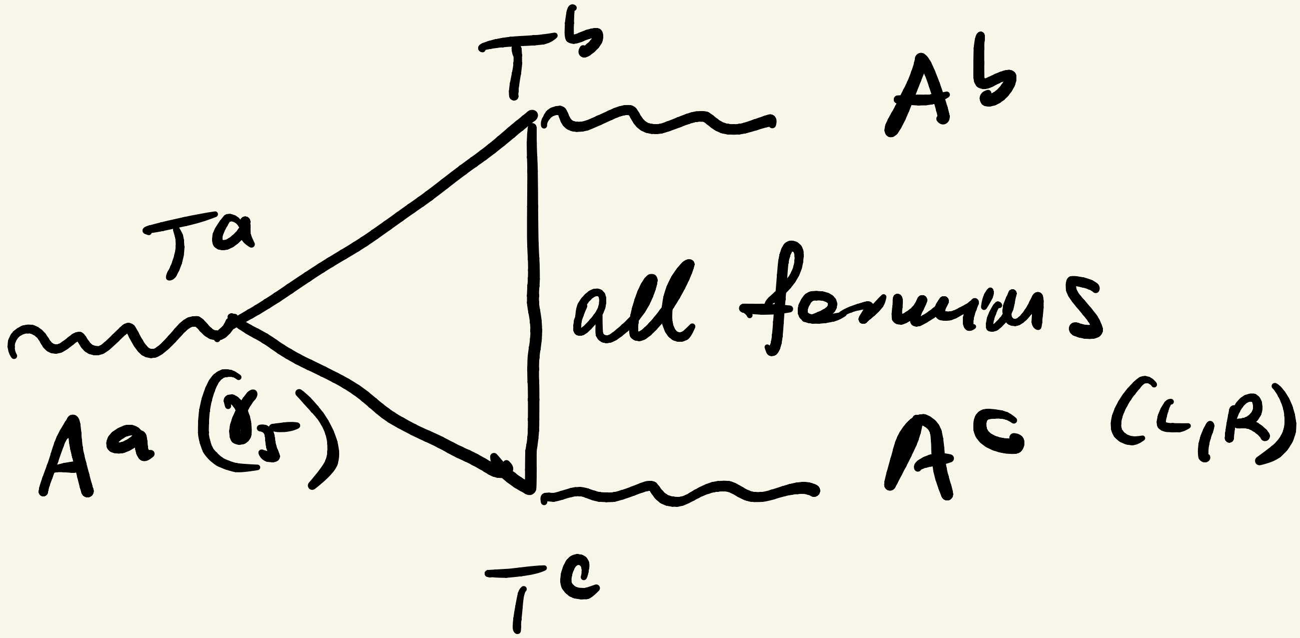
LMU GUT Course
Lecture X

4/12/2020

LMU

Fall 2020

Anomalies



$$A_{abc} \propto Tr\{T_a, T_b\} T_c$$

$$Tr\{T_a^R, T_b^R\} T_c^R = d_{abc}(R)$$

R = representation

universal form

$$[T_a^R, T_b^R] = \underbrace{i f_{abc}}_{\text{group universel}} T_c^R$$

<u>SU(N)</u>

$$T_a = T_a^\dagger ; \quad \bar{T}_a = -T_a$$

$$U = e^{i\theta_a T_a} \qquad a = 1, \dots, N^2-1$$

$$T_a' = S T_a S^\dagger \qquad \boxed{SS^\dagger = 1}$$

$$\boxed{T_a' \Leftrightarrow T_a}$$

$$(T_a')^\dagger = (S T_a S^\dagger)^\dagger = S T_a^\dagger S^\dagger$$

$$= S T_a S^\dagger = T_a'^\dagger$$

$$\Rightarrow \quad (T_a')^\dagger = T_a'$$

$$\mathrm{Tr}\, T_a' = 0$$

$$\left(\mathrm{Tr}\, T_a' = \mathrm{Tr}\, S^\dagger S\, T_a = \mathrm{Tr}\, T_a = 0 \right)$$

$$[T_a', T_b'] = [S T_a S^\dagger, S T_b S^\dagger]$$

$$= S\, [T_a, T_b] S^\dagger =$$

$$= S\, i f_{abc}\, T_c S^\dagger = i f_{abc}\, T_c'$$

$$\Rightarrow \boxed{\mathrm{Tr}\, \{T_a', T_b'\} T_c' = \mathrm{Tr}\, \{T_a, T_b\} T_c}$$

