Neutrino Mass
and
Grand Unification

$\boxed{\text{Lecture XIII}}$

30/11/2021

LMU

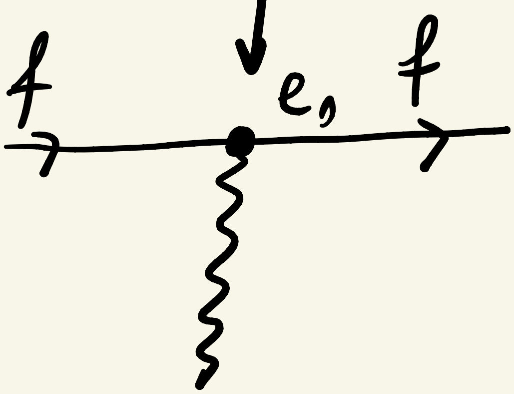
Fall 2021

SU(5) and Unification of Gauge Couplings

QED

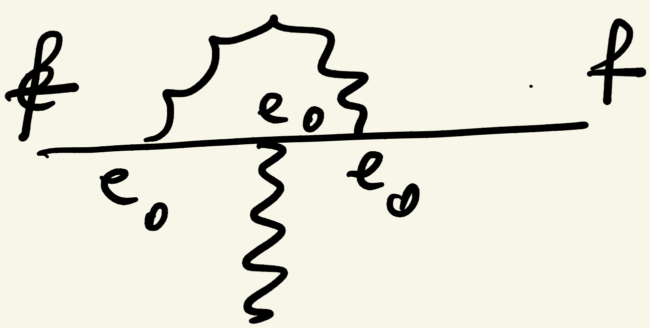
$Q_{em} = 1 \, (-)$

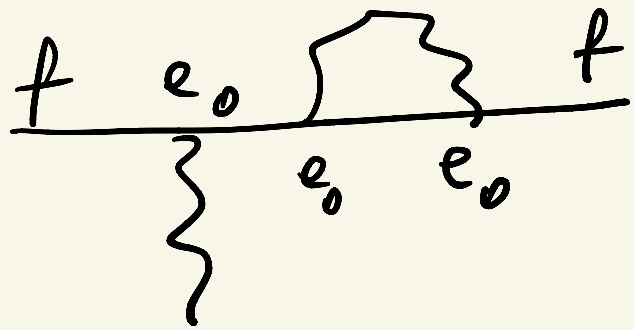
$$\mathcal{L}_{int} = e_0 \bar{f} \gamma^\mu Q_{em} A_\mu f \quad (e)$$

