LMU GUT Course
Fall 2020

Lecture V

17/11/2020

- monopole

$\exists \quad q_e g_m = 2\pi n$ (Dirac)

$\boxed{\text{charge is quantized}}$

- theory with quantized charge

$SU(2)$

$Q = T_3$

$q = n\frac{1}{2} \quad (\pm)$

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu D_\mu \Psi + \frac{1}{2}(D_\mu \Sigma)(D^\mu \Sigma)$$

$$- V(\Sigma) - \frac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu a} \quad (1)$$

$$F_{\mu\nu}{}^{3} = \partial_\mu A_\nu{}^{3} - \partial_\nu A_\mu{}^{3} + g\left(A_\mu{}^{1} A_\nu{}^{2} - A_\mu{}^{2} A_\nu{}^{1}\right) \quad (2)$$

$\boxed{\text{vacuum}}$

$$\Sigma_0 = v T_3$$

$$V = \frac{\lambda}{4}\left(2\,\mathrm{Tr}\,\Sigma^2 - v^2\right)^2 \quad (3)$$

$$\mathrm{Tr}\,\Sigma^3 = 0$$

$$\boxed{A_\mu = A_\mu{}^{3}} \text{ (photon)} \quad (4)$$

$\Rightarrow$ $F_{\mu\nu} W^{+\mu} W^{-\nu}$ from (2)

$$W^{\pm} = \frac{A_1 \pm i A_2}{\sqrt{2}} \quad (5)$$

$$\boxed{e = g} \qquad J_\mu^W = \gamma_\mu$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{2 M_W^2} = \frac{e^2}{2 M_W^2} \quad (6)$$

$$\Rightarrow M_W = 80 \text{ GeV}$$

$\Downarrow$ $\boxed{\text{monopoles}}$

$\boxed{\text{Vacuum}}$

$$F_{\mu\nu} = ?$$

$$\boxed{\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \quad (7) \\ &\neq F_{\mu\nu}^3 \end{aligned}}$$

$\boxed{SM}$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2 \sin^2\theta_W}$$

$$J_\mu = J_{\mu L} = \gamma_\mu \frac{1+\gamma_5}{2}$$

$$8 = 2 \cdot (2)^2$$

$$\sin\theta_W = \frac{1}{2}$$

$$(\sin^2\theta_W = 0.23)$$

U(1)

$$\tan\theta_W = g'/g$$

SU(2)

GUT

I need a gauge invariant definition of $F_{\mu\nu}$

$$F_{\mu\nu}^3 = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \; [= F_{\mu\nu}]$$

$$+ \boxed{g\,(A_\mu^1 A_\nu^2 - A_\mu^2 A_\nu^1)} \quad (*)$$

$$\Downarrow$$

$$F_{\mu\nu} = F_{\mu\nu}^3 - \text{''} -$$

$$\| $$

$$f(\phi_a^0)$$

$$\Sigma = T_a \phi_a$$

$$\Downarrow$$

$$\Sigma_0 = T_a \phi_a^0$$

$$\phi_a^0 = v\,\delta_{a3} \Rightarrow \text{above}$$

$$\boxed{F_{\mu\nu} = F_{\mu\nu}^a \frac{\phi_a^0}{v} - (*)}$$

gauge invariant form

(8)

$$-\frac{1}{g}\epsilon^{abc}\,\frac{\phi_0^a (D_\mu \phi_0)^b (D_\nu \phi_0^c)}{v^3}$$

$$D_\mu = \partial_\mu - i g A T \quad \boxed{\text{check}}$$

Exercise: Rewrite as a f-n of Σ_0

*~~xx~~

$$D_\mu \phi_0 = \cancel{\partial_\mu \phi_0} - \cdots -$$

$\Downarrow$

generalise to any value of Σ_0 (ϕ_0^a) or Σ

$\Downarrow$ $\boxed{\text{monopole}}$

$\boxed{\text{Finite energy, } \underline{\text{static}} \text{ solution}}$

$$\boxed{\partial_0 = 0 , \quad A_0^a = 0}$$

$$E = \int d^3x \left[|D_i \Sigma|^2 \frac{1}{2} + V(\Sigma) + \frac{1}{2} (B^a)^2 \right]$$

$\downarrow \infty$ (9)

