Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 10, discussed January 7 - January 10, 2019

Exercise 58 - Short questions

(i) Name at least four physical effects which can be explained with quantum mechanics and not with classical mechanics.

(ii) Suppose you are given a Hamiltonian of a free particle. What are the main things you should do in order to quantize the system?

(iii) Which of the following operators are Hermitian? Position operator, momentum operator, annihilation operator (defined in the analysis of harmonic oscillator), Hamiltonian. Give another example of a non-Hermitian operator.

(iv) Simplify the following expression

$$[\hat{p}\hat{q}, \hat{q}^2] \tag{1}$$

(v) What property does an operator have to satisfy to correspond to a physical observable? Why?

(vi) Give an example of compatible operators (i.e. those who's commutator vanishes). What does this imply for the eigenvectors of the operators if the operators have non-degenerate eigenvalues? Why?

(vii) Show that the hermicity of Hamilton operator follows from the unitarity of the time-evolution operator.

(viii) Consider a system of two pendulums of length l on whose ends are attached balls of mass m. Suppose that these two balls are connected via weak spring k. (Figure 1.) The Lagrangian of the system is then given by

$$L = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 - \frac{1}{2}mgl\theta_1^2 - \frac{1}{2}mgl\theta_2^2 - \frac{1}{2}kl^2(\theta_2 - \theta_1)^2 \tag{2}$$

Find the normal coordinates in terms of θ_1 and θ_2. What is the spectrum of the Hamiltonian after quantization?

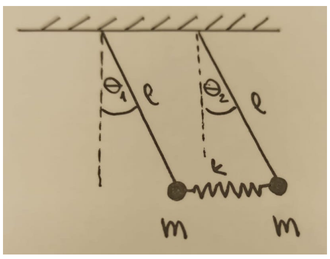

Figure 1: Pendulums connected via spring.

(ix) Suppose we have 2 systems which in the beginning do not interact. Then they interact and at some point stop interacting again. Can we write the wave function of the full system as a product of wave functions of first and second system after they stop interacting?

(x) Consider a particle in a potential $\hat{V}$. How can you interpret the zeroth, first and second order of the perturbation expansion of the propagator $K(f, i)$? Write them down.

Exercise 59 - Quantization

Consider a classical scalar field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu} - m^2\phi^2\right). \tag{3}$$

(i) Find the conjugate momenta and the Hamiltonian density.

(ii) Find the equation of motion and by Fourier expanding the field as

$$\phi(t,\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}.\boldsymbol{x}}\phi_{\boldsymbol{k}}(t), \tag{4}$$

show that the dispersion relation is $\omega_k^2 = k^2 + m^2$. What property does $\phi_{\boldsymbol{k}}(t)$ have under complex conjugation?

We can define the equal time Poisson bracket for two functionals f and g as

$$\{f(t,\boldsymbol{x}), g(t,\boldsymbol{y})\} = \int d^3z \left(\frac{\delta f(t,\boldsymbol{x})}{\delta\phi(t,\boldsymbol{z})}\frac{\delta g(t,\boldsymbol{y})}{\delta\pi(t,\boldsymbol{z})} - \frac{\delta g(t,\boldsymbol{y})}{\delta\phi(t,\boldsymbol{z})}\frac{\delta f(t,\boldsymbol{x})}{\delta\pi(t,\boldsymbol{z})}\right). \tag{5}$$

(iii) Find the equal time Poisson brackets of all combinations of ϕ and π.

(iv) We can define

$$a_{\boldsymbol{k}}(t) = \int \frac{d^3x}{(2\pi)^{3/2}} a(\boldsymbol{x})e^{-i\boldsymbol{k}.\boldsymbol{x}} = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\boldsymbol{k}.\boldsymbol{x}}\sqrt{\frac{\omega_k}{2}}\left(\phi(t,\boldsymbol{x}) + \frac{i}{\omega_k}\pi(t,\boldsymbol{x})\right) \tag{6}$$

and the corresponding complex conjugate $a_{\boldsymbol{k}}^*(t)$. Use your result of the previous question to find the equal time Poisson brackets of all combinations of $a_{\boldsymbol{k}}(t)$ and $a_{\boldsymbol{k}}^*(t)$.

