Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 14, discussed February 3 - February 7, 2020

Exercise 76

Consider a Stern-Gerlach Experiment oriented in the z-direction connected to a detector. Before the measurement the detector is in the state $\left|\psi^A[0]\right\rangle$. If a silver atom with spin up $|\uparrow\rangle$ is measured the detector is in the state $\left|\psi^A[\uparrow]\right\rangle$ and and for spin down $|\downarrow\rangle$ the state of the device is $\left|\psi^A[\downarrow]\right\rangle$. Let the silver atom be in a general spin state $\left|\psi^S\right\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$, with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

(i) Write down the general state $\left|\Psi^1\right\rangle$ of the whole system before and after the measurement.

(ii) Write the states as $|\uparrow\rangle = (1,0)^T$, $|\downarrow\rangle = (0,1)^T$,
$\left|\psi^A[\uparrow]\right\rangle = (1,0,0)^T$, $\left|\psi^A[0]\right\rangle = (0,1,0)^T$, $\left|\psi^A[\downarrow]\right\rangle = (0,0,1)^T$
and express $\left|\Psi^1\right\rangle$ in that basis.

Now we want to repeat this experiment N times with silver atoms being in the same state $\left|\psi^S\right\rangle$. Define new states of the detector as $\left|\psi^A_+[n]\right\rangle$ and $\left|\psi^A_-[m]\right\rangle$ where n and m are the numbers of silver atoms measured with spin up and down respectively.

(iii) What is the corresponding spin state for $\left|\psi^A_+[1]\right\rangle \otimes \left|\psi^A_-[2]\right\rangle$ if $N = 3$?

(iv) Write the state of the whole system before and after the measurement and bring it in the normalized form

$$\left|\Psi^N\right\rangle = \sum_{n=0}^{N} c_n\,|\uparrow_1\rangle \otimes ... \otimes |\uparrow_n\rangle \otimes |\downarrow_{n+1}\rangle \otimes ... \otimes |\downarrow_N\rangle \otimes \left|\psi^A_+[n]\right\rangle \otimes \left|\psi^A_-[N-n]\right\rangle \tag{1}$$

(v) Find the expectation values of n and m in the limit $N \to \infty$. Interpret your result.
(*Hint:* Use the fact that the binomial distribution becomes the normal distribution in the large N limit.)

Solution:

(i) Before measurement: $\left|\Psi^1_0\right\rangle = \left|\psi^S\right\rangle \otimes \left|\psi^A[0]\right\rangle = \alpha\,|\uparrow\rangle \otimes \left|\psi^A[0]\right\rangle + \beta\,|\downarrow\rangle \otimes \left|\psi^A[0]\right\rangle$
After measurement: $\left|\Psi^1_t\right\rangle = \alpha\,|\uparrow\rangle \otimes \left|\psi^A[\uparrow]\right\rangle + \beta\,|\downarrow\rangle \otimes \left|\psi^A[\downarrow]\right\rangle$

(ii) $$\left|\Psi^1_0\right\rangle = \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix};\ \left|\Psi^1_t\right\rangle = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{pmatrix}$$

(iii) The particles are distinguishable as they are sent into the Stern-Gerlach apparatus after each other and therefore the order matters. The corresponding spin state contains all the states with two spins down and one up:
$|n=1, m=2\rangle = \alpha\beta^2(|\uparrow_1\rangle \otimes |\downarrow_2\rangle \otimes |\downarrow_3\rangle + |\downarrow_1\rangle \otimes |\uparrow_2\rangle \otimes |\downarrow_3\rangle + |\downarrow_1\rangle \otimes |\downarrow_2\rangle \otimes |\uparrow_3\rangle)$

(iv) Before the measurement the state of the whole system is:
$$\left|\Psi^N_0\right\rangle = \bigotimes_{i=0}^{N} (\alpha\,|\uparrow_i\rangle + \beta\,|\downarrow_i\rangle) \otimes \left|\psi^A_+[0]\right\rangle \otimes \left|\psi^A_-[0]\right\rangle$$

As the detector only counts the number of atoms with spin up or down the information about the order is lost and we are only interested in the total count. The state after the measurement is therefore:

$$\left|\,\Psi_t^N\right\rangle = \sum_{n=0}^{N} \frac{1}{\sqrt{\mathcal{N}_n}} \binom{N}{n} \alpha^n \beta^{N-n} \left|\uparrow\right\rangle^{\otimes n} \left|\downarrow\right\rangle^{\otimes N-n} \otimes \left|\,\psi_+^A[n]\right\rangle \otimes \left|\,\psi_-^A[N-n]\right\rangle$$