$$\boxed{A(R)\, d_{abc} = \mathrm{Tr}\, \{T_a^R, T_b^R\} T_c^R}$$

$$\boxed{\begin{array}{c} \mathcal{A}(F) = \mathcal{A}(N) = 1 \\ \parallel \\ \text{fundamental } (N \text{ dim}) \end{array}}$$

$$\boxed{F, \quad T_a \qquad (N)}$$

$$T_a(\text{product}) = T_a \otimes \mathbb{1} + T_a \otimes \mathbb{1}$$

$$\mathrm{Tr}\{T_a, T_b\} T_c(\text{product}\} =$$

$$= \mathrm{Tr}\{T_a, T_b\} T_c \cdot \underbrace{\mathrm{Tr}\,\mathbb{1}}_{N} + (\text{same})$$

$$= 2N \; \mathrm{Tr}\{T_a, T_b\} T_c$$

product $N \times N = S + A$

$\phi_i \; \phi_j \qquad \underset{\phi_i\phi_j + \phi_j\phi_i}{\downarrow}$

$(\phi_i\phi_j - \phi_j\phi_i)$

$$\boxed{\text{Anomaly} = 0}$$

(theories of nature)

- $\psi_L, \quad \psi_R \qquad (f_L, f_R)$

$\uparrow \qquad \uparrow$

$T_a^L \qquad T_a^R$

$$\psi_L \to U_L \psi_L \qquad \psi_R \to U_R \psi_L$$

$$U_{L,R} = e^{i\Theta T_a^{L,R}}$$

(i) $T_a^L = T_a^R \Rightarrow \mathcal{A} = 0$

<u>Proof:</u>

$$\mathcal{L} = i \bar{\psi}_L \gamma^\mu D_\mu^L \psi_L + i \bar{\psi}_R \gamma^\mu D_\mu^R \psi_R$$

$$D_\mu^{L,R} = \partial_\mu - i g T_a^{L,R} A_a$$

$T_a^L = T_a^R$

$$\Rightarrow i\left(\bar{\psi}_L \gamma^\mu D_\mu \psi_L + \bar{\psi}_R \gamma^\mu D_\mu \psi_R\right)$$

$\boxed{\gamma_5^2 = 1}$ $\qquad \frac{1 \cancel{\pm \gamma_5}}{2} \qquad \frac{1 \cancel{- \gamma_5}}{2}$

$= i \bar{\Psi} \gamma^\mu D_\mu \Psi$ NO γ_5

$\Rightarrow$ $\boxed{\text{NO anomaly}}$

$A(\psi_L) = A(\psi_R)$

$A = A(\psi_L) - A(\psi_R) = 0$

Q.E.D.

- $\psi_L + \psi_R \leftarrow$ SM

$\Downarrow$ instead

$\psi_L \, , \, (\psi^c)_L \equiv C \bar{\psi}_R^T$

- $\psi + \psi^c$ $\boxed{\text{equivalent}}$

$$(\psi^c)_L = C\,\bar{\psi}_R^{\,T} = C\left(\psi_R^\dagger \gamma^0\right)^T$$

$$\| \qquad = C\gamma_0\,\psi_R^*$$

$$\psi^c \to U_R^*\,\psi^c$$

$$= e^{-i\theta_a T_a^{R*}} \quad (*)$$

$$\Rightarrow \boxed{\begin{array}{l} A(\psi_L) = A(\psi_R) \\ A(\psi) + A(\psi^c) = 0 \end{array}}$$

$(*)$ $T_a^c = -T_a^{R*}$

$$T_a^R = T_a^L \Rightarrow T_a^c = -T_a^*$$

$$= -T_a^T \quad (T_a = T_a^\dagger)$$

$$\mathrm{Tr}\{T_a^T, T_b^T\} T_c^T =$$

$$\mathrm{Tr}\{T_a, T_b\} T_c$$

$$\mathrm{Tr}\{T_a^c, T_b^c\} T_c^c = - \quad \text{–}\|\text{–}$$

$$\boxed{\mathcal{A}(\psi) + \mathcal{A}(\psi_c) = 0}$$

$$\boxed{\mathcal{A}(N) = 1 \,, \quad \mathcal{A}(N^*) = -1}$$

$$T_a(N) \equiv T_a \qquad T_a(N^*) = -T_a^*$$

$$\boxed{\begin{array}{c}\text{vector-like} \\ \equiv \quad A(L) = A(R) \\ T_a{}^L = T_a{}^R\end{array}}$$

$$\boxed{\begin{array}{c}T_a{}^R = S\, T_a{}^L S^+ \Rightarrow \\ A = 0\end{array}}$$