(v) Find the equations of motion of $a_{\boldsymbol{k}}(t)$ and $a_{\boldsymbol{k}}^*(t)$ and show that

$$a_{\boldsymbol{k}}(t) = a_{\boldsymbol{k}}e^{-i\omega t} \quad \text{and} \quad a_{\boldsymbol{k}}^*(t) = a_{\boldsymbol{k}}^*e^{i\omega t} \tag{7}$$

Use this to show that we can write the field as

$$\begin{aligned}\phi(t,\boldsymbol{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left(a_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}.\boldsymbol{x}} + a_{\boldsymbol{k}}(t)^*e^{-i\boldsymbol{k}.\boldsymbol{x}}\right)\\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left(a_{\boldsymbol{k}}e^{-ik.x} + a_{\boldsymbol{k}}^*e^{ik.x}\right),\end{aligned} \tag{8}$$

where $k.x = \omega t - \boldsymbol{k}.\boldsymbol{x}$.

(vi) So far everything has been done classically. What do we have to do to move to the quantum picture? Write down all the Poisson brackets you calculated in the quantum picture. What is different?

(vii) Show that the Heisenberg equation reproduces the equations of motion of the classical system for ϕ and π in the quantum mechanical picture.

Exercise 60 - Density matrix in 2-dimensions and Bloch Sphere

We are going to discuss the density matrix formalism in the case of a 2-dimensional Hilbert space; as a physical example, it corresponds to the Hilbert space of a spin 1/2 particle.

(i) First of all, let's see how a (pure) state looks like. Prove that a generic normalized state (i.e. a ray-vector of the Hilbert space) can be written as

$$|\,\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\,0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\,1\rangle \quad \text{with} \quad \theta\in[0,\pi] \quad , \quad \phi\in[0,2\pi] \tag{9}$$

where $|\,0\rangle, |\,1\rangle$ are the eigenvectors of the Pauli matrix $\hat{\sigma}_3$ (the choice of $\hat{\sigma}_3$ is purely conventional). Show that this means that the 2-dimensional Hilbert space of normalized states is isomorphic to the 2-Sphere S^2. In this geometric picture every state $|\,\psi\rangle$ is a point of S^2: in this context, the sphere is called **Bloch Sphere**.

(ii) Let's try now to enlarge our view considering also mixed states. To do that, we want to study first how a general density matrix looks like. Recalling that $\{\mathbb{1}, \boldsymbol{\sigma}\}$ (where $\boldsymbol{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$) form a basis for the operators in a 2-dimensional Hilbert space, it means that the most general operator is of the form:

$$\hat{\rho}_{\alpha,\boldsymbol{\beta}} = \alpha\mathbb{1} + \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \tag{10}$$

Recalling the properties of a density matrix, find conditions on $\alpha, \boldsymbol{\beta}$ to make $\hat{\rho}$ a density matrix.

You should find:

$$\hat{\rho}_{\boldsymbol{r}} = \frac{1}{2}(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma}) \quad \text{with} \quad |\boldsymbol{r}| \leq 1 \quad \boldsymbol{r} \in \mathbb{R}^3 \tag{11}$$

Therefore, all states (pure and mixed) live within the interior of the Bloch Sphere.

(iii) For which $\boldsymbol{r}$ does $\hat{\rho}$ represent a pure state? And a mixed one? Would you expect this from what you had found in part (i)?

(iv) Show that any two orthonormal vectors correspond to antipodal points on the Bloch sphere.

(v) Consider the Hamilton operator $\hat{H} = a\boldsymbol{B} \cdot \boldsymbol{\sigma}$. Compute its expectation value in the state given by $\hat{\rho}_{\boldsymbol{r}}$

(vi) For which value of $\boldsymbol{r}$ does $\hat{\rho}_{\boldsymbol{r}}$ correspond to a spin one-half particle which is randomly produced with probability 1/2 in the state $|\,0\rangle$ and with probability 1/2 in the state $|\,1\rangle$? What is the the entropy in this case?

(vii) A density matrix is said to be describing a maximally entangled state if it has maximum entanglement entropy (or Von Neuman entropy). What is the maximum entanglement entropy in this case? To which $\boldsymbol{r}$ (i.e. which point(s) in the Bloch Sphere) does it correspond?

(viii) Consider now spin one-half particles which are produced with spins in any direction with equal probability. Calculate the density matrix.

Exercise 61 - Time dependent two level system

Consider a two level system with orthonormal basis (ONB) $|\,1\rangle$,$|\,2\rangle$. The Hamiltonian is

$$H = H_0 + V(t) \tag{12}$$

where

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 & \delta e^{iwt} \\ \delta e^{-iwt} & 0 \end{pmatrix} \tag{13}$$

(i) Write down an equation for the time evolution in the interaction picture. Spell it out in components. You obtain a system of two coupled differential equations.

Hint: Write $|\,\psi(t)\rangle_I$ as $|\,\psi(t)\rangle_I = c_1(t)\,|\,1\rangle + c_2(t)\,|\,2\rangle$ (two level system).

(ii) Show that one can eliminate $c_1(t)$ to obtain a differential equation for $c_2(t)$.

