Neutrino Mass and Grand Unification

Lecture IX

16/11/2021

LMU
Fall 2021

SU(5) GUT: Interactions and Masses

$\Sigma \to U \Sigma U^{+}$ = adjoint Higgs repr.

$\Sigma^{+} = \Sigma, \; Tr\, \Sigma = 0$

$$\underbrace{\Sigma_0 = v\,(1,1,1,-3/2,-3/2)}$$

vev = vacuum expectation value

$$SU(5) \xrightarrow[\Sigma_0]{} SU(3)_c \times SU(2)_L \times U(1)_Y$$

$$V(\Sigma) = -\frac{\mu_\Sigma^2}{2} \mathrm{Tr}\,\Sigma^2 + \frac{a}{4} (\mathrm{Tr}\,\Sigma^2)^2 + \frac{b}{2} \mathrm{Tr}\,\Sigma^4 \qquad (\Sigma \to -\Sigma)$$

no cubic for simplicity

$\Downarrow$

Guth, Weinberg
1982
full = + cubic

$$\boxed{2\mu^2 = (15a + 7b) v^2}$$

extremum condition

- gauge boson masses

$$\mathcal{L}_{kin}(\Sigma) = \mathrm{Tr}(D_\mu \Sigma)(D^\mu \Sigma)$$

$$D_\mu \Sigma = \partial_\mu - ig\, T_a A_\mu^a$$

$$T_a = \frac{\lambda_a}{2} \qquad a = 1, \dots, 24$$

$$\mathrm{Tr}\, T_a T_b = \frac{1}{2} \delta_{ab}$$

1, 2, ..., 8 — $SU(3)_c$

9, 10, 11, $\boxed{12}$ — $SU(2)_L \times U(1)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}, \quad \lambda_2 \; (1 \to -i)$$

example

$\lambda_{13}, \ldots, 24 \longrightarrow$ 12 new generators

$\alpha = 1,2,3$

$$5_F = \begin{pmatrix} d^\alpha \\ e^c \\ -\nu^c \end{pmatrix}_R \qquad 5_H = \begin{pmatrix} ? \end{pmatrix}$$

$$\boxed{Q_{em} = \left(-1/3, -1/3, -1/3; \ 1, 0\right)}$$

$$\underbrace{\frac{Y}{2} = \left(-1/3, -1/3, -1/3; \ 1/2, 1/2\right)}_{\text{\underline{\underline{not}} normalised !}}$$

$\Downarrow$

$$T_1 (Y/2)^2 = 1/9 \cdot 3 + 1/4 \cdot 2 = 5/6$$

$$\Rightarrow \boxed{Y/2 = \sqrt{5/3}\, T_1} \qquad (T_1 = T_{12})$$

$$T_r\, T_1^2 = \frac{1}{2}$$

U(1) generator = normalised

$$D_\mu = \partial_\mu - i g\, T_1\, A_{12} - \cdots$$

$$= \partial_\mu - i g' \frac{Y}{2} B_\mu + \cdots$$

$$\Rightarrow \boxed{A_{12} = B}$$

$$\Downarrow$$

$$g' \frac{Y}{2} = g\, T_1 \qquad (T_1 = T_{12})$$

$$g' \sqrt{\frac{5}{3}} \cancel{T_1} = g\, \cancel{T_1}$$

$$\Rightarrow \boxed{g' = \sqrt{3/5}\; g}$$

$$\tan^2\theta_W = g'^2/g^2 = 3/5$$

weak mixing angle

$$\Rightarrow \boxed{\underset{(M_{GUT})}{\sin^2\theta_W} = \frac{\tan^2\theta_W}{1+\tan^2\theta_W} = 3/8}$$