0

$$\boxed{\Sigma_\infty \to \Sigma_0 \quad (M_0)} \qquad (10)$$

$$\Rightarrow \boxed{\varphi_a^\infty \varphi_a^\infty = v^2} \qquad (11)$$

$$\Rightarrow \varphi_a(r \to 0) \to 0$$

$$M_\infty = S_2 \ (r \to \infty)$$

$$M_0 = S_2 \ (\varphi_a^0 \varphi_a^0 = v^2)$$

$$\boxed{\begin{aligned} \phi_a^3 &= v \cos\theta \\ \phi_a^1 &= v \sin\theta \cos n\varphi \\ \phi_a^2 &= v \sin\theta \sin n\varphi \end{aligned}} \qquad \begin{aligned} \varphi &\in [0, 2\pi] \\ \theta &\in [0, \pi] \end{aligned} \qquad (12)$$

Last time: $\boxed{n = 1}$

$$\Rightarrow \varphi_a^\infty = v \frac{x^a}{r}$$

$$k_\mu = \frac{1}{2e} \epsilon^{\mu\nu\alpha\beta} \epsilon^{abc}$$ (13)

$$\partial_\nu \phi_\infty^a \partial_\alpha \phi_\infty^b \partial_\beta \phi_\infty^c \frac{1}{v^3}$$

$$\boxed{\partial^\mu k_\mu = 0}$$

$\Downarrow$

$$k_0 = \frac{1}{2} \epsilon^{ijk} \epsilon^{abc} \frac{\partial_i \phi_\infty^a \partial_j \phi_\infty^b \partial_k \phi_\infty^c}{v^3}$$

$$Q = \int d^3x \, k_0 = \text{number}$$

$$= \frac{1}{2} \frac{\epsilon^{ijk} \epsilon_{abc}}{v^3} \int \partial_i \left(\phi_\infty^a \partial_j \phi_\infty^b \partial_k \phi_\infty^c \right)$$

$$- \quad 0 \quad]$$

$\Downarrow$

$$\left[Q = \frac{1}{2e}\int d^3x\, \partial_i \frac{\epsilon^{ijk}\epsilon_{abc}\,\phi_\infty^a \partial_j \phi_\infty^b \partial_k \phi_\infty^c}{v^3} \right.$$

$$= \frac{1}{2e}\int dS_i\, \epsilon_{ijk}\,\epsilon_{abc} \frac{\phi_\infty^a \partial_j \phi_\infty^b \partial_k \phi_\infty^c}{v^3}$$

$$= \frac{1}{2e}\int d\theta\, d\varphi\, \underbrace{\epsilon_{abc}}_{\epsilon_{\alpha\beta}}\, \phi_\infty^a \partial_\alpha \phi_\infty^b \partial_\beta \phi_\infty^c$$

$$\alpha, \beta = (\theta, \varphi)$$

$\epsilon_{\alpha\beta}$ = anti-symmetric 2 tensor

$$\overset{?}{=} \frac{1}{2}\int d\theta\, d\varphi\; 2\,\phi_\infty^3 \left(\partial_\theta \phi_\infty^1 \partial_\varphi \phi_\infty^2 - \partial_\theta \phi_\infty^2 \partial_\varphi \phi_\infty^1 \right)$$

$\propto$ M $\boxed{\text{Homework}}$

$$\phi_\infty' = v \sin\theta \cos(n)\varphi$$

$g = e$

$$F_{\mu\nu} = F^a_{\mu\nu}\frac{\phi^a}{v} - \frac{\epsilon_{abc}}{e v^3}\phi^a (D_\mu\phi)^b (D_\nu\phi)^c$$

$$\Sigma \to \Sigma_\infty \Rightarrow \phi \to \phi_\infty \Rightarrow \boxed{D_\mu \phi_\infty = 0}$$

finite energy

$$\boxed{F_{ij} = F^a_{ij}\frac{\phi^a_\infty}{v}}$$

$n = 1$

$$\phi^a_\infty = v\frac{x^a}{r}, \quad Q = T_a\frac{x_a}{r}$$

$$\Sigma_\infty = T_a v \frac{x_a}{r}$$

$$A_i = A^a_i \frac{x^a}{r}$$

$$F_{ij} = F^a_{ij}\frac{x^a}{r}$$

$\boxed{\text{Connection between } g_m \text{ and the basic SU(2) charge?}}$

Topology = essential

finite energy $\Rightarrow \Sigma_\infty \in \mathcal{M}_0$

maps from $\mathcal{M}_\infty$ to $\mathcal{M}_0$

- $\phi_a^\infty \phi_a^\infty = v^2$
- $D_i \phi_a^\infty = 0 \quad (D_i \Sigma_\infty = 0)$

