Prof. Dr. Stefan Hofmann Winter term 2019/20

Exercises on General Relativity TVI TMP-TC1

Problem set 10

Exercise 1 – Transformation properties of tensors

(i) Show that $\{\partial_\mu\}$ form a basis of the tangent space and show that they transform like a vector.

(ii) Repeat this analysis with $\{\mathrm{d}x^\mu\}$ and show that they transform like a 1-form.

(iii) Given a tensor T of type $(2,2)$, calculate the function $T(v,w,h,k)$ for arbitrary vectors v and w and arbitrary 1-forms h and k.

(iv) How do the components of the tensor T transform?

(v) It was explicitly shown how the measure of the action of electrodynamics changes when a coordinate transformation is performed. Show that

$$\int \mathrm{d}\mu := \int \mathrm{d}^4x\sqrt{g} \tag{1}$$

with $g = -\det(g_{\mu\nu})$, is invariant under diffeomorphisms by using that the metric transforms as

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta} \ . \tag{2}$$

Exercise 2 – Examples of 1-forms and tensors

(i) Calculate the 1-forms $\mathrm{d}h$ and $\mathrm{d}k$, where

$$h(x,y,z) = 4x^2y + x^3z, \quad k(x,y) = \sqrt{x^2+y^2} \ . \tag{3}$$

(ii) In 3-dimensional Euclidean space, decide whether the following maps are tensors:

$$T : (v,w) \mapsto 2v \times w - v(n\cdot w) \tag{4}$$
$$S : (v,w) \mapsto 2v \times w - (v\cdot v)(n\cdot w) \tag{5}$$
$$R : (v,w) \mapsto n\cdot(v \times w) - (n\cdot v)(n\cdot w) \tag{6}$$

with $v,w,n \in \mathrm{Vec}(\mathbb{R}^3)$ and n is fixed.
For each $K \in \{T,S,R\}$ for which it is possible determine the components $K^a{}_{bc}$ in a given basis.

Exercise 3 – Commutative properties of vector fields

Consider the Lie bracket of two smooth vector fields X and Y on a manifold M:

$$[X,Y](f) := XY(f) - YX(f)\,, \tag{7}$$

where $f \in C^\infty(M)$.

(i) Show that the vector fields fulfil the Jacobi identity:

$$\big[[X,Y],Z\big] + \big[[Z,X],Y\big] + \big[[Y,Z],X\big] = 0 \tag{8}$$

(ii) Furthermore, show

$$[fX,Y] = f[X,Y] - Y(f)X\ . \tag{9}$$

(iii) Let from now on $M = \mathbb{R}^n$. Show that the bracket of the coordinate vector fields $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \in \mathrm{Vec}(\mathbb{R}^n)$ vanishes.

(iv) Show that, for $X, Y \in \mathrm{Vec}(\mathbb{R}^n)$, if

$$X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^n g_j \frac{\partial}{\partial x_j} \tag{10}$$

then the bracket is given by

$$[X,Y] = \sum_{j=1}^n \left(\sum_{i=1}^n \left(f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i} \right) \right) \frac{\partial}{\partial x_j}\ . \tag{11}$$

(v) As an example calculate the bracket of the vector fields $X, Y \in \mathrm{Vec}(\mathbb{R}^2 \setminus \{0\})$ with

$$X = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\ , \tag{12}$$

where $r := \sqrt{x^2 + y^2}$.

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`