To fix the normalization constant we demand:

$$1 = \left\langle \Psi_t^N \,\middle|\, \Psi_t^N \right\rangle = \sum_{n=0}^{N} \frac{1}{\mathcal{N}_n} \binom{N}{n}^2 |\alpha|^{2n} |\beta|^{2(N-n)} = (|\alpha|^2 + |\beta|^2)^N$$

The last step is true if we choose $\mathcal{N}_n = \binom{N}{n}$. And therefore:

$$\left|\,\Psi_t^N\right\rangle = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \alpha^n \beta^{N-n} \left|\uparrow\right\rangle^{\otimes n} \left|\downarrow\right\rangle^{\otimes N-n} \otimes \left|\,\psi_+^A[n]\right\rangle \otimes \left|\,\psi_-^A[N-n]\right\rangle$$
$$\Rightarrow c_n = \sqrt{\binom{N}{n}} \alpha^n \beta^{N-n}$$

(v) The state where exactly n atoms are measured with spin up is given by:

$$\left|\,\Phi_n^N\right\rangle = \sqrt{\binom{N}{n}} \alpha^n \beta^{N-n} \left|\uparrow\right\rangle^{\otimes n} \left|\downarrow\right\rangle^{\otimes N-n} \otimes \left|\,\psi_+^A[n]\right\rangle \otimes \left|\,\psi_-^A[N-n]\right\rangle$$
$$P(n) = |c_n|^2 = \frac{N!}{n!(N-n!)} |\alpha|^{2n} |\beta|^{2(N-n)} \overset{N\to\infty}{\longrightarrow} \frac{1}{2\pi\sqrt{N\,|\alpha|\,|\beta|}} \exp\left\{-\frac{(n-N\,|\alpha|^2)^2}{2n\,|\alpha|\,|\beta|}\right\}$$

The derivation of why $|c_n|^2$ can be interpreted as the probability $P(n)$ is given in:
`https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.29.454`
This is just the normal distribution where we can read off the expectation value:
$\langle n\rangle = N\,|\alpha|^2$
$\langle m\rangle = \langle N-n\rangle = \langle N\rangle - \langle n\rangle = N(1-|\alpha|^2) = N\,|\beta|^2$
This means that α and β are just the square roots of the probabilities.

Exercise 77 - Wave packets

Consider a gaussian wave packet as in exercise 67:

$$\psi(q,t) = N \exp\left[\alpha(t) + \frac{i}{\hbar} p_{cl}(t)(q - q_{cl}(t)) - \frac{1}{2\sigma^2(t)}(q - q_{cl}(t))^2\right]. \tag{2}$$

where $p_{cl}(t) = m\frac{d}{dt}q_{cl}(t)$.

(i) Determine the normalization coefficient N (assuming that $q_{cl}(t)$ is real).

(ii) Determine the expectation value of $\hat{q}$ in the state $\psi(q,t)$ and show that in the same state

$$\Delta q^2 \equiv \langle(\hat{q} - \langle\hat{q}\rangle)^2\rangle = \frac{|\sigma(t)|^4}{2\Re\sigma^2(t)}. \tag{3}$$

(iii) Find the expression for the wave function in the momentum space.

(iv) Using the previous result, evaluate the expectation value of $\hat{p}$ and Δp^2.

(v) Verify that the Heisenberg uncertainty relations are satisfied. For which values of $\sigma(t)$ do we get an equality?

(vi) Until now we considered the Gaussian packet purely kinematically, without any dynamics. Consider now the Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{4}$$

Remember that in Exercise 61 it was shown that $q_{cl}(t)$ satisfies the classical Newton's equations

$$\dot{p}_{cl}(t) = -V'(q_{cl}(t)) \tag{5}$$

while the function $\sigma^2(t)$ satisfied equation

$$\frac{d}{dt}\sigma^2(t) = \frac{i\hbar}{m} + \frac{i}{\hbar}V''(q_{cl}(t))\sigma^4(t). \tag{6}$$

Solve this equation for the free particle and a particle in a homogeneous (constant) force field.

(vii) Analyze the spreading of the wave packet in the momentum space.

(viii) Analyze the spreading of the wave packet in the position space.