- "real" representation

$$T_a^* = -T_a \rightarrow A = 0$$

(generalise) $T_a^* = -S\, T_a S^+$

$$\Downarrow$$

$$\mathcal{A}=0$$

$$\Downarrow$$

$$\boxed{\mathcal{A}(SO(2))=0}$$

$$T_a = \frac{\sigma_a}{2}$$

$$T_a^* = -S\,T_a\,S^\dagger$$

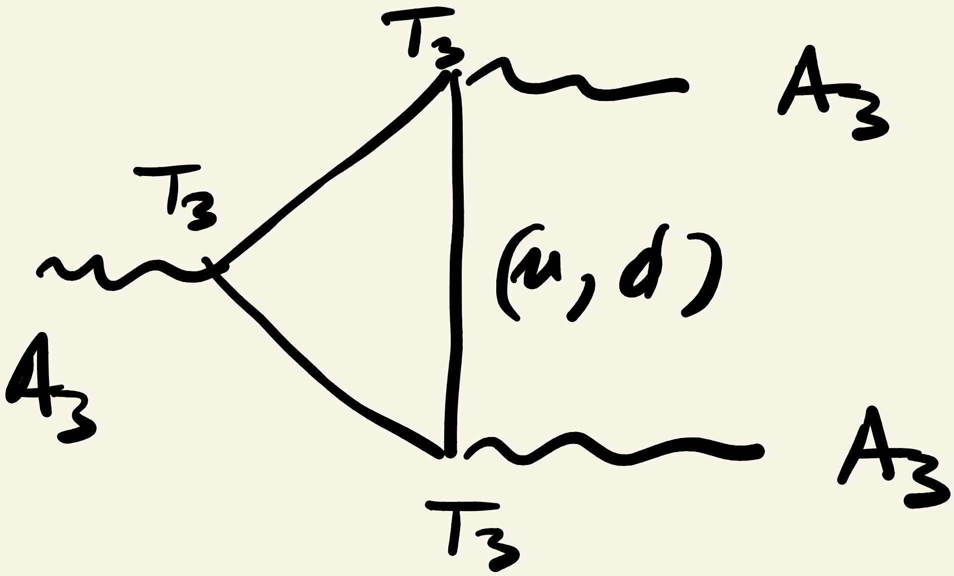
$$\boxed{S=\sigma_2} \Leftarrow \sigma_2\,\sigma_a\,\sigma_2 = -\sigma_a^*$$

$$a=1,3 \quad \sigma_2\sigma_a\sigma_2 = -\sigma_a\sigma_2^2$$

$$= -\sigma_a^* \checkmark$$

$$a=2 \quad \sigma_2^{\,3} = \sigma_2 = -\sigma_2^* \checkmark$$

$$\Rightarrow \boxed{\mathcal{A}(SU(2))=0}$$



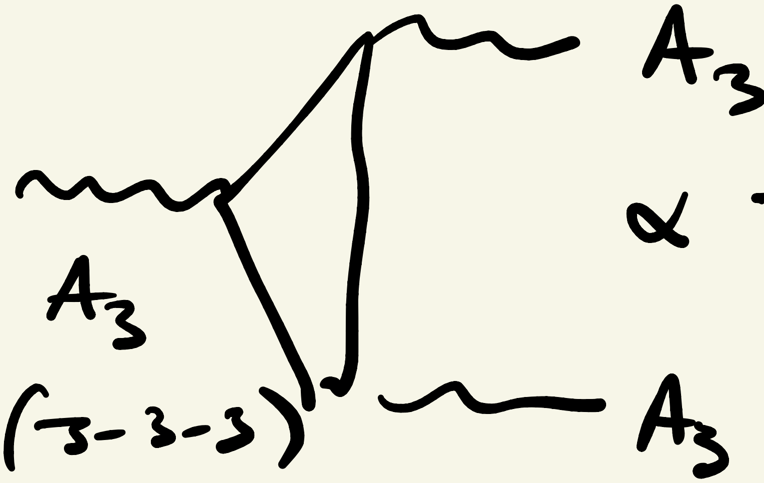
$$\mathcal{A} \propto \mathrm{Tr}\, \sigma_3^{\,3} \propto \mathrm{Tr}\, \sigma_3 = 0$$