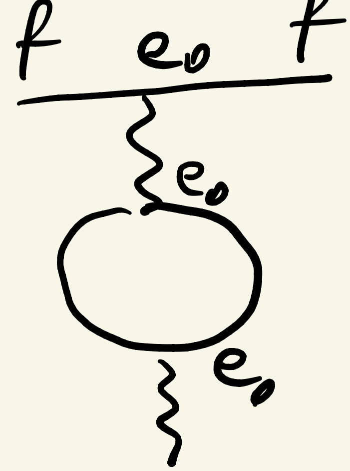


$\boxed{\text{tree - level}}$

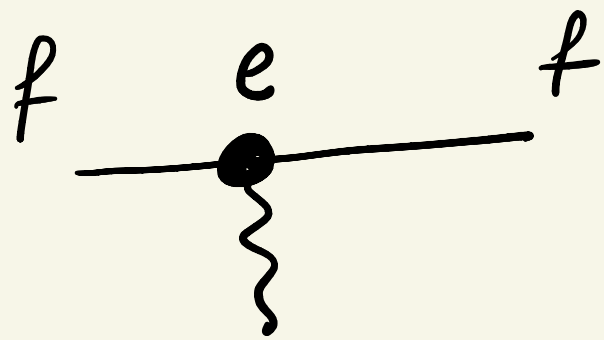
1- loop



$+$ 

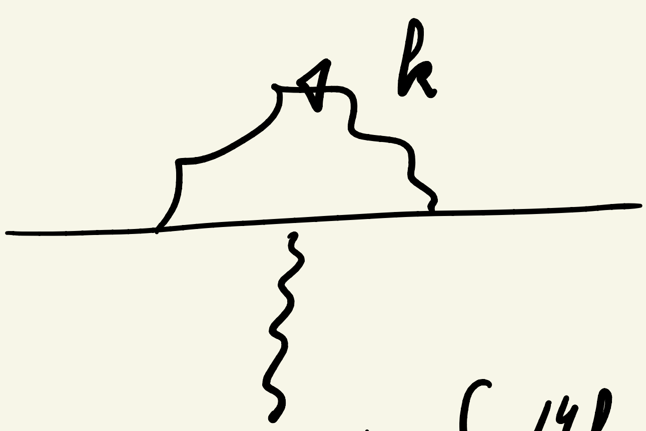
$+$ 

tree + loop



physical

$$e(E) = e_0 + e_0^3\, c \ln \Lambda/E$$



$$\propto \int d^4k \, \frac{1}{k^2} \, \frac{1}{\not{k} - m_f} \, \frac{1}{\not{k} - m_f}$$

photon

$$\propto (m_f = 0) \int d^4k \, \frac{1}{k^2 \not{k} \not{k}} = \int^{\Lambda} d^4k \, \frac{1}{k^4}$$

UV divergence

$$\propto \ln \Lambda$$

$$e(E_1) = e_0 + e_0^3 \, c \ln \Lambda/E_1$$

$$e(E_2) = e_0 + e_0^3 \, c \ln \Lambda/E_2$$

$$\Downarrow$$

$$\boxed{e(E_2) - e(E_1) \simeq e^3 \, c \ln E_1/E_2}$$

Nature: $\Lambda \to M_{pl}$

$$G_N = \frac{1}{M_{pl}^2}$$

$$\Lambda_{\text{cut-off}} \simeq M_{pl}$$

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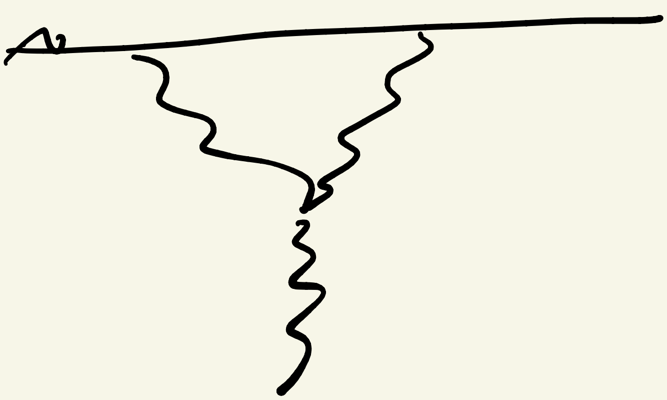
summon large log

$$\boxed{\frac{1}{\alpha(E_2)} = \frac{1}{\alpha(E_1)} + \frac{b}{2\pi} \ln E_2/E_1}$$

· valid for $\alpha \ll 1$ $(\alpha \leq 1/10)$

$$b = \frac{11}{3} T_{GB} - \frac{2}{3} T_F - \frac{1}{3} T_S$$

gauge bosons (YM)   fermion   scalar



$+ \; - - - -$

$$\boxed{\delta_{ab}\, T(R) = \mathrm{Tr}\, T_a(R)\, T_b(R)}$$

$\boxed{\text{particle mass: } E \geq m}$

$E < m \Rightarrow$ stop heavy

decoupling theorem:

heavy particles ($m \gg E$)

= decouple $\Longleftrightarrow$

effects $\propto (E/m)^n$

$\boxed{\text{Appelquist, Carrazone '75}}$

Computing $T(R)$

Fundemental: $\quad T(F) = \frac{1}{2}$

$$\Leftrightarrow \quad SU(2): \quad T_a = \sigma_a/2$$

$$\mathrm{Tr}\, T_a T_b = \frac{1}{4} \mathrm{Tr}\, \sigma_a \sigma_b$$

$$= \frac{1}{4} \cdot 2\, \delta_{ab} = \frac{1}{2} \delta_{ab}$$

$$\boxed{\begin{gathered} SU(2) \subseteq SU(N) \\ \Downarrow \\ T(F)\Big|_N = \frac{1}{2} \end{gathered}}$$