Solution

(i) The wave function is Gaussian so to fix N we need to evaluate the integral

$$1 = \int_{-\infty}^{+\infty} |\psi(q,t)|^2 dq = |N|^2 \int_{-\infty}^{+\infty} \exp\left[2\Re\alpha(t) - \frac{1}{2}\left(\frac{1}{\sigma(t)^2} + \frac{1}{\bar{\sigma}(t)^2}\right)(q - q_{cl}(t))^2\right] dq \tag{7}$$

$$= |N|^2 e^{2\Re\alpha(t)} \sqrt{\frac{2\pi}{\frac{1}{\sigma^2} + \frac{1}{\bar{\sigma}^2}}} = |N|^2 e^{2\Re\alpha(t)} \sqrt{\frac{2\pi|\sigma(t)|^4}{2\Re\sigma(t)^2}}. \tag{8}$$

Up to an overall (irrelevant) phase we normalization constant N can be chosen to be

$$N = e^{-\Re\alpha(t)} \sqrt[4]{\frac{\Re\sigma^2(t)}{\pi|\sigma(t)|^4}}. \tag{9}$$

The real part of $\alpha(t)$ is therefore irrelevant and the Gaussian wave function takes the form

$$\psi(q,t) = \sqrt[4]{\frac{\Re\sigma^2(t)}{\pi|\sigma(t)|^4}} \exp\left[i\Im\alpha(t) + \frac{i}{\hbar}p_{cl}(t)(q - q_{cl}(t)) - \frac{1}{2\sigma^2(t)}(q - q_{cl}(t))^2\right]. \tag{10}$$

(ii) The expectation value of $\hat{q}$ is given by the integral

$$\langle\hat{q}\rangle = \int_{-\infty}^{+\infty} q|\psi(q,t)|^2 dq = \sqrt{\frac{\Re\sigma^2(t)}{\pi|\sigma(t)|^4}} \int_{-\infty}^{+\infty} q \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^2(t)} + \frac{1}{\bar{\sigma}^2(t)}\right)(q - q_{cl}(t))^2\right] dq \tag{11}$$

$$= q_{cl}(t) \tag{12}$$

which we see immediately by shifting the integration variable and noticing that the remaining integral is an odd function of q. By the same shift of the integration variable we can reduce the evaluation of Δq^2 to

$$\Delta q^2 = \int_{-\infty}^{+\infty} (q - q_{cl}(t))^2 |\psi(q,t)|^2 dq \tag{13}$$

$$= \sqrt{\frac{\Re\sigma^2(t)}{\pi|\sigma(t)|^4}} \int_{-\infty}^{+\infty} q^2 \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^2(t)} + \frac{1}{\bar{\sigma}^2(t)}\right)q^2\right] dq = \frac{1}{\frac{1}{\sigma^2(t)} + \frac{1}{\bar{\sigma}^2(t)}} = \frac{|\sigma(t)|^4}{2\Re\sigma^2(t)}. \tag{14}$$

as can be seen by a differentiation of the basic Gaussian integral with respect to a:

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}aq^2} dq = \sqrt{\frac{2\pi}{a}} \qquad \longrightarrow \qquad \int_{-\infty}^{+\infty} q^2 e^{-\frac{1}{2}aq^2} dq = \sqrt{\frac{2\pi}{a}} \times \frac{1}{a}. \tag{15}$$

(iii) To find the momentum space wave function, we have by the definition

$$\tilde{\psi}(p,t) = \langle p \,|\, \psi(t)\rangle = \int_{-\infty}^{+\infty} \langle p \,|\, q\rangle \langle q \,|\, \psi(t)\rangle \, dq. \tag{16}$$

Remembering that

$$\langle p \,|\, q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}pq}, \tag{17}$$

we have

$$\tilde{\psi}(p,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\hbar}} \sqrt[4]{\frac{\Re\sigma^2(t)}{\pi|\sigma(t)|^4}} \exp\left[-\frac{i}{\hbar}pq + i\Im\alpha(t) + \frac{i}{\hbar}p_{cl}(t)(q - q_{cl}(t)) - \frac{1}{2\sigma^2(t)}(q - q_{cl}(t))^2\right] dq. \tag{18}$$