$\Downarrow$

$$D_i = \partial_i - ig \hat{T}_a A_i^a$$

$$(\hat{T}_a)_{jk} = -i \varepsilon_{ajk}$$

when acting on ϕ^a

$$D_i \Sigma = \partial_i \Sigma - ig [T_a, \Sigma] A_i^a$$

$$T_a \equiv \frac{\sigma_a}{2}$$

$$(D_i \phi_\infty)^a = 0 \Longrightarrow$$

$$A_i^a = a \delta_i^a + b \hat{x}_i \hat{x}_a + \varepsilon_{aij} x_j c$$

$$\Downarrow$$

$$A_i^a = f(\partial_i \phi_a^a)$$

$$F_{ij} = F_{ij}^a \frac{\phi^a_a}{v} = \varepsilon_{ijk} B_k$$

Show

$$\nabla \cdot \vec{B} = 4\pi Q = \frac{4\pi}{e} n$$
$$= g_m$$

$$\int \nabla \cdot \vec{B}\, dV = \int d\vec{s}\, \vec{B} = g_m$$

$$\vec{B} = \frac{g_m}{4\pi r^2} \hat{r}$$

$$Q \propto g_m$$

$$g_m = \frac{4\pi}{e} \qquad (n=1)$$

$q_e = -1$

$$g_m \, q \, e = 2\pi n \qquad \text{(Dirac)}$$

$\Downarrow$

$$\boxed{g_m = \frac{2\pi}{e \, q_{min}} \qquad \text{(minimal)}}$$

Agree (us and Dirac)

(scalars)

I can have fermions which are doublets

$$\Rightarrow q_{min} = \frac{1}{2}$$

$$g_m = \frac{2\pi}{\frac{1}{2} e} = \frac{4\pi}{e} \quad \checkmark$$

checks!

$q_{min} = q_{basic}$

$\boxed{\text{Adjoint}}$ $\quad \Sigma \to U \Sigma U^{+}$

$$\hat{T}_a \Sigma = [T_a, \Sigma]$$

$$(\hat{T}_a)_{ij} = -i\varepsilon_{aij} \quad (\phi^a)$$

$$\Sigma = T_a \phi^a$$

$$T_3 = \left(\begin{array}{cc|c} 0 & -i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$\mathrm{Tr}\, T_3 = 0$

$\det = -1$

$\Downarrow$

eigenvalues $1, -1, 0$

$$T_3 \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = + \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$T_3 \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$T_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$\boxed{\text{Adjoint} = \text{vector of } SO(3)}$

$$\boxed{(\hat{T}_a)_{ij} = -i\,\varepsilon_{aij}}$$

Lecture IV

$$\phi^a_\infty = v\,\frac{x^a}{r} \Rightarrow A^a_i \rightarrow -i\,\varepsilon_{aij}\,\frac{x_j}{r^2}$$

$$D_i \phi_a = 0$$

$$F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g\,\epsilon_{abc} A_i^b A_j^c$$

$$F_{ij}^a =$$

$$\partial_i A_j^a = \partial_i \left(-i\,\varepsilon_{aju} \frac{x_u}{r^2} \right)$$

$$= -i\,\varepsilon_{aju} \left[\frac{\delta_{iu}}{r^2} - 2\,\frac{x_i x_u}{r^4} \right]$$

$$F_{ij} = F_{ij}^a\, \frac{x_a}{r} \qquad \cancel{\varepsilon_{aju}\, x_i x^a x_u}$$