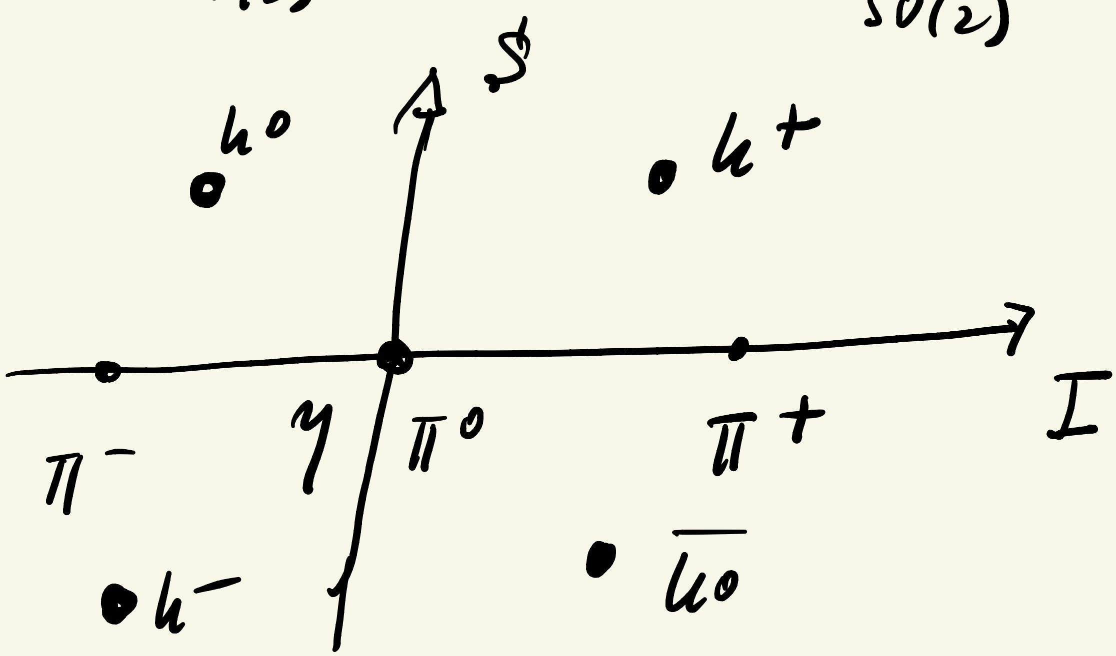
gauge boson = adjoint repr.

. $T(\text{Adjoint}) = ?$ SU(N)

<u>SU(2)</u> $T_2(A) = 2$

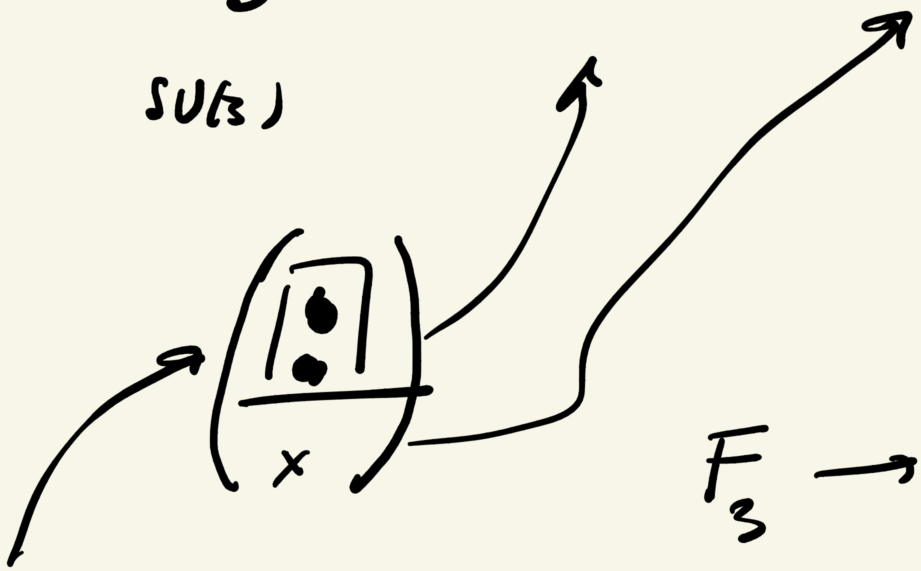
$$T_3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