(iii) At $t = 0$ the system is in state $|\,1\rangle$. Show that the above equation is solved by

$$c_2(t) = Ae^{-it(\omega-\omega_{21})/2}\sin(\Omega t) \tag{14}$$

where A is a normalization constant, $\omega_{21} = (E_2 - E_1)/\hbar$ and

$$\Omega^2 = \delta^2/\hbar^2 + \frac{(\omega - \omega_{21})^2}{4} \tag{15}$$

(iv) Compute $c_1(t)$ and A.

(v) What is the probability to find the system in state $|\,2\rangle$ after the time t? Determine also the maximum (over t) probability.

(vi) Compute the previous transition probability in first order perturbation theory, taking into account again that at $t = 0$ the system is in state $|\,1\rangle$. Then compare to the exact result. When do the results agree?

Exercise 62 - Scattering

The scattering amplitude in the first Born approximation is given by

$$f(\vartheta,\varphi) = -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} V(\boldsymbol{r}). \tag{16}$$

Here $\boldsymbol{k}$ is the wave vector of the incoming wave and $\boldsymbol{k}'$ is the wave vector of the outgoing wave.

(i) Consider a spherically symmetric potential $V(|\boldsymbol{r}|)$. Simplify the formula in the first Born approximation, i.e. integrate over all angles.

(ii) What is the energy of the incoming wave in terms of $\boldsymbol{k}$? Consider the low energy limit of the scattering amplitude. Show that the expression for it is to the leading order

$$f = -\frac{2m}{\hbar^2}\int_0^\infty r^2 V(r)dr. \tag{17}$$

(iii) Consider now the spherically symmetric potential well of depth V_0 and radius a, i.e. a potential

$$V(\boldsymbol{r}) = \begin{cases} -V_0 & |\boldsymbol{r}| < a \\ 0 & |\boldsymbol{r}| > a. \end{cases} \tag{18}$$

What is the low energy scattering amplitude in this case and what is the total cross section?

Exercise 63 - Propagator of the harmonic oscillator

In this exercise we want to calculate the propagator of the harmonic oscillator explicitly. Remember that the propagator is given by:

$$\mathcal{K}(q_F,t_F;q_I,t_I) = \lim_{N\to\infty}\int \mathrm{d}q_2...\mathrm{d}q_N\mathrm{d}p_1...\mathrm{d}p_N \prod_{j=1}^{N} \langle q_{j+1}\,|\,p_j\rangle\,\langle p_j\,|\,q_j\rangle\, \mathrm{e}^{-\frac{i}{\hbar}\epsilon H(p_j,q_j,t+(j-1)\epsilon)} \tag{19}$$

with the Hamiltonian of the harmonic oscillator given by:

$$H(p,q) = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \tag{20}$$

(i) Consider the sequence of $N\times N$ matrices M_{N+1} given by

$$M_{N+1} = \begin{pmatrix} \alpha & -1 & 0 & 0 & ... & 0 \\ -1 & \alpha & -1 & 0 & ... & 0 \\ 0 & -1 & \alpha & -1 & ... & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & -1 & \alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix} \tag{21}$$

Show that $\det(M_{N+1}) = \alpha\det(M_N) - \det(M_{N-1})$.

(ii) Now assume that $\det(M_N) = f(\epsilon N)$ and $\alpha = 2-\epsilon^2\omega^2$.
Show that for $\epsilon\to 0$,

$$f''(\epsilon N) = -\omega^2 f(\epsilon N) \text{ with } f(0)=0, f'(0) = \frac{1}{\epsilon} \tag{22}$$

follows from the result of (i).

(iii) Setting $\epsilon = \frac{t_F - t_I}{N}$ show that in the limit $N\to\infty$ we have

$$\epsilon\det(M_N) \to \frac{1}{\omega}\sin(\omega(t_F-t_I)); \qquad \det(M_N)-\det(M_{N-1}) \to \cos(\omega(t_F-t_I)) \tag{23}$$

(iv) Perform the gaussian integration of p in (19) to write the propagator as

$$\mathcal{K}(q_F,t_F;q_I,t_I) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N/2}\int \mathrm{d}q_2...\mathrm{d}q_N \mathrm{e}^{\frac{im}{2\hbar\epsilon}(q_F^2+q_I^2-2q_Nq_F-2q_2q_I+\sum_{ij=2}^{N}(M_N)_{ij}q_iq_j-\epsilon^2\omega^2q_I^2)} \tag{24}$$

Hint: You may use the formula for n dimensional gaussian integration given in exercise 45.

(v) Perform the Gaussian integration over q and check if the result coincides with the one obtained in Exercise 45.