$$\sigma = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

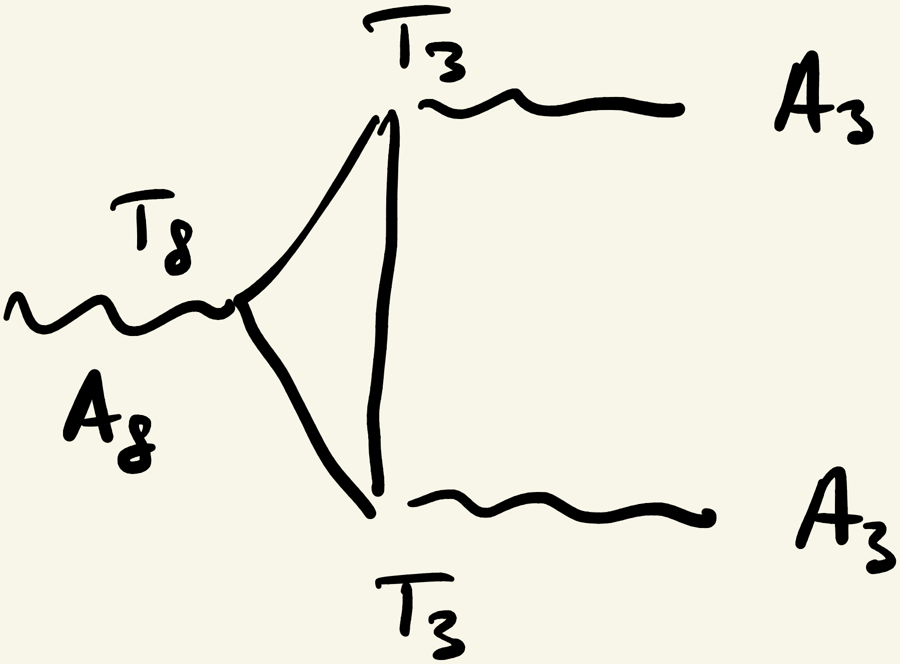
$\boxed{SU(3)\ ?}$

$$T_3 \propto \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$T_8 \propto \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$



$$A(3-3-3) \propto T_3{}^3 = 0$$



$$A(8-3-3) \propto \mathrm{Tr}\; T_8 T_3^2$$

$$\propto \mathrm{Tr} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

$$\propto 2 \neq 0$$

Anomaly free representation in SO(N)

$$\underline{N} \longrightarrow \underline{N} \times \underline{N} = S + A$$

$$\underline{N}^* \longrightarrow \underline{N}^* \times \underline{N}^* = S^* + A^*$$

- $\boxed{N \times N^*} \neq \boxed{\text{adjoint}}$

$$\boxed{\Sigma \longrightarrow U \Sigma U^+}$$

$$N \times N^* = \underset{(\mathrm{Tr}=0)}{\text{adjoint}} + \text{singlet}$$

$$\mathrm{Tr}\,\Sigma \longrightarrow \mathrm{Tr}\, U \Sigma U^+ = \mathrm{Tr}\, U^+ U \Sigma = \mathrm{Tr}\,\Sigma$$

($\mathrm{Tr}\,\Sigma = 0$ - irreducible)

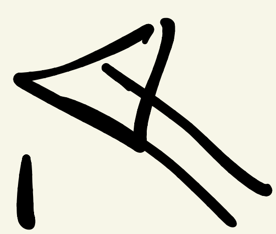
$$\underbrace{N \times N^{*}}_{N^2 \text{ elements}} = \underbrace{N^2 - 1}_{\text{adjoint}} + \underbrace{1}_{\text{singlet}}$$