<u>SU(3)</u> $\underset{SU(3)}{8} = \underbrace{3 + 2 + \bar{2} + 1}_{SU(2)}$



$$3 = 2 + 1$$

SU(3)



SU(2)

$$F_3 \longrightarrow U_3 F_3$$

$$U_3 U_3^\dagger = 1, \quad \det U_3 = 1$$

$$U_3 = \begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U_2 = e^{i\theta_a \sigma_a/2} \quad (SU(2))$$

- $A \longrightarrow U A U^\dagger$ (Adjoint)

$$\Updownarrow$$

$$\boxed{A \simeq F \times \bar{F}}$$

$$\mathrm{Tr}\, A = 0$$

$SU(3)$:

$$8 = \overbrace{3 \times \bar{3}}^{\sim \text{ Adjoint of } SU(3)} =$$

$$= \underbrace{(2+1) \times (\bar{2}+1)}$$

$$= \underbrace{\underset{\substack{\uparrow \\ \text{adjoint of } SU(2)}}{3} + 2 + \bar{2} + 1}_{\mathrm{Tr}\, 8 = 0} + 1$$

$$T(8) = T(3) + T(2) + T(\bar{2}) + T(1)$$

$$= 2 + \frac{1}{2} + \frac{1}{2} + 0$$

$$= 3$$

$$T_3(A) = 3$$

$\Downarrow$ guess

$$\boxed{T_N(A) \stackrel{?}{=} N}$$

Induction

assume: $T_N(A) = N$

$\Rightarrow$ Prove: $T_{N+1}(A) = N+1$

must

$\Downarrow$

Proof:

$$A(N+1) = (N+1) \otimes \overline{(N+1)}$$

$$= (N+1) \otimes (\overline{N}+1) =$$

$$= \underbrace{N \otimes \overline{N}}_{A(N)} + N + \overline{N} + 1$$

$$\Downarrow \qquad \begin{array}{c} T(F) \\ \equiv \end{array}$$

$$T_{N+1}(A) = T_N(A) + T(N) + T(\overline{N})$$

$$= N + \frac{1}{2} + \frac{1}{2}$$

assume

$$= N+1$$

Q.E.D.

$\Downarrow$ $SU(N)$

complex

$$\boxed{b_N = \frac{11}{3} \cdot N - \frac{4}{3} T_F - \frac{1}{3} T_S}$$

if $b > 0$ (g.b. dominate)

$\Downarrow$

$$\frac{1}{\alpha(E_2)} > \frac{1}{\alpha(E_1)} \Longrightarrow$$

$$\boxed{\alpha(E_2) < \alpha(E_1)}$$

Asymptotic Freedom (AF)

apply to SM !.

$b_3 > 0, \quad b_2 > 0$

$(b_1 < 0)$

$$\boxed{\bullet \; T_N(A) = N, \quad N \geq 2}$$

$T_2 = 0 \iff$ photon does not interact itself

$SU(5)$ GUT

$SU(5) \supseteq SU(3) \times SU(2) \times U(1)$

$\boxed{\text{follow } \alpha_3, \alpha_2, \alpha_1}$

$$\frac{1}{\alpha_U} \equiv \frac{1}{\alpha_3(M_X)} = \frac{1}{\alpha_3(M_W)} + \frac{b_3}{2\pi} \ln \frac{M_X}{M_W}$$

$$\frac{1}{\alpha_U} \equiv \frac{1}{\alpha_2(M_X)} = \frac{1}{\alpha_2(M_W)} + \frac{b_2}{2\pi} \ln \frac{M_X}{M_W}$$

$$\frac{1}{\alpha_U} \equiv \frac{1}{\alpha_1(M_X)} = \frac{1}{\alpha_1(M_W)} + \frac{b_1}{2\pi} \ln \frac{M_X}{M_W}$$

