Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 7

Exercise 1 – Gravitational waves

Due to the non-linearity of gravity a gravitational wave itself serves as a source. Therefore we consider a weak wave in order to neglect any non-linearities and work with Fierz-Pauli without self-interaction.
In this theory the equations of motion are as discussed before,

$$\Box \bar{h}_{\mu\nu} = 0 \,. \tag{1}$$

We are familiar with this differential equation from electrodynamics, therefore we take as an ansatz the plane-wave expansion:

$$\bar{h}_{\mu\nu}(x) = \int \mathrm{d}\omega_k \sum_s \epsilon^s_{\mu\nu}(k) a^s(k) \exp(ik_\alpha x^\alpha) + \text{c.c.} \tag{2}$$

where $\epsilon^s_{\mu\nu}(k)$ is a momentum-dependent symmetric rank-2 tensor, $a^s(k)$ is a complex function and $\mathrm{d}\omega_k$ is the abbreviation of $\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega}$ with $\omega := k^0$. The sum runs over all linear independent components of the tensor.

(i) Using this ansatz and inserting it into (1) what properties of the wave can you deduce?

(ii) Since no further conditions on ϵ were imposed this ansatz possesses 10 degrees of freedom and therefore has to be gauged. Show that using the de-Donder gauge introduces 4 conditions:

$$k_\mu \epsilon^{\mu\nu}_s = 0 \,. \tag{3}$$

As we discussed in a previous exercise there are 4 degrees of freedom left which are removed by a coordinate transformation with an arbitrary vector field e^μ satisfying $\Box e^\mu = 0$. It can be shown that this freedom of choosing e^μ and of choosing a Lorentz frame allows us to make ϵ^s traceless and $\epsilon^s_{0\nu} = 0$. This choice is called the transverse traceless gauge.

(iii) Considering a monochromatic wave-mode propagating in the z direction with $k_\alpha = (k_0, 0, 0, k_3)$, determine the components of ϵ_s in terms of the remaining two degrees of freedom. How can these degrees of freedom be interpreted physically?

(iv) Find how a ring with radius R lying in the (x, y) plane is distorted during the propagation of the gravitational wave through the ring. Sketch your results and discuss how you could build a detector of gravitational waves.

Exercise 2 – Sources of gravitational waves

The goal of this exercise is to derive the amplitude of a gravitational wave generated by a specific source which could be measured on earth.
Analogously to electrodynamics we find the solutions for an arbitrary source with the Green's function method. The defining equation for the Green's function G is given by

$$\Box\, G(x^\alpha - y^\alpha) = \delta^{(4)}(x^\alpha - y^\alpha)\ . \tag{4}$$

Then the general solution to (1) for a source given by the energy-momentum tensor T is

$$\bar{h}_{\mu\nu}(x^\alpha) = -16\pi G_N \int \mathrm{d}^4 y\, G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha)\ . \tag{5}$$

Recapitulate that the retarded Green's function for the d'Alembert operator is given by

$$G(x^\alpha - y^\alpha) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta\left(|\boldsymbol{x}-\boldsymbol{y}| - (x^0 - y^0)\right) H(x^0 - y^0) \tag{6}$$

where H is the Heaviside step function.

(i) Insert the retarded Green's function into the general solution and perform the y^0 integration. Further go into frequency space using the Fourier transformation

$$\hat{\bar{h}}_{\mu\nu}(\omega, \boldsymbol{x}) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x^0\, \mathrm{e}^{i\omega x^0}\, \bar{h}_{\mu\nu}(x^0, \boldsymbol{x})\ . \tag{7}$$

(ii) Assuming that the distance r between the source and the earth is huge compared to the extent of the source a, we can approximate $|\boldsymbol{x}-\boldsymbol{y}|$ as the constant r. Additionally, using the de-Donder gauge in frequency space should give you as a result

$$\hat{\bar{h}}_{ij}(\omega, \boldsymbol{x}) = \frac{4G}{r}\mathrm{e}^{i\omega r} \int \mathrm{d}^3 y\, \hat{T}_{ij}(\omega, \boldsymbol{y})\ . \tag{8}$$

(iii) Show that the integral can be rewritten as

$$\int \mathrm{d}^3 y\, \hat{T}_{ij}(\omega, \boldsymbol{y}) = -\frac{\omega^2}{2}\hat{Q}_{ij}(\omega) \tag{9}$$

with the quadrupole tensor

$$\hat{Q}^{ij}(\omega) = \int \mathrm{d}^3 y\, y^i y^j \hat{T}^{00}(\omega, \boldsymbol{y})\ , \tag{10}$$

by performing the following steps.

- since the source we want to describe is compact, boundary terms vanish; use partial integration
- exploit the conservation of the source $\partial_\mu T^{\mu\nu} = 0$ in frequency space
- consider the vanishing boundary term $\partial_k \left(y^i y^j \hat{T}^{0k}\right)$

(iv) Show that using the result of iii), restoring the speed of light c and transforming back equation (8) the metric perturbation becomes

$$\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{2G_N}{rc^4}\frac{\mathrm{d}}{\mathrm{d}t^2}Q_{ij}(t - r) \tag{11}$$

with $t := x^0$.

(v) As an example consider a binary system with two point masses circling around each other in the (x,y) plane with masses $M_1 = 2M_\odot$, $M_2 = M_\odot$ and a distance between them of $a = 100\,\mathrm{km}$.
Compute the $\bar{h}_{11}$ component of the gravitational wave and estimate the amplitude as it could be measured on earth with the distance between the source and the detector $r = 40\,\mathrm{Mpc}$.
Optional: Despite the direct detection by LIGO, already over 20 years ago gravitational waves were detected indirectly, can you imagine how?

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`