Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 9, discussed December 16 - December 20, 2019

Exercise 53 (central tutorial)

In this exercise we will analyse another way of doing time-dependent perturbation theory, in which we will use the interaction picture. Let's consider a system with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$ where $\hat{V}(t)$ is small.

(i) How is the time-evolution operator in the interaction picture defined?

(ii) Show that the time-evolution operator in the interaction picture satisfies the integral equation

$$\hat{U}_I(t, t_i) = 1 - \frac{i}{\hbar} \int_{t_i}^{t} \hat{V}_I(t') \hat{U}_I(t, t_i) dt' \tag{1}$$

with initial conditions $\hat{U}_I(t_i, t_i) = \mathbb{1}$.

(iii) Show that solving this equation iteratively, one obtains the *"Dyson series"*:

$$\hat{U}_I(t, t_i) = \mathbb{1} - \frac{i}{\hbar} \int_{t_i}^{t} \hat{V}_I(t') dt' + \left(-\frac{i}{\hbar}\right)^2 \int_{t_i}^{t} \hat{V}_I(t_1) dt_1 \int_{t_i}^{t_1} \hat{V}_I(t_2) dt_2 + ... \tag{2}$$

(iv) Consider now the following situation. Suppose that for $t < t_i$ and $t > t_f$, the system is described by the free Hamiltonian which satisfies $\hat{H}_0 \,|\, \psi_n \rangle = E_n \,|\, \psi_n \rangle$, and, for $t_i < t < t_f$ the system is described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$. What is the probability that if the system was initially in the state $|\, \psi_n \rangle$, it will be found in the state $|\, \psi_m \rangle$ with $m \neq n$, after a time interval $T = t_f - t_i$? *Hint:*

$$P(n \to m) = \left| \langle \psi_m \,|\, \hat{U}_I(t_f, t_i) \,|\, \psi_n \rangle \right|^2 \tag{3}$$

(v) Let now H_0 be the Hamiltonian for a harmonic oscillator, and $\hat{V}(q, t) = V_0 \hat{q}^3 e^{-t/\tau}$ where V_0 is constant. Calculate the probability for a transition from the ground state at $t_i = 0$ to n'th excited state for $t_f \to \infty$. *Hint: You may use the following integral*

$$\left| \int_0^{\infty} e^{-\left(\frac{1}{\tau} - inw\right)t} dt \right|^2 = \frac{1}{n^2 w^2 + \frac{1}{\tau^2}} \tag{4}$$

Exercise 54 (central tutorial)

Let $|\, \phi_n \rangle$ be the eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ which has no degenerate eigenvalues. The complete system shall be described by $\hat{H} = \hat{H}_0 + \hat{V}$. The correction to a state can generally be written as

$$|\, \bar{\phi}_n \rangle = |\, \phi_n \rangle + \sum_l c_n^{\ l} \,|\, \phi_l \rangle \tag{5}$$

Calculate $c_n^{\ l}$, at first order in $\hat{V}$.

Exercise 55

The transition amplitude for the two level system is defined as in the lecture by

$$P_T(a \to b) = |\lambda_{ba}^{(1)}|^2 = \frac{1}{\hbar^2} \left| \int_0^T dt_I V_{ba}(t_I) e^{i\omega_{ba} t_I} \right|^2 \tag{6}$$

where

$$V_{ba}(t_I) = \int dq_I \psi_b^*(q_I) V(q_I, t_I) \psi_a(q_I) \quad \text{and} \quad \omega_{ba} = \frac{E_b - E_a}{\hbar} \tag{7}$$

is the matrix between the eigenstates of the energy levels. Show that

$$P_T(a \to b) = P_T(b \to a) \tag{8}$$

Exercise 56

Prove that

$$\lim_{T \to \infty} \frac{\sin^2(\alpha T)}{\pi \alpha^2 T} = \delta(\alpha) \tag{9}$$

Hint: you may find useful the following result, $\int_{\mathbb{R}} dx \frac{\sin^2(x)}{x^2} = \pi$.

Exercise 57

Show for an isotropic stochastic electromagnetic field, that

$$\left\langle |\vec{E}|^2 \right\rangle = 3 \left\langle E_z^2 \right\rangle . \tag{10}$$