$$\boxed{\Sigma^{+} = U \Sigma^{+} U^{+} = \Sigma}$$

irreducible

$N^2 - 1$ Traceless, hermitean matrices

$$\mathcal{A}(\text{adjoint}) = 0$$



$$\mathcal{A}(N) = 1, \quad \mathcal{A}(N^{*}) = -1$$

$$\mathcal{A}(N \times N^*) = 1 - 1 = 0$$

$$\mathcal{A}(\text{singlet}) = 0$$

$$T_a(\text{singlet}) = 0$$

· $\Sigma = T_a \phi_a$

generators

$$\Sigma \to U \Sigma U^\dagger$$

$$T_a \phi_a \to (1 + i\Theta_b T_b)(T_c \phi_c)$$

$$(1 - i\Theta_b T_b) + \ldots$$

$$= T_c \phi_c + i \Theta_b [T_b, T_c] \phi_c$$

$$= T_a \phi_a + i\theta_b \, i f_{bca} T_a \phi_c$$

$$= T_a \left[\phi_a - f_{abc} \theta_b \phi_c \right]$$

$$= T_a \phi_a'$$

$$\Rightarrow \boxed{\phi_a' = \phi_a - f_{abc} \theta_b \phi_c}$$

$$\phi_a' = \phi_a + i(\theta_b \hat{T}_b)_{ac} \phi_c$$

$$= \phi_a + i\theta_b (\hat{T}_b)_{ac} \phi_c$$

$$(\hat{T}_b)_{ac} = i f_{abc} =$$

$$= -i f_{bac}$$

$$\Downarrow$$

$$\boxed{(\hat{T}_a)_{bc} = -i f_{abc}}$$

$$\Rightarrow \boxed{T_a^* = -T_a} \qquad S = 1$$

$$\boxed{\begin{array}{l}\underline{\text{Theorem:}} \quad T_a^* = -S T_a S^\dagger \\ \qquad\qquad \Rightarrow d = 0\end{array}}$$

$$\boxed{\bar{N} = N^* = N^\dagger}$$

anti-fundamental

$$\psi \rightarrow \underbrace{U}_{\text{fund.}} \psi$$

$$\psi^c \longrightarrow U^* \psi^c$$

anti-fundamental

- $N \times N^* = N^2-1 + 1$

Adjoint singlet

$$A = 1 - 1 = 0 + 0$$

- $N \times N$

$$T_a(N \times N) = T_a \otimes 1 + 1 \otimes T_a$$

$$\boxed{A(N \times N) = 2N}$$

$$\Downarrow$$

- $N \times N = S + A$

$$A(S) = N+4$$

$$A(A) = N-4$$

Proof: $\boxed{\text{induction ?}}$

(i) $SU(3) \Rightarrow 3 \times 3 = 6 + 3^*$

$$3 \times 3 \times 3 = \cdots + \left(1 = 3 \times 3^*\right)$$

$$A(3^*) = -1$$

$$A(3) = 1$$

$$\Rightarrow A(S+A) = 6$$

$$\Rightarrow A(S) = 7 = 3+4$$

$$\overset{(N)}{A}(s) = N+4$$

$$\Rightarrow \overset{N+1}{A}(s) = N+1+4$$

$\boxed{\text{To be proven}}$

$\boxed{\text{Weak}}$

$$\boxed{\begin{array}{c} (\psi + \psi^c)_L \\ \Downarrow \text{ MUST} \\ A(\psi + \psi_c) = 0 \end{array}}$$