This is again a Gaussian integral so one way to evaluate it is to find a stationary point of the exponent:

$$0 = \frac{d}{dq}\left[-\frac{i}{\hbar}pq + i\Im\alpha(t) + \frac{i}{\hbar}p_{cl}(t)(q - q_{cl}(t)) - \frac{1}{2\sigma^2(t)}(q - q_{cl}(t))^2\right] \tag{19}$$

$$0 = -\frac{i}{\hbar}(p - p_{cl}(t)) - \frac{1}{\sigma^2(t)}(q - q_{cl}(t)) \tag{20}$$

$$q - q_{cl}(t) = -\frac{i\sigma^2(t)}{\hbar}(p - p_{cl}(t)). \tag{21}$$

Evaluating the square root prefactor as before, we find for the momentum space wave function

$$\tilde{\psi}(p,t) = \sqrt[4]{\frac{\Re\sigma^2(t)}{\pi|\sigma(t)|^4}} \frac{\sigma(t)}{\sqrt{\hbar}} \exp\left[-\frac{\sigma^2(t)}{2\hbar^2}(p - p_{cl}(t))^2 - \frac{i}{\hbar}pq_{cl}(t) + i\Im\alpha(t)\right] \tag{22}$$

which is again gaussian and it is easy to check that it is properly normalized.

(iv) Using again the odd/even integration argument we find that

$$\langle \hat{p}\rangle = \int_{-\infty}^{+\infty} p|\tilde{\psi}(p,t)|^2 = p_{cl}(t). \tag{23}$$

For the uncertainty (variance) we find

$$\Delta p^2 = \int_{-\infty}^{+\infty} p^2|\tilde{\psi}(p,t)|^2 = \frac{1}{\frac{\sigma^2(t)}{\hbar^2} + \frac{\bar{\sigma}^2(t)}{\hbar^2}} = \frac{\hbar^2}{2\Re\sigma^2(t)}. \tag{24}$$

(v) The Heisenberg uncertainty relations tell us that

$$\Delta q^2 \Delta p^2 \geq \frac{\hbar^2}{4}. \tag{25}$$

From our results we have

$$\Delta q^2 \Delta p^2 = \frac{|\sigma(t)|^4}{2\Re\sigma^2(t)} \frac{\hbar^2}{2\Re\sigma^2(t)} = \frac{\hbar^2}{4} \frac{(\Re\sigma^2(t))^2 + (\Im\sigma^2(t))^2}{\Re\sigma^2(t)} \tag{26}$$

which saturates the Heisenberg relations if and only if the imaginary part of $\sigma^2(t)$ vanishes.

(vi) If the particle is free, we have $V = 0$ so the equation for $\sigma^2(t)$ is

$$\frac{d}{dt}\sigma^2(t) = \frac{i\hbar}{m} \tag{27}$$

If we have a constant field, the second derivatives of the potential still vanish so we find the same equation. The solution of this equation are simple:

$$\sigma^2(t_f) - \sigma^2(t_i) = \frac{i\hbar}{m}(t_f - t_i). \tag{28}$$

(vii) The real part of $\sigma^2(t)$ is not changing in time. Since Δp^2 is given only in terms of the real part, the momentum space wave packet does not spread. This is consistent with the fact that by the translation symmetry of the system the Hamiltonian commutes in the momentum (for the free particle). For the constant force the momentum space wave packet translates but the shape (and so the variance Δp^2) stays the same.

(viii) We see that the imaginary part of $\sigma^2(t)$ is linearly increasing in time, so at one point in time t_{min} the imaginary part vanishes and from the previous questions we know that at this point in time we had a Gaussian packet minimizing the uncertainty relations. The width of the position space wave packet is

$$\Delta q^2 = \frac{(\Re\sigma^2)^2 + (\Im\sigma^2)^2}{2\Re\sigma^2} = \frac{\hbar^2}{4\Delta p^2} + \frac{(\Im\sigma^2)^2\Delta p^2}{\hbar^2} \tag{29}$$

Since $\Im\sigma^2$ is linearly increasing with zero at t_m, we can write this as

$$\Delta q^2(t) = \frac{\hbar^2}{4\Delta p^2} + \frac{\Delta p^2(t-t_{min})^2}{m^2} = (\Delta q^2)_{min} + \Delta v^2(t-t_{min})^2 = (\Delta q^2)_{min} + \frac{\hbar^2}{4m^2(\Delta q^2)_{min}}(t-t_{min})^2. \tag{30}$$

We see that we have two sources of spreading of the particle at time t: first of all, if we start with the minimizing wave packet at time t_{min}, it has certain initial width at t_{min}. This initial $(\Delta q^2)_{min}$ determines via the Heisenberg uncertainty relations the initial distribution of momenta/velocities Δv^2:

$$\Delta v^2 = \frac{\Delta p^2}{m^2} = \frac{\hbar^2}{4m^2(\Delta q^2)_{min}} \tag{31}$$

The second term describes how the packet spreads in time due to this distribution of velocities.

Exercise 78

A Stern-Gerlach experiment oriented in the z-direction is fed with silver atoms. The result is that half of the atoms is measured with spin up and the other half with spin down.