$\boxed{\text{unification assumption}}$

$\boxed{\text{Georgi, Quinn, Weinberg '74}}$

UNIF $\Downarrow$

$$\frac{1}{\alpha_2(M_W)} - \frac{1}{\alpha_3(M_W)} = (b_3 - b_2) \ln \frac{M_X}{M_W} \quad (1)$$

$$\frac{1}{\alpha_1(M_W)} - \frac{1}{\alpha_2(M_W)} = (b_2 - b_1) \ln \frac{M_X}{M_W} \quad (2)$$

$$\boxed{\begin{array}{ll} \alpha_2^{-1}(M_W) \simeq 30 & (27.?) \\ \alpha_3^{-1}(M_W) \simeq 10 & (8.44) \\ \alpha_{em}^{-1}(M_W) \simeq 130 & (128) \\ \text{perturbative!} & \end{array}}$$

n_g = # of generations

$$b_3 = \frac{11}{3}\cdot 3 - \frac{2}{3}\left[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right] n_g$$

$$5_F = \begin{pmatrix} d \\ e^c \\ \nu^c \end{pmatrix}_R \qquad 10_F = \begin{pmatrix} u^c & u & d \\ & & e^c \end{pmatrix}_L$$

$$5_H = \begin{pmatrix} T \\ D = \Phi \end{pmatrix}$$ color triplet

p decay $\Rightarrow$ $\boxed{m_T \simeq M_X}$

$\boxed{\text{T does not run}}$

$\Downarrow$

$$\boxed{b_3 = \frac{33}{3} - \frac{4}{3} n_g} = 7 \quad (n_g = 3)$$

quark doublet

$$b_2 = \frac{11}{3}\cdot 2 - \frac{2}{3}\left[\frac{1}{2} + \frac{1}{2}\cdot 3\right] n_g - \frac{1}{3}\cdot\frac{1}{2}$$

lepton doublet

Higgs

$$\boxed{b_2 = \frac{22}{3} - \frac{4}{3} n_g - \frac{1}{6}} \boxed{= \frac{19}{6} \quad (n_g = 3)}$$

$$\Downarrow$$

Higgs

$$b_3 - b_2 = \frac{33-22}{3} - \underbrace{0}_{\text{fermions}} + \frac{1}{6}$$

$$= \frac{11}{3} + \frac{1}{6} = \frac{23}{6}$$

Why do fermions cancel?

$$\ln \frac{M_X}{M_W} \longleftarrow \boxed{\text{split of masses in a representation}}$$

$$5_H = \begin{pmatrix} T \\ D \end{pmatrix} \begin{matrix} \leftarrow M_X \\ \leftarrow M_W \end{matrix} \Big\} \text{ split}$$

$$24_V = (X, Y) + (\bar{X}, \bar{Y}) \simeq M_X$$

$$+ (W, Z, \text{gluons}) \leq M_W$$

$$5_F = \begin{pmatrix} d \\ e^c \\ \nu^c \end{pmatrix}_R \leftarrow \text{light } (\leq M_W)$$

$$\boxed{\text{NO split}}$$

$$\Downarrow \quad {}^{\sim 27.?}$$

$$2\pi\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_3}\right)\Bigg|_{M_W} = \frac{23}{6}\ln\frac{M_X^{(2-3)}}{M_W}$$

8.5

$$\Rightarrow \boxed{M_X^{(2-3)} \simeq 10^{17}\,\mathrm{GeV}}$$

$$\Uparrow$$

$$M_{p\,stability} \leq M_X \ll M_{pe}$$

$$(\sim 10^{16}\,\mathrm{GeV})$$

2-3 meeting =

= beautiful

1 - 2 meeting

$b_1 = ?$

$\alpha_1 \longleftrightarrow T_1$ (gen. of $SU(5)$)

$\therefore \; \mathrm{Tr}\, T_1^2 = \frac{1}{2}$

$$\boxed{\Rightarrow \; \alpha_1 \neq \alpha_Y}$$

$Q = T_3 + \frac{Y}{2}$

$$5_F = \begin{pmatrix} d \\ e^c \\ \begin{pmatrix} \nu^c \end{pmatrix} \end{pmatrix}_R$$

