) &= -v \end{aligned}\right\}$ $V=0$ at ∞

$\boxed{\text{DOMAIN WALL}}$

(c) $\phi_{cl}(-\infty) = v$

$\phi_{cl}(+\infty) = -v$

(d) $\phi_{cl}(+\infty) = \phi_{cl}(-\infty) = -v$

$$\boxed{\text{vacuum}}$$

$$E/S = \int dz \left[\frac{1}{2}\left(\frac{d\phi_{cl}}{dz}\right)^2 + V \right]$$

$$\frac{L}{S} = \int dt \left[-\frac{1}{2}\left(\frac{d\phi_{cl}}{dt}\right)^2 - V \right]$$

$$\Downarrow$$

$$\frac{d^2\phi_{cl}}{dz^2} = \frac{dV}{d\phi_{cl}} \Big/ \frac{d\phi_{cl}}{dz}$$

$$\Downarrow$$

$$\frac{1}{2}\frac{d}{dz}\left(\frac{d\phi_{cl}}{dz}\right)^2 = \frac{dV}{d\phi}\frac{d\phi_{cl}}{dz} = \frac{dV}{dz}$$

$$\Downarrow$$

$$\boxed{\frac{1}{2}\left(\frac{d\phi_{cl}}{dz}\right)^2 = V + \text{const.}}$$

$$\downarrow \qquad\qquad \downarrow_{\infty} \qquad\qquad \Downarrow$$

$$\infty \qquad\qquad 0 \qquad\qquad \text{const} \xrightarrow[\infty]{} 0$$

$$\Downarrow$$

$$\boxed{\text{const} = 0}$$

$$\Downarrow \qquad V = \frac{\lambda^2}{2}(\phi^2 - v^2)^2$$

$$\boxed{\frac{d\phi_{cl}}{dz} = \pm\sqrt{2V}}$$

$$= \pm \lambda\,(\phi_{cl}^2 - v^2)$$

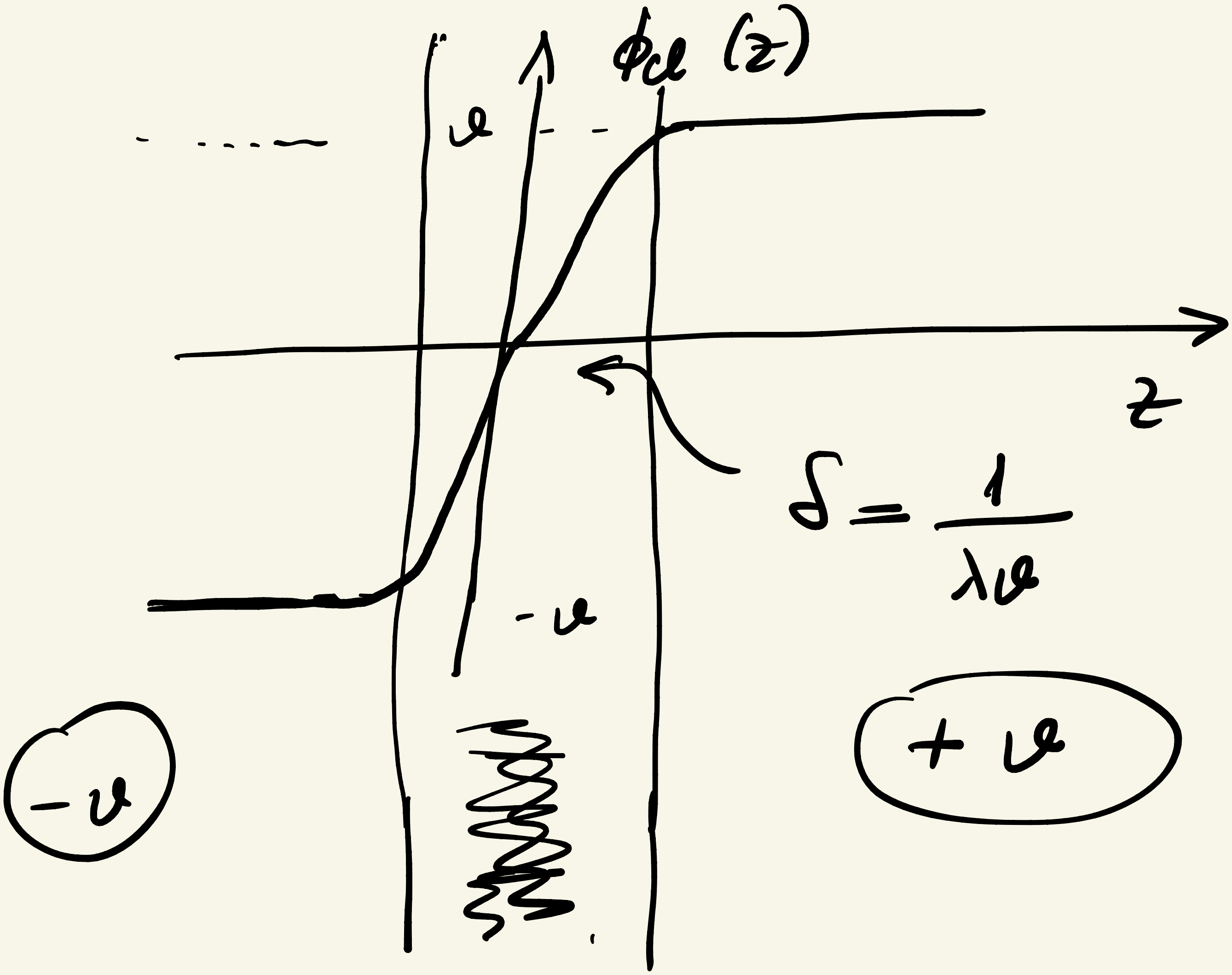
$$\left.\begin{aligned} \phi_{cl}(+\infty) &= +v \\ \phi_{cl}(-\infty) &= -v \end{aligned}\right\}$$