$$\int A(s) = N+4 \qquad N \times N$$

$$\mathcal{A}(A) = N-4 \qquad = S + A$$

$\underline{SU(5)}$: $\mathcal{A}(A) = 1$

$$\mathcal{A}(\bar{A}) = -1$$

$\Rightarrow$ <u>no</u> gravitational anomaly

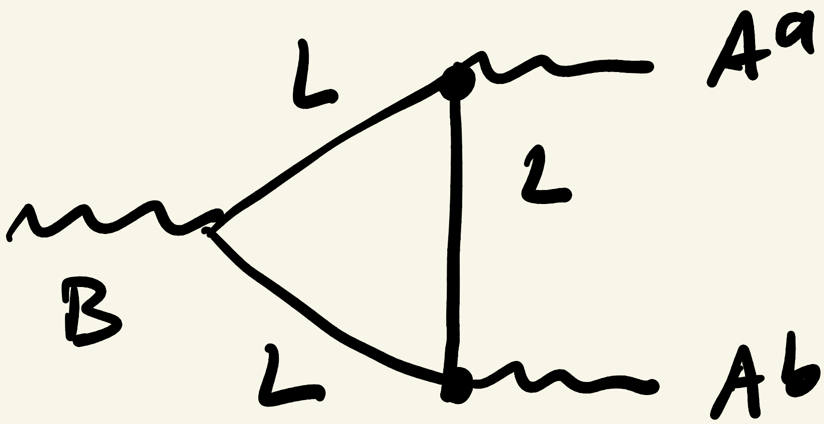
$$\mathrm{Tr}\, T_a = 0$$

$$\boxed{\text{SM}}$$

$A^a \, (T_a) \; a = 1, 2, 3$

$B \, (Y)$

$$\boxed{d_{abc} \, (A^a) = 0}$$



$[Y, T_a] = 0$

$\Downarrow$

$Y =$ fixed for a multiplet

$$\left.\begin{aligned} Y\left(q = \begin{pmatrix} u \\ d \end{pmatrix}_L\right) &= \frac{1}{3} \\ Y\left(l = \begin{pmatrix} \nu \\ e \end{pmatrix}_L\right) &= -1 \end{aligned}\right\} \; 3 \cdot \frac{1}{3} - 1 = 0$$

$\Uparrow$

$$Y(\Phi) = +1 \Rightarrow \phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix}$$

$$\Downarrow$$

$$A \propto \mathrm{Tr}\, Y_L T_a^2 \propto \mathrm{Tr}\, Y_L \overset{\downarrow}{=} 0$$

$$Q = T_3 + Y/2$$

$$\mathrm{Tr}\, Q_L = \mathrm{Tr}\, T_3 + \mathrm{Tr}\, \frac{Y_L}{2} \propto \mathrm{Tr}\, Y_L \overset{\downarrow}{=} 0$$

$$\Updownarrow$$

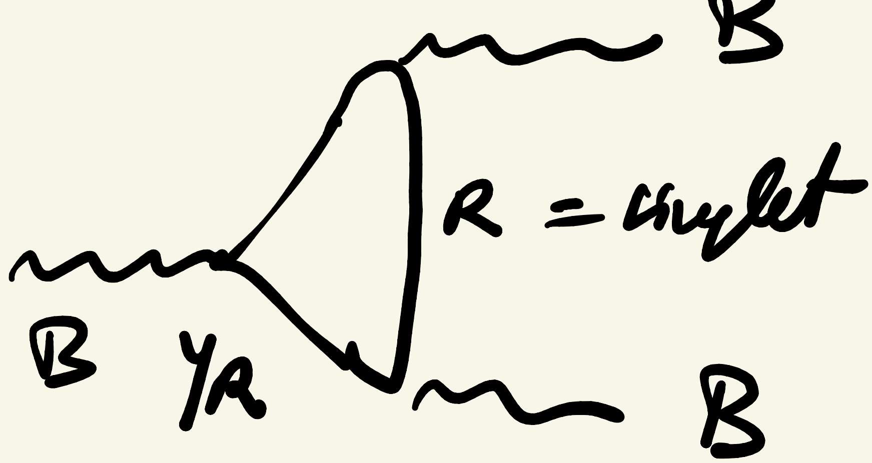
$$\boxed{(q_u^L + q_d^L)\cdot 3 + q_\nu^L + q_e^L = 0}$$

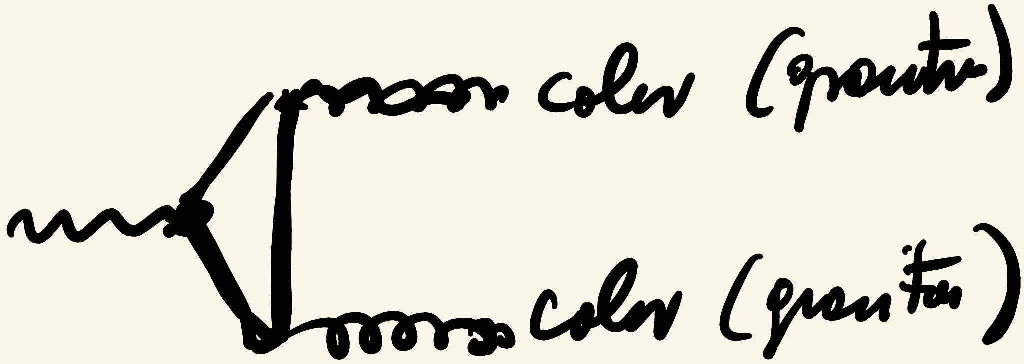
Anomaly $= 0 \Rightarrow$ charge quantization

$$\Downarrow$$

- $\mathrm{Tr}\, Y_L = 0$ $\boxed{\text{proven}}$

- $\mathrm{Tr}\, Y_R = ?$





$$\boxed{\mathrm{Tr}\, Y_R = 0} \not\Leftarrow \boxed{\mathrm{Tr}\, Y_L = 0}$$

- $Q = T_3 + \frac{Y}{2}$
  $\mathrm{Tr}\, Q_L = 0 = \mathrm{Tr}\, Y_L$ $\Big\}$ proven

