Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 11, discussed January 13 - January 17, 2019

Exercise 64 (Central Tutorial)

Let's consider the hydrogen atom ignoring the spins of the electron and of the proton. In order to have non-zero transition probability from an energy state $|\psi_a\rangle$ to another one $|\psi_b\rangle$, the matrix element of the dipole moment operator $\boldsymbol{D}_{ba} \equiv e\langle\psi_b|\hat{\boldsymbol{r}}|\psi_a\rangle$ must be non-zero too ($\hat{\boldsymbol{r}} = (\hat{x}, \hat{y}, \hat{z})$). The conditions to have non-zero transition probabilities are called *selection rules*.

(i) Which are good quantum numbers that characterize the state $|\psi\rangle$ of the hydrogen atom? To what do they physically correspond?

(ii) Which values can those quantum numbers have? What is the degeneracy of the state for a given energy level?

(iii) Recalling the definition of the angular momentum operator $\hat{\boldsymbol{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$, calculate

$$\left[\hat{L}_z, \hat{x}\right] \quad , \quad \left[\hat{L}_z, \hat{y}\right] \quad , \quad \left[\hat{L}_z, \hat{z}\right] \tag{1}$$

(iv) Using the results of the previous point and taking the expectation values of those between two different states $|\psi_a\rangle$ and $|\psi_b\rangle$, derive the selection rules to have $\boldsymbol{D}_{ab} \neq 0$.

(v) These are not the only selection rules. Recall the definition of the Casimir operator $\hat{L}^2$; what is the action of this operator on an eigenstate $|\psi\rangle$? Prove that

$$\left[\hat{L}^2, \left[\hat{L}^2, \hat{\boldsymbol{r}}\right]\right] = 2\hbar^2(\hat{\boldsymbol{r}}\hat{L}^2 + \hat{L}^2\hat{\boldsymbol{r}}) \tag{2}$$

(vi) Use the previous results to find other selection rules for the transition between $|\psi_a\rangle$ and $|\psi_b\rangle$.

(vii) What do these selection rules correspond physically to?

Exercise 65

Show that the life time τ of an atom in an excited state is inversely proportional to the Einstein-coefficient A of spontaneous emission.

Exercise 66

Consider a two-level system as in the lecture. Write the equations for the occupation number of the lower level, $\frac{dN_a}{dt}$, and upper level, $\frac{dN_b}{dt}$. Using these, show that $N_a + N_b = const.$

Exercise 67

Consider the general Schrödinger equation

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2\psi(q,t)}{\partial q^2} + V(q)\psi(q,t) \tag{3}$$

where V is at most quadratic in q. Validate that the Ansatz

$$\psi(q,t) = \frac{1}{N}\exp\left[\alpha(t) + \frac{i}{\hbar}p_{cl}(t)(q-q_{cl}(t)) - \frac{(q-q_{cl}(t))^2}{2\sigma^2(t)}\right] \tag{4}$$

where $p_{cl} = m\frac{dq_{cl}}{dt}$ leads to an equation of the form

$$F_1(t) + F_2(t)(q-q_{cl}(t)) + F_3(t)(q-q_{cl}(t))^2 = 0 \tag{5}$$

Show that

$$\begin{aligned}
F_1 = 0 &\equiv \frac{d\alpha}{dt} = \frac{i}{\hbar}(\frac{p_{cl}^2}{2m} - V(q_{cl})) - \frac{i\hbar}{2m\sigma^2(t)} \\
F_2 = 0 &\equiv \frac{dp_{cl}}{dt} = -\frac{\partial V}{\partial q}(q_{cl}) \\
F_3 = 0 &\equiv \frac{d\sigma^2}{dt} = \frac{i\hbar}{m} - \frac{i}{\hbar}\frac{\partial^2 V}{\partial q^2}\sigma^4
\end{aligned} \tag{6}$$

General information

The *lecture* takes place on:
- Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
- Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)

The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`