$$\Downarrow$$

$$\boxed{F_{ij} = \varepsilon_{ijk} \frac{x_k}{e r^2}}$$

$$\boxed{\nabla \cdot \vec{B} = \frac{4\pi}{e} Q}$$

$$\vec{B} = \frac{\hat{r}}{e r^2} \qquad (n=1)$$

$$\nabla \cdot \vec{B} = 0 \quad (r \to \infty)$$

$$\Downarrow$$

monopole is inside a core

finite size

$\boxed{\text{energy finiteness}}$

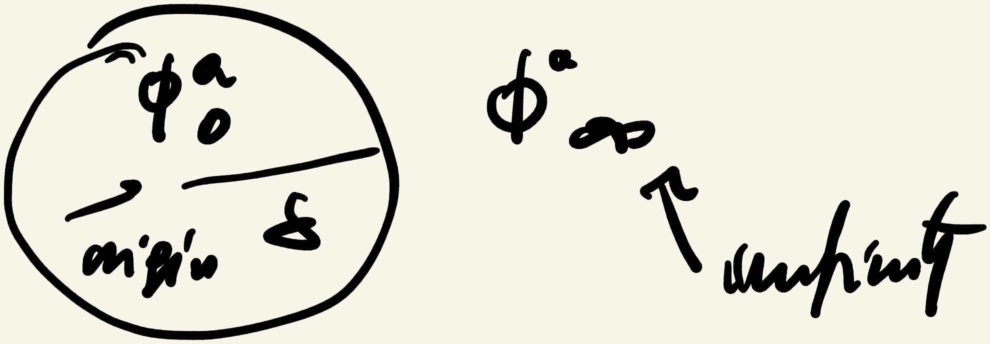
$$\phi_a^{\infty} \xrightarrow[r\to\infty]{} v\,\frac{x^a}{r} \qquad (n=1)$$

$$\Rightarrow \quad \phi_a \xrightarrow[r\to 0]{} 0$$

$$\boxed{\begin{aligned} &\phi_a(\text{monopole}) = v\,\frac{x^a}{r}\,H(evr) \\ &\therefore \quad H(evr) \xrightarrow[r\to\infty]{} 1 \\ &\qquad\; H(evr) \xrightarrow[r\to 0]{} 0 \end{aligned}}$$

$r \to 0$ $\boxed{\phi_a(\text{monopole}) \to 0 \quad \text{maximum!}}$

$$\Rightarrow V(r=0) = \frac{\lambda}{4} v^4$$



$$\boxed{(D_i \phi) \xrightarrow[r \to 0]{} \text{finite}}$$

$$E = \int dV \left[V + |D_i \phi|^2 + \vec{B}^2 \right]$$

$$= \int_0^\delta dV + \int_\delta^\infty$$

$$\downarrow$$

$$\geq \int_0^\delta \underset{\text{(origin)}}{V(\max)} + \int_\delta^\infty \underset{(\infty)}{\vec{B}^2}$$

$$\boxed{E \geq \frac{\lambda}{4} v^4 \delta^3 + \int_{\delta}^{\infty} dr \, \frac{1}{e^2 r^4}}$$

$$(\delta \to 0) \qquad\qquad \int_{\delta}^{\infty} dr \, \frac{1}{e^2 r^4} = \frac{1}{e^2 \delta} \qquad (\delta \to \infty)$$

$$\frac{\partial E}{\partial \delta} = 0 \Rightarrow$$

$$\boxed{\lambda v^4 \delta^2 = \frac{1}{e^2 \delta^2}}$$

• $\lambda = e^2$ exemple

$$\boxed{\delta \simeq \frac{1}{e v} = \frac{1}{M_W}}$$

$(q = e)$

$\lambda = 0$ Problem !?!?

BPS limit

Bogomolny Prasad Sommerfield

$$E = \int_0^{\delta} \left[V + |D_i \phi|^2 + \vec{B}^2 \right] + \cdots$$

$$\lambda = 0 \Longrightarrow \boxed{D_i \phi \simeq e v}$$

$$\Longrightarrow E_{\text{inside}} \simeq e^2 v^2 \delta^3$$

$$\Downarrow$$

$$\boxed{\lambda = 0} \quad \boxed{\delta^{BPS} = \frac{1}{e v}}$$

$$\Downarrow$$

$$M_m = E(t)$$

$$M_W = ev$$
$$= gv$$

$$= g_m v \quad (?)$$

$$= \frac{1}{e} v$$

$$\boxed{M_m / M_W \simeq \frac{1}{e^2} \gtrsim 10}$$

- $$g_m = \frac{4\pi}{e} = \frac{2\pi}{(e/2)}$$