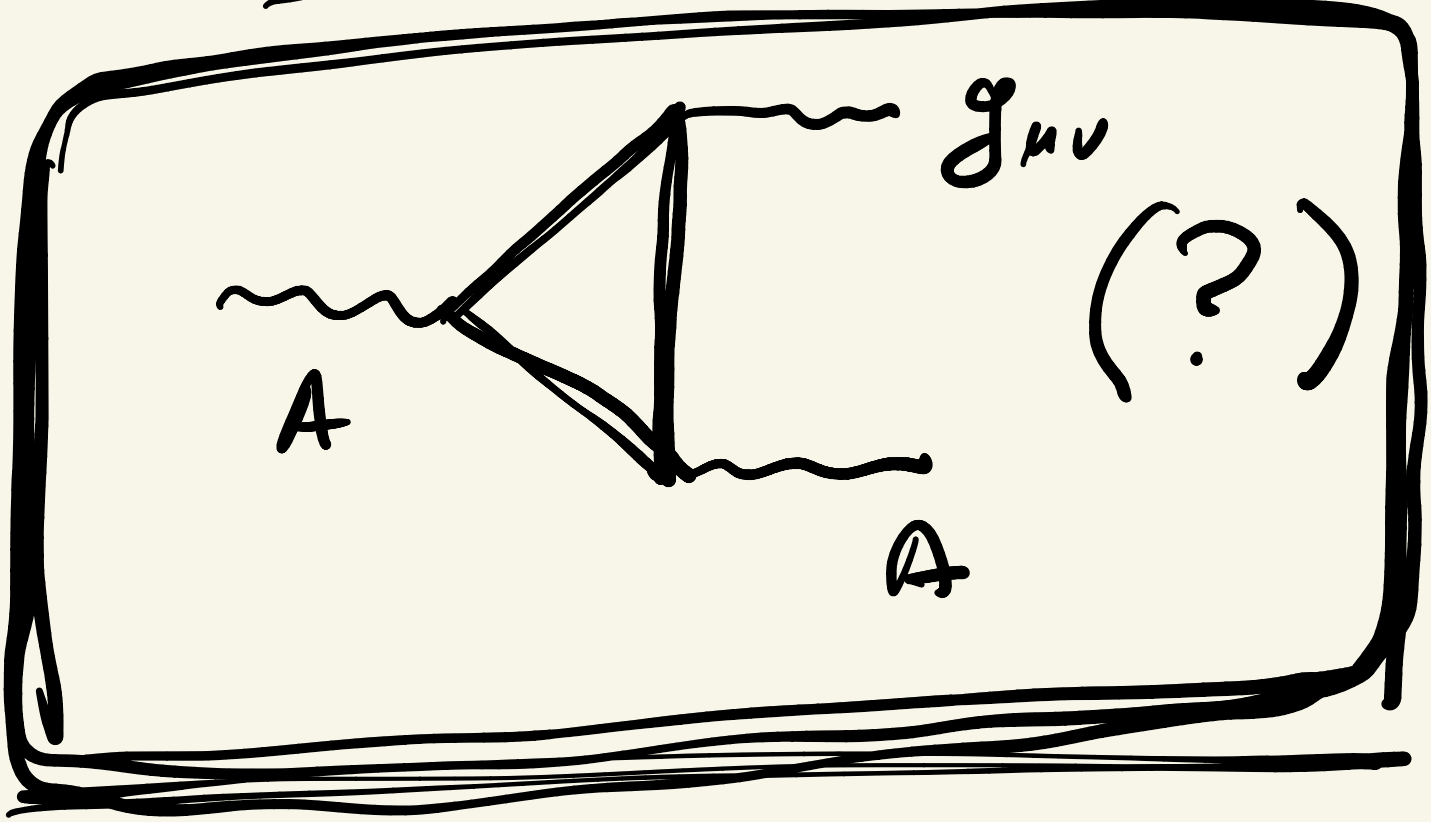
$$\boxed{Q_R = Q_L}$$ (built in)

