Exercises on General Relativity TVI TMP-TC1

Problem set 1

Exercise 1 – Lorentz transformations

(i) Starting from the assumption that in all inertial frames of reference the laws of physics should be the same and the speed of light has the same value $c = 1$, show that the line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is invariant. This can be written using the Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$ with $\mu, \nu \in \{0, 1, 2, 3\}$, as $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where $dx^\mu = (dt, dx, dy, dz)$.

(ii) We define a transformation from one frame of reference to another as a Lorentz transformation and write it as $x'^{\mu} = \Lambda^\mu_{\ \nu} x^\nu$. Show that the invariance of ds^2 implies the following condition on the Lorentz-transformation matrices Λ,

$$\eta = \Lambda^T \eta \Lambda \ . \tag{1}$$

Furthermore show that

$$\Lambda(\xi) = \begin{pmatrix} \cosh(\xi) & -\sinh(\xi) & 0 & 0 \\ -\sinh(\xi) & \cosh(\xi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

is a possible solution of Eq. (1).

(iii) Consider a particle at rest in the lab frame described by $x^\mu = (ct, 0, 0, 0)$. Perform the Lorentz transformation in Eq. (2) and find a relation between the velocity v of the new reference frame and the parameter ξ.

(iv) Perform two subsequent transformations $\Lambda(\xi_1)$ and $\Lambda(\xi_2)$. Show that $\Lambda(\xi_2)\,\Lambda(\xi_1)$ is again a Lorentz transformation and determine a general expression for the addition of velocities.

Exercise 2 – Lorentz transformations and their generators

(i) Using Eq. (1) show that $\det(\Lambda) = \pm 1$. Furthermore show that $\Lambda^0_{\ 0} = \pm\sqrt{1 + \sum_i \left(\Lambda^i_{\ 0}\right)^2}$. In the following we will consider only transformations for which $\det(\Lambda) = +1$ and $\Lambda^0_{\ 0} \geq 1$.

(ii) Show that $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R_{3\times 3} \end{pmatrix}$ fulfils Eq. (1), where $R_{3\times 3}$ represents a rotation matrix in 3 dimensions.

(iii) To find other transformations fulfilling Eq. (1) and mixing time and space coordinates (we will call such transformations “boosts”) consider the generators of such transformations. The generator of a transformation Λ is defined by $\Lambda = e^{\mathrm{i}\xi K}$. Take an inifinitesimal transformation, i.e. $\Lambda^\mu_{\ \rho} = \delta^\mu_\rho + \mathrm{i}\xi K^\mu_{\ \rho}$ and show that

$$K^T \eta = -\eta K \ . \tag{3}$$

(iv) Solve Eq. (3) to find the three generators of boosts.

(v) Show that the generator for boosts in the x-direction leads to the Lorentz transformation given in Eq. (2).

Exercise 3 – Light cone and relativistic particles

(i) Consider the Minkowski spacetime $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ and discuss the light cone of this spacetime. Use Lorentz transformations to investigate why spacelike separated events cannot be causally connected in contrast to timelike and lightlike separation. Additionally, calculate whether the following events are timelike, spacelike or lightlike seperated to each other, respectively: $A = (1, 1, 0, 0), B = (0, 1, 1, 0), C = (0, 1, 0, 1)$.

(ii) The action of a free relativistic particle is given by

$$S = -m \int_C \mathrm{d}\tau = -m \int_C \sqrt{-\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} \tag{4}$$

where τ is the proper time along the path C the particle takes. Show that this describes a free particle by deriving the equations of motion. Does this change if we replace the Minkowski metric with a position-dependent metric?

(iii) The particle pion is unstable and therefore decays. If we are in the restframe of the pion its half-time is $2.5 \cdot 10^{-8}$ s. How do we experience its half-time if it is moving with 99.9% of the speed of light with respect to us?

General information

The lecture takes place on Monday at 10:00-12:00 and on Wednesday at 10:00-12:00 in A348.

Presentation of solutions:
Thursday at 08:00 - 10:00 in A 348

There are two tutorials:
Monday at 12:00 - 14:00 in A 249
Friday at 14:00 - 16:00 in A 348

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/tvi_tc1_gr/index.html`