3 colors

$$\mathrm{Tr}\left(\frac{Y}{2}\right)^2\Big|_{5_F} = \left[\frac{1}{9}\cdot 3 + \frac{1}{4}\cdot 2\right]$$

$$= 5/6$$

$$\Rightarrow \boxed{\frac{y}{2} = \sqrt{\frac{5}{3}}\, T_1}$$

$$D_\mu = \cdots - i g' \frac{y}{2} B_\mu$$

$$= \cdots - i g_1 T_1 B_\mu$$

$$\Rightarrow \boxed{\begin{array}{ccc} g' & = \sqrt{\frac{3}{5}} & g_1 \\ \uparrow & & \uparrow \\ \frac{y}{2} & & T_1 \end{array}}$$

$$\Downarrow$$

$$\boxed{\frac{1}{\alpha_1} = \frac{3}{5} \frac{1}{\alpha'} \equiv \frac{3}{5} \frac{1}{\alpha_y}}$$

$$b_1 = \frac{3}{5} b' \equiv \frac{3}{5} b_Y$$

$$b_Y = \underbrace{\frac{4}{3}\cdot 0}_{\substack{U(1)\ \leftarrow\ \text{no} \\ \text{gauge boson} \\ \text{self. int.}}} - \frac{2}{3}\cdot 4\left[\underset{\uparrow\, 5_F}{\frac{5}{6}} + \underbrace{1 + 4/3 + \frac{1}{6}}_{\overset{\uparrow}{10_F}}\right]$$

$10/3$

$$-\frac{1}{3}\cdot\frac{1}{2}$$

$\uparrow$ Higgs

$$10_F = (u^c + \overbrace{u + d}^{} + e^c)$$

$$\mathrm{Tr}\left(\frac{Y}{2}\right)^2\Big|_{10_F} = \frac{4}{9}\cdot 3 + \frac{1}{36}\cdot 3\cdot 2 + 1$$

$$= 4/3 + 1/6 + 1$$

$$\Downarrow$$

$$b_1 = 0 - \frac{2}{3}\cdot\cancel{\frac{3}{5}}\cdot\cancel{\frac{\overset{2}{10}}{3}}\,y - \frac{1}{2}\cdot\frac{\cancel{3}}{5}\,\frac{1}{\cancel{3}}$$

$$\boxed{b_1 = -\frac{4}{3}y - \frac{1}{10}}$$

$$\Rightarrow\quad b_2 - b_1 = \frac{22}{3} \cancel{- \frac{4}{3}y} - \frac{1}{6}$$

$$\cancel{+\frac{4}{3}y} + \frac{1}{10}$$

$$= \frac{22}{3} - \frac{1}{15} = \frac{109}{15}$$

$$\Downarrow$$

$$2\pi\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right)\Big|_{M_W} = \frac{109}{15}\ln\frac{M_X^{(1-2)}}{M_W}$$

?

$$\frac{1}{\alpha_1} = \frac{3}{5}\frac{1}{\alpha_Y} = \frac{3}{5}\cos^2\theta_W\frac{1}{\alpha_{em}}$$

$$\alpha_{em} = \sin^2\theta_W\,\alpha_2 = \cos^2\theta_W\,\alpha_Y$$

$$\Downarrow$$

$$\frac{1}{\alpha_1} \cong \frac{3}{5}\,\frac{3}{4}\,\frac{1}{\alpha_{em}} \simeq 9/20\,\frac{1}{\alpha_{em}} \simeq \frac{1}{2}\frac{1}{\alpha_{em}}$$

$$\Rightarrow \boxed{M_X^{(1\to2)} \simeq 10^{13}\,\mathrm{GeV}}$$

TODAY: $M_X^{(2\text{-}3)} = 10^{17}$ GeV ←

$M_X^{(1\text{-}2)} = 10^{13}$ GeV

Disaster!