(i) Write down the spin state of one silver atom.

(ii) The Stern-Gerlach experiment can be interpreted as a spin operator $\hat{S}_z$ with eigenstates $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ and corresponding eigenvalues $\pm\frac{\hbar}{2}$. Write $\hat{S}_z, \hat{S}_x$ and $\hat{S}_y$ in terms of their eigenstates.

(iii) Use the fact that $|\uparrow_x\rangle$ measured in z-direction gives eigenvalues $\pm\frac{\hbar}{2}$ with $\frac{1}{2}$ probability but never $-\frac{\hbar}{2}$ if measured in x-direction to write $\hat{S}_x$ in terms of $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$. There should be one free parameter left.

(iv) Repeat the same for the y-direction.

(v) Use the fact that $|\uparrow_x\rangle$ measured in y-direction gives eigenvalues $\pm\frac{\hbar}{2}$ with $\frac{1}{2}$ probability to fix the remaining free parameter.

(vi) You now have $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in the eigenbasis of $\hat{S}_z$. What do they remind you of?

Solution:

(i) $|\phi\rangle = \frac{1}{\sqrt{2}}(e^{i\alpha}|\uparrow_z\rangle + e^{i\beta}|\downarrow_z\rangle)$ with $\alpha, \beta \in \mathbb{R}$

(ii) $\hat{S}_i = \frac{\hbar}{2}(|\uparrow_i\rangle\langle\uparrow_i| - |\downarrow_i\rangle\langle\downarrow_i|)$ with $i \in \{x, y, z\}$

(iii) $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(e^{i\phi}|\uparrow_z\rangle + e^{i\psi}|\downarrow_z\rangle) = \frac{1}{\sqrt{2}}e^{i\phi}(|\uparrow_z\rangle + e^{i\theta_1}|\downarrow_z\rangle)$ where $\theta_1 = \psi - \phi$.
We can drop the prefactor $e^{i\phi}$ as it is just an overall phase with no physical relevance.
Repeating this for $|\downarrow_x\rangle$ we arrive at:
$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + e^{i\theta_2}|\downarrow_z\rangle)$
To eliminate θ_2 we use that:
$0 = \langle\downarrow_x \mid \uparrow_x\rangle = \frac{1}{2}\left(1 + e^{i(\theta_1-\theta_2)}\right) \Leftrightarrow e^{i(\theta_1-\theta_2)} = -1 \Rightarrow \theta_2 = \theta_1 - \pi$
And therefore $|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle - e^{i\theta_1}|\downarrow_z\rangle)$
$\hat{S}_x = \frac{\hbar}{2}\left(e^{i\theta_1}|\downarrow_z\rangle\langle\uparrow_z| + e^{-i\theta_1}|\uparrow_z\rangle\langle\downarrow_z|\right)$

(iv) Repeating the same for y we get:
$|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + \mathrm{e}^{i\gamma_1}\langle\downarrow_z|\right)$
$|\downarrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle - \mathrm{e}^{i\gamma_1}\langle\downarrow_z|\right)$
$\hat{S}_y = \frac{\hbar}{2}\left(\mathrm{e}^{i\gamma_1}|\downarrow_z\rangle\langle\uparrow_z| + \mathrm{e}^{-i\gamma_1}|\uparrow_z\rangle\langle\downarrow_z|\right)$

(v) $\frac{1}{2} = |\langle\downarrow_x \mid \uparrow_y\rangle|^2 = \frac{1}{4}\left|1 - \mathrm{e}^{i(\gamma_1-\theta_1)}\right|^2 = \frac{1}{2} - \frac{1}{4}\mathrm{e}^{-i(\gamma_1-\theta_1)}\left(1 + \mathrm{e}^{2i(\gamma_1-\theta_1)}\right) \Leftrightarrow \gamma_1 - \theta_1 = \frac{\pi}{2}$
As we can choose an overall phase freely, we can set $\theta = 0$. With this choice the spin operators are given by:
$\hat{S}_x = \frac{\hbar}{2}\left(|\downarrow_z\rangle\langle\uparrow_z| + |\uparrow_z\rangle\langle\downarrow_z|\right)$
$\hat{S}_y = \frac{\hbar}{2}\left(i|\downarrow_z\rangle\langle\uparrow_z| - i|\uparrow_z\rangle\langle\downarrow_z|\right)$

(vi) Representing the spin states by two dimensional vectors $|\uparrow_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ then the spin operators are:

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_z \qquad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_x \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_y$$

where $\hat{\sigma}_i$ are the Pauli matrices which are the generators of $SU(2)$.

