Exercises on General Relativity TVI TMP-TC1

Problem set 8

Exercise 1 – Geodesic equation

In the lecture the field Γ was introduced as

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta} \left(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu} \right) , \tag{1}$$

together with the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \ , \tag{2}$$

where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$.

(i) Show that the geodesic equation is equivalent to the following expression:

$$\dot{x}^\alpha \left(\partial_\alpha \dot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\beta \right) =: \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0 \ . \tag{3}$$

(ii) Consider the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and evaluate the geodesic equation in the Newtonian limit, i.e. Φ is sufficiently small and time-independent and geodesics are considered in the non-relativistic limit $|\dot{x}^0| \gg |\dot{x}^i|$ with $i \in I(3)$. Use for $h_{\mu\nu}$ the result from problem set 5 for a point mass ($h_{\mu\nu} = -2\Phi\delta_{\mu\nu}$ with the gravitational potential Φ).

Exercise 2 – De-Donder gauge

The De-Donder gauge or harmonic gauge was used in the lecture and in the exercise sheets. Nevertheless, the form of the de Donder gauge condition is not unique. Show the equivalence of the following expressions of this gauge:

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0 \Longrightarrow \partial_\mu h^{\mu\alpha} = \tfrac{1}{2}\partial^\alpha \mathrm{tr}_\eta(h) \, , \tag{4}$$

where $g = \eta + h$ with h being small compared to η; therefore, we consider only the linear order in h.

Exercise 3 – Metric and geodesics of the 2-sphere

Consider the metric of a 2-sphere of radius R:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{5}$$

with $\mu, \nu \in I(2)$. Why does this describe the metric of a 2-sphere?
The metric encodes all information on the geometry of the manifold. Choosing $x^1 = \theta$ and $x^2 = \phi$, read off the matrix $g_{\mu\nu}$.
Compute the non-vanishing Christoffel symbols for the two-sphere.
(Hint: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, so only a few components have to be computed explicitly.)
Find the geodesic equations on a sphere using these Christoffel symbols.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`