Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 3

Exercise 1 – Ricci calculus

(i) Given a function $f\left(x^\alpha\right)$ the corresponding one-form $\mathrm{d}f$ is

$$\mathrm{d}f = \sum_\alpha \frac{\partial f}{\partial x^\alpha}\mathrm{d}x^\alpha \ . \tag{1}$$

Determine the one-forms $\mathrm{d}f_i$ for the functions

$$f_1\left(x,y\right) = xy \quad \text{and} \quad f_2\left(x,y\right) = \sqrt{x^2+y^2} \tag{2}$$

and compute the corresponding vector fields.

(ii) Consider a rank 2 tensor X and a vector V which have in a coordinate neighborhood the following components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \ , \ V^\mu = (-1,2,0,-2) \ . \tag{3}$$

Find the components of $X^\mu{}_\nu$, $X_\mu{}^\nu$, $X^\lambda{}_\lambda$, $V^\mu V_\mu$, and $V_\mu X^{\mu\nu}$. Use the Minkowski metric $\eta_{\mu\nu}$ to raise and lower the indices, e.g. $A^\mu = \eta^{\mu\nu}A_\nu$ and $A_\mu = \eta_{\mu\nu}A^\nu$, where $\eta^{\mu\nu}$ are the components of the inverse of the metric.

Exercise 2 – (Lorentz)Tensors

Consider the following expressions and explicitly determine whether they are Lorentz tensors and or general tensors.

(i) $\partial_\mu\phi$, (ii) $\partial_\mu A_\nu$, (iii) $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$, (iv) $S_{\mu\nu} := \partial_\mu A_\nu + \partial_\nu A_\mu$

where ϕ is a scalar field and A is a Lorentz vector.
Hint: Bear in mind, that a proper tensor is a general geometrical quantity with components having the following transformation properties under coordinate transformations $\boldsymbol{x} \to \tilde{\boldsymbol{x}}(\boldsymbol{x})$:

$$\tilde{T}^{\tilde{\mu_1}\cdots\tilde{\mu_k}}_{\tilde{\nu_1}\cdots\tilde{\nu_l}} = T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}\frac{\partial\tilde{x}^{\tilde{\mu_1}}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\nu_l}}{\partial\tilde{x}^{\tilde{\nu_l}}} \tag{4}$$

Exercise 3 – From oscillators to field theory

Consider a system of N point particles, each with mass m, which are connected with harmonic oscillators, imaginable as springs with spring constant k. At rest, two particles connected with a spring are separated by a distance d. The kinetic energy of the whole system is given by

$$T = \frac{1}{2}\sum_{j=1}^{N} m\dot{x}_j^2, \tag{5}$$

with $x_j := x_j(t)$ denoting the position of the j-th particle with respect to its position at rest. We assume the maximum elongation of the springs to be small such that we have for the potential energy

$$V = \frac{1}{2}\sum_{j=1}^{N} m\omega^2(x_{j+1} - x_j)^2, \tag{6}$$

with $\omega = \sqrt{\frac{k}{m}}$.

(i) Show that we can express the Lagrange function as

$$L = \frac{1}{2}\sum_{j=1}^{N} d\left(\dot{x}_j^2 - \omega^2 d^2 \frac{(x_{j+1} - x_j)^2}{d^2}\right). \tag{7}$$

(ii) Perform the limit $d \to 0$ and $N \to \infty$ and regard the positions of the particles to be continuous. Furthermore, rewrite $x_j(t) \to \phi(t, x) =: \phi(x)$.

(iii) We write $d^2\omega^2 \to v^2$ and demand it to be the speed of light to obtain a Lorentz-invariant action. Show that the action becomes

$$S = -\frac{1}{2}\int \mathrm{d}^4x \sqrt{-\eta}\ \ \partial_\mu\phi\partial^\mu\phi \tag{8}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu = -\frac{1}{c^2}\partial_t + \partial_x$ and $\eta_{\mu\nu}$ is the Minkowski metric and η is its determinant.

(iv) Construct for the real scalar field $\phi(x)$ the most general action in four dimensional spacetime. In order to achieve this use only positive powers of the following objects: the real scalar field ϕ, a constant Λ with mass dimension $[\Lambda] = [\phi]$ and the partial derivative ∂_μ.
Hint: You do not need to specify dimensionless constants and signs. Work in natural units for particle physics: $c = \hbar = k_B = 1$.

(v) Derive the equation of motion of the previously found action and discuss the physical meaning of each term.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`