Exercise 79

Assume that a system S is coupled to a measuring device M. The state before the measurement is

$$|\Psi_0\rangle = |s\rangle \otimes |M_0\rangle\,, \qquad |s\rangle = \sum_n a_n |s_n\rangle \tag{32}$$

and the state after the measurement is

$$|\Psi\rangle = \sum_n a_n |s_n\rangle \otimes |M_n\rangle \tag{33}$$

where the device states $|M_n\rangle$ are orthogonal. Find the density operator $\hat{\rho}$ for the subsystem S after the measurement.

Solution:

After measurement the density matrix of the whole system is:

$$\hat{\rho}_{S+M} = |\Psi\rangle\langle\Psi| = \sum_{m,n} a_m^* a_n |s_n\rangle \otimes |M_n\rangle\langle M_m| \otimes \langle s_m|$$

The density matrix of the subsystem S is is obtained by tracing over the subsytem M:

$$\begin{aligned}\hat{\rho}_S = \mathrm{tr}_M(\hat{\rho}_{M+S}) &= \sum_k \langle M_k|\left(\sum_{m,n} a_m^* a_n |s_n\rangle \otimes |M_n\rangle\langle M_m| \otimes \langle s_m|\right)|M_k\rangle \\ &= \sum_k\sum_{m,n} a_m^* a_n |s_n\rangle\langle s_m|\delta_{kn}\delta_{km} = \sum_{m,n} a_m^* a_n \delta_{nm}|s_n\rangle\langle s_m| = \sum_n |a_n|^2 |s_n\rangle\langle s_n|\end{aligned}$$

Exercise 80

Consider the same physical system as in exercise 73.

(i) Show that the classical equation of motion for $\varphi(t)$ is

$$T\ddot{\varphi} = \dot{p_\varphi} = -g(t)p_q - Ig(t)^2\varphi. \tag{34}$$

(ii) Quantize the Hamiltonian found the last week (exercise 73). How does the equation of motion change after the quantization?

(iii) Assume that the driving current is constant, i.e. $g(t) = g_0$. Find the general (operator) solution of the equations of motion with the initial conditions $\hat{\varphi}(t=0) = \hat{\varphi}_0$ and $\dot{\hat{\varphi}}(t=0) = T^{-1}\hat{p}_\varphi^{(0)}$. It is useful to introduce the frequency $Ig_0^2 = T\omega^2$. The result you should get is

$$\hat{\varphi}(t) = -\frac{\hat{p}_q}{Ig_0} + \left(\hat{\varphi}_0 + \frac{\hat{p}_q}{Ig_0}\right)\cos(\omega t) + \frac{\hat{p}_\varphi^{(0)}}{T\omega}\sin(\omega t). \tag{35}$$

(iv) Verify the commutation relation

$$[\hat{\varphi}(t), \hat{\varphi}(t+\tau)] = \frac{i\hbar}{T\omega}\sin(\omega\tau). \tag{36}$$

Solution:

(i) The Hamiltonian of the system is: $H = \frac{p_q^2}{2I} + \frac{p_\varphi^2}{2T} + \frac{Ig^2(t)\varphi^2}{2} + g(t)\varphi p_q$
From here the equation of motion is just the Hamiltonian equations: $T\ddot{\varphi} = \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi}$

(ii) Quantizing this Hamiltonian we define $\hat{\varphi}, \hat{p}_\varphi, \hat{q}, \hat{p}_q$ as hermitian operators and impose the commutation relations $[\hat{\varphi}, \hat{p}_\varphi] = i\hbar$ and $[\hat{q}, \hat{p}_q] = i\hbar$. The equations of motion for the operators are given by the Heisenberg equation: $\dot{\hat{p}}_\varphi = \frac{i}{\hbar}[\hat{H}, \hat{p}_\varphi]$.
Using the canonical commutation relations we see that this is just the calssical equation of motion. This was to be expected as there are no terms in the Hamiltonian which mix non-communting operators.