$$\boxed{\phi_{cl}(z) = \mp v\,\mathrm{th}\,\lambda v z}$$

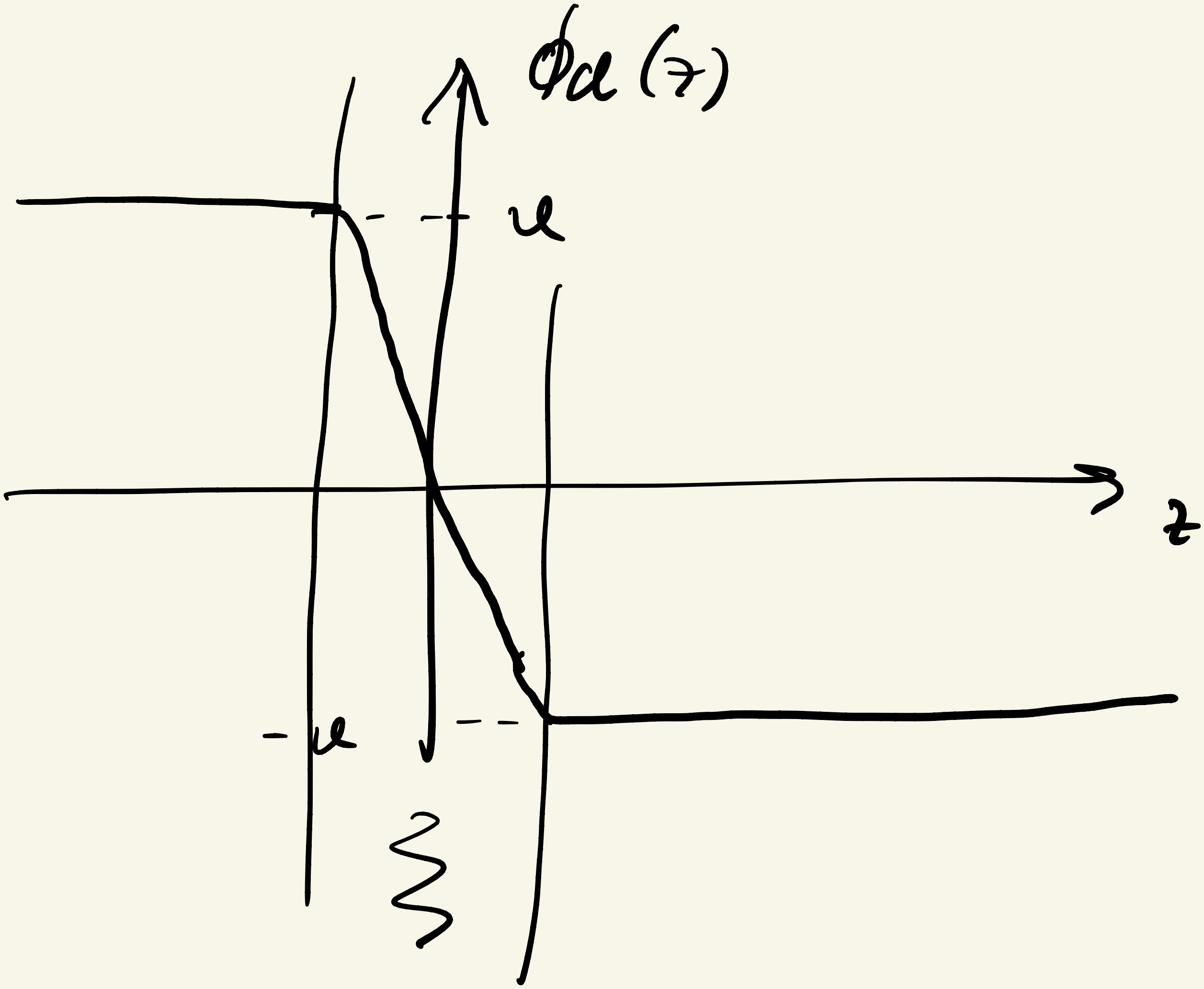
$+$

$$\phi_{cl}^2 - v^2 = v^2\left(\frac{\mathrm{sh}^2}{\mathrm{ch}^2} - 1\right) = -\frac{v^2}{\mathrm{ch}^2 \lambda v z}$$

$$(\mathrm{th}\,\lambda v z)' = +\frac{\lambda v}{\mathrm{ch}^2 \lambda v z}$$



$\boxed{\text{domain wall}}$



anti wall

$$\cdot \; \phi_{cl}(\pm\infty) = \pm v$$

domain wall

$$\phi_{cl}(z) = \phi_{dw}(z)$$

$$\mathcal{M}_0 = \{\pm v\}$$

$$\mathcal{M}_\infty = \{\pm \infty\}$$

$$\boxed{\begin{cases} \phi_{cl}(\pm\infty) = \pm v & (\text{map}) \\ \quad \parallel & \\ \phi_{cl} & \text{non-trivial} \end{cases}}$$

$$\phi_{cl}(\pm\infty) = + v \qquad (\text{map})$$

$$\phi_{cl}(\pm\infty) = - v \qquad \text{trivial}$$

$$\Longrightarrow \quad \phi_{cl}(t) = \pm\phi_0 = \pm v$$

$$E/S = \int \left[\frac{1}{2}\left(\frac{d\phi_{dw}}{dz}\right)^2 + V(\phi_{dw}) \right] dz$$

$$= \int 2V \, dz = \int 2V \, d\phi \frac{dz}{d\phi}$$

$$\left\{ \left(\frac{d\phi}{dz}\right)_{dw} = \sqrt{2V} \right\}$$

$$= \int 2V \frac{1}{\sqrt{2V}} \, d\phi =$$

$$= \int_{-v}^{+v} \sqrt{2V} \, d\phi = \int_{-v}^{+v} \lambda(\phi^2 - v^2) \, d\phi$$

$$\parallel$$

$$\frac{1}{\cot^2 \phi}$$

$\propto \lambda v^3$

dim. grounds

compute