SM: $\sin^2\theta_W = 0.23 \; (M_W)$

at this case

$\boxed{\Rightarrow \text{ stay tuned}}$

Reminder: $W^\pm = \dfrac{A_9 \mp A_{10}}{\sqrt{2}}$

Expect: $X^\pm = \dfrac{A_{13} \mp A_{14}}{\sqrt{2}}$

keep in mind

$\Downarrow$

$$A_\mu = T_a A_\mu^a$$

$$\rightarrow \underbrace{U A_\mu U^+}_{\text{adjoint repr.}} + \underbrace{i/g (\partial_\mu U) U^+}_{\text{gauge}}$$

$$\boxed{A_\mu \simeq 5 \times \bar{5}}$$

$\Downarrow$ $T_a = \lambda_a/2$

$\Downarrow$

$$A_\mu = \frac{1}{2}\left(\begin{array}{c|c} \begin{matrix}\text{gluons}\\ (\lambda_a)\end{matrix} & A_{13} - i A_{14} \\ \hline \sqrt{2} X_1 = A_{13} + i A_{14} & \begin{matrix}SU(2)\\ \text{weak}\\ (\lambda_i)\end{matrix} \end{array}\right) + \frac{1}{2}\lambda_{12} B_\mu$$

$$\left(T_1 = \frac{\lambda_{12}}{2}\right) \qquad \|$$

$$\frac{1}{2}\sqrt{3/5}\ \text{diag}\left(-1/3, -1/3, -1/3, 1/2, 1/2\right) B_\mu$$

$$A \longrightarrow U A U^+ + ---$$

$$\Downarrow$$

$$\hat{T}_a A = [T_a, A]$$

$$\Uparrow$$

$$U = e^{i\theta_a T_a} = 1 + i\theta_a T_a + \cdots$$

$$U^+ = 1 - i\theta_a T_a - - - -$$

$$\Downarrow$$

$$\hat{Q}_{em} A = [Q_{em}, A]$$

$$\Downarrow$$

$$\boxed{Q A_{ij} = (q_i - q_j)}$$

$q_1 = 1/3, \; q_2 = -1/3, \; q_3 = -1/3, \; - -$

$$\Downarrow$$

$$\frac{A_{13} + i A_{14}}{\sqrt{2}} = X_1$$

$$\boxed{Q_{em}(X_1) = \frac{1}{3} + 1 = 4/3}$$

eFc

$$\Downarrow$$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} & \text{glue} & & \bar{X} & \bar{Y} \\ & & & \bar{X} & \bar{Y} \\ & & & \bar{X} & \bar{Y} \\ X & X & X & \frac{1}{\sqrt{2}} A_3 & W^+ \\ Y & Y & Y & W^- & -\frac{1}{\sqrt{2}} W_3 \end{pmatrix}$$

new

$+ \; U(1)$ piece

$$Q(\bar{x}_1) = -\frac{1}{3} - 1 = -4/3$$

$$S = \begin{pmatrix} d \\ e^+ \\ -\nu c \end{pmatrix} \rightarrow (-1/ \text{in } \bar{5}$$

$$Q(x_1) = 4/3$$

$$Q(\bar{y}_1) = -1/3 + 0 = -1/3$$

$$\Downarrow$$

$$\boxed{Q(x) = 4/3, \quad Q(y) = 1/3}$$

(check agreement w/ int. of lecture VIII)

$$\Downarrow$$

$$D_\mu \Sigma_0 = \partial_\mu - ig\,[T_a, \Sigma_0]\, A_\mu^a$$

$$\Sigma_0 = v\,(1, 1, 1, -3/2, -3/2)$$

$$\Rightarrow [T_{color}\ (8), \Sigma_0] = 0$$

$$[T_{weak}\ (3), \Sigma_0] = 0$$

$$[T_1\ (1), \Sigma_0] = 0$$

$$\Downarrow$$

$$\boxed{m_{gluon} = m_W = m_Z = m_A = 0}$$

$$\bar{X}_1 \longleftrightarrow \bar{T}_x \equiv \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$[\bar{T}_x, \Sigma_0] \neq 0$$

$$[\bar{T}_x, \Sigma_0] = \left[\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2 & \\ & & & & -3/2 \end{pmatrix} \right]$$