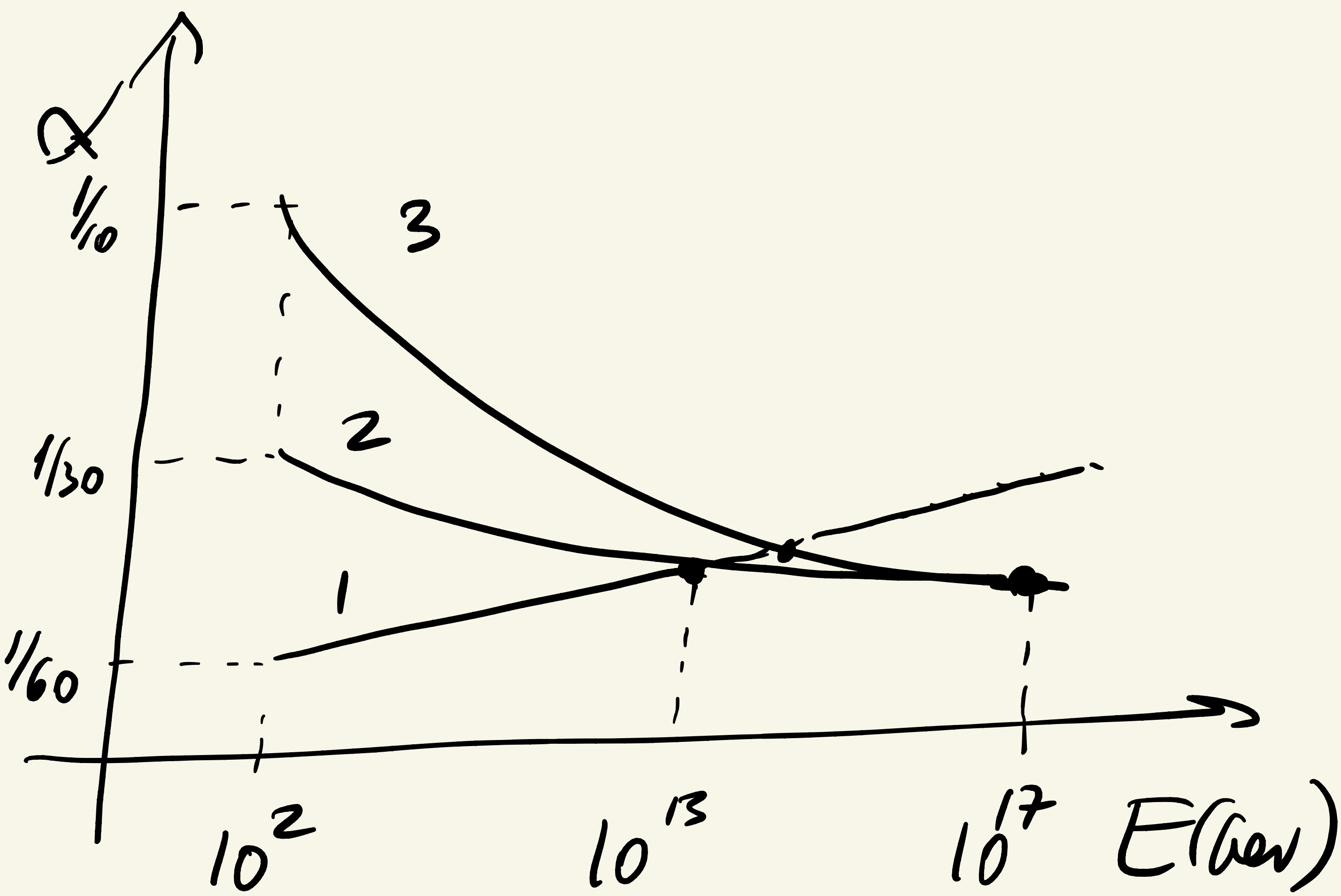
often called M_{GUT}

if: $M_{GUT} \equiv M_X^{12} \Rightarrow$

bad
luggage

M_{GUT} = too low

$\Rightarrow$ p already decayed



$\boxed{\text{NO unification}}$

$\boxed{1974}$

$\theta_W \Leftarrow$ smaller

$\alpha_3 \Leftarrow$ smaller

GQW

$\Downarrow$

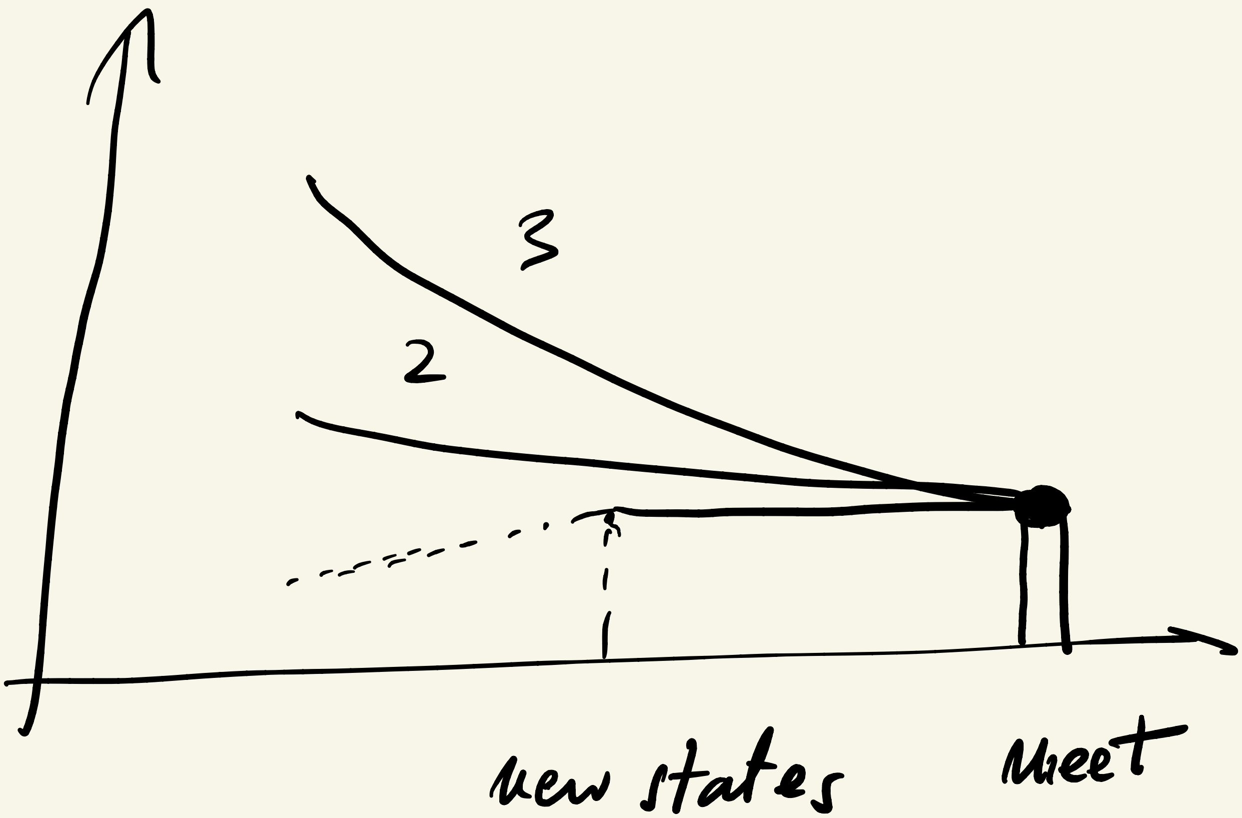
$$\Rightarrow \boxed{M_X^{1-2} = M_X^{2-3} \simeq 10^{15}\,\text{GeV} \quad \Rightarrow \quad \tau_p \simeq 10^{30}\,\text{yr}}$$

BUT

what about 24_H?

Colw 8 ; weak 3

$$\text{GUT} \Longleftrightarrow \underbrace{M_X^{12} = M_X^{23}}_{\text{all meet at point}}$$



- $y_t(M_W) = 1 \Rightarrow y_t(M_X) = ?$
- $\lambda(M_W) = ? \Rightarrow \lambda(M_X) = \text{large?}$

- iff GUT exists $\Leftarrow$

(i) α_i meet $(i = 1, 2, 3)$

(ii) $\lambda^2/4\pi\,(M_X) < 1$

$y_t^2/4\pi\,(M_X) < 1$