Exercises on General Relativity TVI TMP-TC1

Problem set 13

Exercise 1 – Parallel transport

A useful formula to find the components of a vector field V parallel transported along a curve is

$$U^\alpha \nabla_\alpha V^\beta = 0 \tag{1}$$

with $U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}$ and x^α the parametrization of the curve. This expression and its derivation will be discussed in the tutorials.

(i) Show explicitly that parallel transport preserves the length of the vector.

(ii) Perform a parallel transport of a vector field V along a full circle (R, φ) with a fixed radius R in $\mathbb{R}^2$ by solving (1) and find the angle δ between $V(\varphi = 0)$ and $V(\varphi = 2\pi)$:

$$\cos\delta = \frac{V^\alpha(0)V_\alpha(2\pi)}{V^\alpha(0)V_\alpha(0)} \tag{2}$$

Convince yourself of the result with a picture.

Now consider a parallel transport on a 2-sphere. The metric g of a 2-sphere of radius R is given as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{3}$$

with $\mu, \nu \in I(2)$.

(iii) Draw a picture of the parallel transport of an arbitrary tangent vector along the following closed curve: $A \to B \to C \to A$ connected via geodesics with $A = (\theta = \pi/2, \varphi = 0)$, $B = (\theta = \pi/2, \varphi = \pi/2)$ and $C = (\theta = 0, \varphi = 0)$.

(iv) Now take as the curve a circle on the sphere with fixed angle $\theta = \theta_c$ parameterized as $x^\alpha = (\theta_c, \varphi)$. Convince yourself that the unit vector tangent to this curve is

$$U^\alpha = \frac{1}{R\sin\theta_c}(0, 1)\ . \tag{4}$$

Solve (1) and take as the starting point $\varphi = 0$ for which the vector field is given by $V^\alpha(0) = (V_0^\theta, V_0^\varphi)$ and as the end point an arbitrary angle φ.
The solutions are

$$V^\theta(\varphi) = V_0^\theta \cos\bar\varphi + V_0^\varphi \sin\theta_c \sin\bar\varphi \tag{5}$$

$$V^\varphi(\varphi) = V_0^\varphi \cos\bar\varphi - \frac{V_0^\theta}{\sin\theta_c}\sin\bar\varphi \tag{6}$$

with $\bar\varphi = \varphi\cos\theta_c$. Find the angle δ between $V(0)$ and $V(2\pi)$.
What is the behavior of δ at the equator and near the poles? What does this imply for analogous parallel transports on a cylinder?

Exercise 2 – Curvature of a 2-sphere

Calculate the Riemann tensor, Ricci tensor and Ricci scalar of the 2-sphere. Use the Christoffel symbols derived on previous sheets.

How do these quantities change, if you compute them using some different spherical coordinates which are rotated with respect to the original ones?

Exercise 3 – Einstein tensor for weak field

Consider the weak field approximation (linearized gravity) and regard the metric as a small perturbation h around Minkowski space η with $g = \eta + h$.

(i) Using the general definition of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{7}$$

find its linearized version neglecting all terms $O(h^2)$

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial^\lambda \bar{h}_{\lambda\nu} + \partial_\nu \partial^\lambda \bar{h}_{\lambda\mu} - \eta_{\mu\nu} \partial^\lambda \partial^\rho \bar{h}_{\lambda\rho} - \Box \bar{h}_{\mu\nu}\right) \tag{8}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ and $h = h_\mu{}^\mu$.

(ii) Recall the harmonic gauge with $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$ and show that the linearized Einstein tensor in this gauge is given by

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu} \,. \tag{9}$$

(iii) Compare the linearized Einstein tensor with the equation of motion of the Fierz-Pauli action.

Exercise 4 – Friedmann-Lemaître-Robertson–Walker spacetime

Consider the Friedmann-Lemaître-Robertson–Walker spacetime with squared line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \mathrm{d}\Omega^2\right) \,, \tag{10}$$

with t the time coordinate, r the radial coordinate, $\mathrm{d}\Omega^2 = \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right)$ with angles θ, φ and $a(t)$ the scaling parameter and $k \in \{+1, 0, -1\}$.

(i) Find the Ricci tensor and Ricci scalar using an algebraic program.

(ii) Consider the case $k = 0$. Introducing a new time coordinate η with $\mathrm{d}t = a(t)\mathrm{d}\eta$, how does (10) change and what implications can you conclude?

(iii) Use previous results and derive the first and second Friedmann equations

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G_N}{3}\epsilon + \frac{1}{3}\Lambda \tag{11}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\epsilon + 3p) + \frac{1}{3}\Lambda \,. \tag{12}$$

To arrive at the result we use the Einstein field equations with a cosmological constant term:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu} \tag{13}$$

where the energy momentum tensor $T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p\, g_{\mu\nu}$ with the velocity u_μ, energy density ϵ and pressure p. Consider in particular the 00 and ij components of these equations.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`