(iii) The homogeneous solution is: $\hat{\varphi}_h(t) = \hat{A}\sin(\omega t) + \hat{B}\cos(\omega t)$
A special solution for the whole equation is: $\hat{\varphi}_s = -\frac{\hat{p}_q}{Ig_0}$
The general solution is therefore: $\hat{\varphi}(t) = \hat{\varphi}_h(t) + \hat{\varphi}_s = \hat{A}\sin(\omega t) + \hat{B}\cos(\omega t) - \frac{\hat{p}_q}{Ig_0}$
Now we impose the boundary conditions:
$\hat{\varphi}(0) = \hat{B} - \frac{\hat{p}_q}{Ig_0} \equiv \hat{\varphi}_0 \Leftrightarrow \hat{B} = \hat{\varphi}_0 + \frac{\hat{p}_q}{Ig_0}$
$\dot{\hat{\varphi}}(0) = \hat{A}\omega \equiv \frac{\hat{p}_\varphi^0}{T} \Leftrightarrow \hat{A} = \frac{\hat{p}_\varphi^{(0)}}{T\omega}$
$\Rightarrow \hat{\varphi}(t) = \left(\hat{\varphi}_0 + \frac{\hat{p}_q}{Ig_0}\right)\cos\omega t + \frac{\hat{p}_\varphi^{(0)}}{T\omega}\sin\omega t - \frac{\hat{p}_q}{Ig_0}$

(iv) The commutator is given by:

$$\begin{aligned}[\hat{\varphi}(t), \hat{\varphi}(t+\tau)] &= [\hat{\varphi}_0, \hat{p}_\varphi^{(0)}]\frac{\cos(\omega t)\sin(\omega(t+\tau))}{T\omega} + [\hat{p}_\varphi^{(0)}, \hat{\varphi}_0]\frac{\cos(\omega(t+\tau))\sin(\omega t)}{T\omega} \\ &= [\hat{\varphi}_0, \hat{p}_\varphi^{(0)}]\frac{\cos(\omega t)\sin(\omega(t+\tau)) - \cos(\omega(t+\tau))\sin(\omega t)}{T\omega} = \frac{i\hbar}{T\omega}\sin\omega\tau\end{aligned}$$

where in the last step it was used that: $\sin x\cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$

Exercise 81

Consider a system of two spin $\frac{1}{2}$ particles. Let us denote $|\uparrow\rangle$ the normalized eigenvector of σ_3 with eigenvalue $+1$ and $|\downarrow\rangle$ the eigenvector with eigenvalue -1.

(i) Express the normalized eigenvectors of σ_1 $|\rightarrow\rangle$ (eigenvalue $+1$) and $|\leftarrow\rangle$ (eigenvalue -1) in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$.

(ii) Find a vector in two particle Hilbert space

$$|\psi\rangle = \alpha_1|\uparrow\uparrow\rangle + \ldots + \alpha_4|\downarrow\downarrow\rangle \tag{37}$$

such that it has total spin zero, i.e. that all components of $\hat{\boldsymbol{J}}$ annihilate this vector. The total spin operator is

$$\hat{\boldsymbol{J}} = \frac{1}{2}\boldsymbol{\sigma}\otimes\mathbb{1} + \frac{1}{2}\mathbb{1}\otimes\boldsymbol{\sigma}. \tag{38}$$

(iii) Express $|\psi\rangle$ in terms of $|\rightarrow\rangle$ and $|\leftarrow\rangle$ basis. How can you interpret the result?

(iv) Show by a direct calculation that $|\psi\rangle$ cannot be expressed as a product state $|\alpha\rangle\otimes|\beta\rangle$

(v) Calculate the reduced density matrix associated to $|\psi\rangle$ if we can no longer perform any measurements on the second particle (i.e. trace over Hilbert space of the second particle).

(vi) Find the entanglement entropy associated to this reduced density matrix and explain how the result proves that $|\psi\rangle$ was not a product state.

Solution

(i) Since the first Pauli matrix acts as $\sigma_1 | \uparrow\rangle = | \downarrow\rangle$ and vice versa, the normalized eigenstate of σ_1 with eigenvalue $+1$ is

$$| \rightarrow\rangle = \frac{1}{\sqrt{2}} (| \uparrow\rangle + | \downarrow\rangle) . \tag{39}$$

The eigenstate with eigenvalue -1 should be orthogonal to this so we choose it to be

$$| \leftarrow\rangle = \frac{1}{\sqrt{2}} (| \uparrow\rangle - | \downarrow\rangle) . \tag{40}$$

(ii) Total spin zero means that the vector $| \psi\rangle$ must be killed by all components of $\hat{\boldsymbol{J}}$. The third component $\hat{J}_3$ counts the number of up arrow minus the number of down arrows so we must have

$$| \psi\rangle \sim | \uparrow\downarrow\rangle + \alpha | \downarrow\uparrow\rangle \tag{41}$$

with some constant α. If it is annihilated by $\hat{J}_1$ and $\hat{J}_2$, it must be also annihilated by $\hat{J}_+$ where

$$\hat{J}_+ = \hat{J}_1 + i\hat{J}_2 = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \mathbb{1} + \frac{1}{2}\mathbb{1} \otimes (\sigma_1 + i\sigma_2) \equiv \sigma_+ \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_+ . \tag{42}$$