$$= \begin{bmatrix} 0 & 0 & 0 & -5/2 & 0 \\ & 0 & & & 0 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$\Downarrow$$

$$[T_{X_1}, \Sigma_0] = \begin{bmatrix} 0 & \\ 0 & 0 \\ 0 & \\ 5/2 & 0 \\ 0 & \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$\Downarrow$$

$$\mathrm{Tr}\,(D_\mu \Sigma_0)(D^\mu \Sigma_0) =$$

$$= g^2 \, \frac{25}{4} v^2 \, X_1 \bar{X}_1 \cdot \frac{1}{2}$$

normalisation of X

$$\Downarrow$$

$$\boxed{M_X^{\,2} = \frac{25}{8} g^2 v^2} \leftarrow (1,2,3)$$

color

$$\Downarrow$$

Φ $SU(2)_L$

$$M_Y^2 = \frac{25}{8} g^2 v^2 \leftarrow (1,2,3 \text{ color})$$

$$\boxed{M_X = M_Y = M_{GUT}}$$

Consistent gauge boson mass spectrum

$\Downarrow$

Prove that Σ_0 = minimum

Proof:

$\Sigma \leftrightarrow 24$ states

$\Sigma = T_a \varphi_a \qquad a = 1, \ldots, 24$

$\Sigma_0 =$ minimum $\Leftrightarrow$ eigenvalues of 2^{nd} derivative > 0

$\Leftrightarrow$ $(\text{masses})^2$ of particles in Σ are positive

$$\Sigma = \left(\begin{array}{ccc|cc} & & & \bar{\Sigma}_x & \bar{\Sigma}_y \\ & \text{scalar "gluons"} & & \bar{\Sigma}_x & \bar{\Sigma}_y \\ & & & \bar{\Sigma}_x & \bar{\Sigma}_y \\ \hline \Sigma_x & \Sigma_x & \Sigma_x & \text{scalar} & \\ \Sigma_y & \Sigma_y & \Sigma_y & \text{"weak bosons"} & \end{array}\right) +$$

$$+ (v + \varphi_0)\, \mathrm{diag}(1\ 1\ 1\ {-\tfrac{3}{2}}\ {-\tfrac{3}{2}})$$

normalise

$\Rightarrow$ must show $(\mathrm{m})^2$ of

scalar gluons, --- > 0

but

$$\boxed{\begin{array}{c} SU(3) \times SU(2) \times U(1) \\ \text{symmetry} \end{array}}$$

$$\Downarrow$$

m_8 = all 8 scalar "gluons"

m_3 = all 3 weak "bosons"

m_0 = Higgs (OUT)

$$\boxed{\text{what about } \Sigma_X, \Sigma_Y \text{ states?}}$$

eaten by X, Y!

$\Downarrow$

① unitary gauge

$\Rightarrow$ gauge away Σ_x, Σ_y

$\Rightarrow$ $\boxed{\text{gone}}$

② ren. gauge

$\Rightarrow$ must include them

$$\underbrace{m_{\Sigma_x} = m_{\Sigma_y}}_{SU(2)} = ?$$

↑ compute

$m_8 = ?$

$$\boxed{m_8 \equiv m\,\varphi_8}$$

$$\Sigma = \Sigma_0 + \frac{1}{2}\begin{pmatrix} \varphi_8 & & & & \\ & -\varphi_8 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

$$= \mathrm{diag}\,\big(v + \tfrac{1}{2}\varphi_8,\ v - \tfrac{1}{2}\varphi_8,\ v,\ -\tfrac{3}{2}v,\ -\tfrac{3}{2}v\big)$$

$$\mathrm{Tr}\,\Sigma^2 = \Big(\frac{15}{2}v^2 + \underline{\frac{1}{2}\varphi_8^2}\Big)$$

$$(\mathrm{Tr}\,\Sigma^2)^2 \longrightarrow \frac{15}{2}\cdot\cancel{\frac{1}{2}\cdot 2}\,v^2\varphi_8^2$$