$\boxed{\text{1961}}$

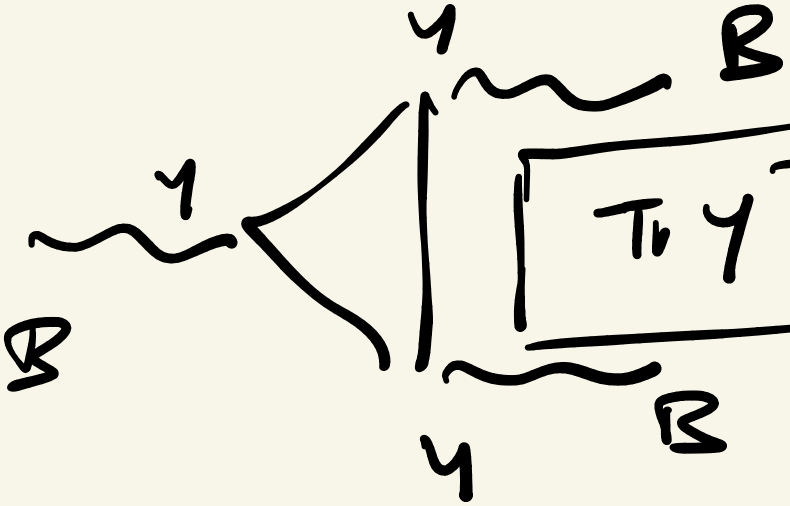
$$Q_R = Y_R \quad (T_{3R} = 0)$$

$$\Downarrow$$

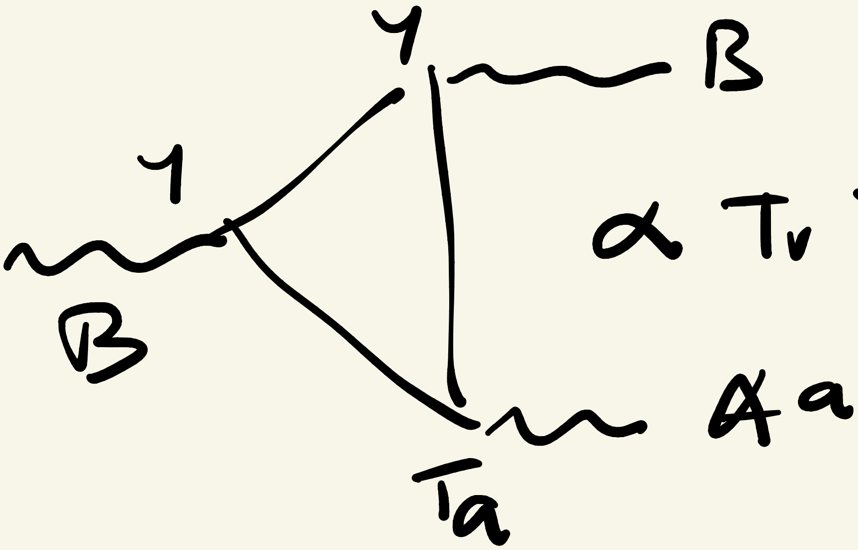
$$\boxed{\mathrm{Tr}\, Y_R = 0}$$



$$\boxed{\begin{gathered} \mathcal{A}(R)\, d_{abc} = \mathrm{Tr}\, \{T_a^R, T_b^R\}\, T_c^R \\ \mathcal{A}(N) = 1 \end{gathered}}$$



$\boxed{\mathrm{Tr}\, Y^3 = 0}$



$\propto \mathrm{Tr}\, T_a = 0$

- $\boxed{\text{Harvey: Anomalies TASI (*posted)}}$
- $\boxed{\text{Srednicki: QFT}}$ **
- $\boxed{\text{Peskin, Schroeder: QFT}}$ **

- A. Zee: QFT
  Weinberg: QFT

superficial look

deep

$\boxed{\text{Tong: QFT}}$

$\boxed{\text{Anomalies}}$

- Fujikawa: -- quantum anomalies

$\boxed{\text{Schwartz: QFT}}$