The matrix σ_+ acts very simply: $\sigma_+ | \uparrow\rangle = 0$ and $\sigma_+ | \downarrow\rangle = | \uparrow\rangle$ so it flips down spin up and annihilates the up spin. Since we want $\hat{J}_+$ to annihilate $| \psi\rangle$, we need to have

$$0 = \hat{J}_+ | \psi\rangle = \hat{J}_+ | \uparrow\downarrow\rangle + \alpha\hat{J}_+ | \downarrow\uparrow\rangle = (1 + \alpha) | \uparrow\uparrow\rangle . \tag{43}$$

We thus find $\alpha = -1$ and the properly normalized state is

$$| \psi\rangle = \frac{1}{\sqrt{2}} (| \uparrow\downarrow\rangle - | \downarrow\uparrow\rangle) . \tag{44}$$

It is easy to see that this state is annihilated by all $\hat{\boldsymbol{J}}$ operators.

(iii) A simple calculation shows that

$$| \psi\rangle = \frac{1}{\sqrt{2}} (| \leftarrow\rightarrow\rangle - | \rightarrow\leftarrow\rangle) . \tag{45}$$

This has the same form as when expressed in terms of σ_3 eigenstates. This makes sense because the condition of having spin 0 (i.e. being a scalar) does not depend on the choice of coordinate axes so $| \psi\rangle$ should look the same independently of the choice of the axes.

(iv) If $| \psi\rangle$ was a linear combination or

$$| \alpha\rangle = \alpha_\uparrow | \uparrow\rangle + \alpha_\downarrow | \downarrow\rangle \qquad \text{and} \qquad | \beta\rangle = \beta_\uparrow | \uparrow\rangle + \beta_\downarrow | \downarrow\rangle \tag{46}$$

we would need to have

$$\frac{1}{\sqrt{2}} (| \uparrow\downarrow\rangle - | \downarrow\uparrow\rangle) = \alpha_\uparrow\beta_\uparrow | \uparrow\uparrow\rangle + \alpha_\uparrow\beta_\downarrow | \uparrow\downarrow\rangle + \ldots \tag{47}$$

which has no solution because we need to have $\alpha_\uparrow\beta_\downarrow \neq 0 \neq \alpha_\downarrow\beta_\uparrow$ which forces all four constants to be non-zero but on the other hand we need to have $\alpha_\uparrow\beta_\uparrow = 0 = \alpha_\downarrow\beta_\downarrow$ which is absurd.

(v) We first write the density matrix associated to $| \psi\rangle$ (before the partial trace)

$$\hat{\rho} = | \psi\rangle \langle\psi | = \frac{1}{2} (| \uparrow\downarrow\rangle \langle\uparrow\downarrow | - | \uparrow\downarrow\rangle \langle\downarrow\uparrow | - | \downarrow\uparrow\rangle \langle\uparrow\downarrow | + | \downarrow\uparrow\rangle \langle\downarrow\uparrow |) . \tag{48}$$

Taking the trace means summing

$$\hat{\rho}_{red} = \mathrm{Tr}_{\mathcal{H}_2}\hat{\rho} = \langle\uparrow |_2 \hat{\rho} | \uparrow\rangle_2 + \langle\downarrow |_2 \hat{\rho} | \downarrow\rangle_2 \tag{49}$$

Only the first and the fourth terms contribute and we find

$$\hat{\rho}_{red} = \frac{1}{2} (| \uparrow\rangle \langle\uparrow | + | \downarrow\rangle \langle\downarrow |) = \frac{1}{2}\hat{\mathbb{1}}. \tag{50}$$

(vi) The eigenvalues of $\hat{\rho}_{red}$ are (two times) $\frac{1}{2}$ so the entanglement entropy is

$$-\sum_{j=1}^{2} p_j \log p_j = -2 \times \tfrac{1}{2} \times \log \tfrac{1}{2} = \log 2. \tag{51}$$

Since this is non-zero, the state $| \psi\rangle$ that we started with was not a product state: if it was a product state, the density matrix would be that of a pure state (even after the partial trace!) and we know that the von Neumann entropy of a pure state is zero.

General information

The *lecture* takes place on:

Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)

The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`