$$\mathrm{Tr}\,\Sigma^4 \longrightarrow 6\,v^2\,\frac{1}{4}\,\varphi_8^2\cdot 2$$

$$\Downarrow$$

$$V = -\frac{\mu_\Sigma^2}{2} \frac{1}{2}\varphi_f^2 + \frac{a}{9}\frac{15}{2}v^2\varphi_f^2$$

$$+ \frac{b}{2} 3 \varphi_f^2$$

$$\Downarrow \quad 2\mu_\Sigma^2 = (15a + 7b)v^2$$

$$m_f^2 = \frac{5b}{4}v^2 > 0$$

$$m_3^2 = 5bv^2 > 0$$

$$m_0^2 = (15a + 7b)v^2 = 2\mu_\Sigma^2$$

$$> 0$$

Prove !!!

$$m_{\Sigma_x} = m_{\Sigma_y} = ?$$

scalar fluons

(a) $\Sigma = \Sigma_0 + \operatorname{diag}\left(\frac{1}{2}\varphi_8, -\frac{1}{2}\varphi_8, 0, \ldots, 0\right)$

field $\Leftrightarrow \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix}$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

$\updownarrow$

φ_8'

(b)

$$\Sigma = \Sigma_0 + \frac{1}{2}\lambda_8 \, \varphi_8'$$

$$\Rightarrow \boxed{m_{8'} = m_8} \quad \boxed{\text{Prove!}}$$

$m_3 = ?$

$$\Sigma = \Sigma_0 + \mathrm{diag}\left(0, 0, 0, \tfrac{1}{2}\varphi_3, -\tfrac{1}{2}\varphi_3\right)$$

$$\Rightarrow \boxed{m_{\varphi_3} = m_3}$$

$\boxed{\text{Comment on } m_0}$

$$\boxed{m_0^2 = 2\mu^2}$$

step back: SM $\quad \phi = $ doublet

$$V = -\frac{\mu^2}{2}\phi^\dagger\phi + \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$$

$$\Rightarrow \boxed{m_h^2 = 2\mu^2}$$

$$\phi_{\text{un}} = \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$\boxed{\text{one more step back}}$

$$V(\varphi) = -\mu^2/2\, \varphi^2 + \frac{\lambda}{4}\varphi^4$$

$\varphi \in \mathbb{R}$ $\quad (D:\ \varphi \to -\varphi)$

$$\varphi = v + h$$

$\Rightarrow \boxed{\lambda v^2 = \mu^2}$ extremum

$$V(\varphi) = -\mu^2/2\,(v^2 + h^2 + 2vh) + \frac{\lambda}{4}(v^4 + h^4 + 6v^2h^2 + 4vh^3 + 4v^3h)$$

$$= \underbrace{\left(-\mu^2/2 + \frac{6\lambda}{4}v^2\right)h^2}_{\parallel} + \frac{\lambda}{4}h^4 + \lambda v h^3 + \ldots$$

$$\frac{2\mu^2}{2}v^2h^2 \Longrightarrow \boxed{\mu^2 > 0 \Longleftarrow \lambda > 0}$$

$$\boxed{m_h^2 = 2\mu^2}$$

<u>normalisation</u> $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2$

$$\Sigma_0 = \text{minimum}$$

$$\text{iff} \quad (15a + 7b) > 0$$

$$b > 0$$

$\mu_\Sigma^2 > 0$

- $\boxed{\text{HW}}$ $\boxed{X, Y \text{ int.}}$

$$5_F = \begin{pmatrix} d \\ e^c \\ -\nu^c \end{pmatrix}_R$$

$$10_F = \begin{pmatrix} u^c & u & d \\ \hline & 0 & e^c \\ & & 0 \end{pmatrix}_L$$

$\Uparrow$

$$\overline{5_F}\,\gamma^\mu D_\mu 5_F + \overline{10_F}\,\gamma^\mu D_\mu 10_F$$

Tr = Trace

$$\boxed{\hat{T}10 = T10 + 10\,T^T}$$