Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 8, discussed December 9 - December 13, 2019

Exercise 48

Consider the motion of a particle in a potential $V(q(t))$. Show that the second order of the perturbation expansion of the propagator $K(f,i) = K(q_f, q_i; t_f, t_i)$ can be written as

$$K^{(2)}(f,i) = -\frac{1}{\hbar^2}\int_{t_i}^{t_f} \mathrm{d}t_{II} \int_{t_i}^{t_{II}} \mathrm{d}t_I \int\limits_{-\infty}^{+\infty} \mathrm{d}q_{II} \int\limits_{-\infty}^{+\infty} \mathrm{d}q_I K^{(0)}(f,II)V(II)K^{(0)}(II,I)V(I)K^{(0)}(I,i)$$

Here II and I represent (q_{II}, t_{II}) and (q_I, t_I) respectively and $t_{II} > t_I$.

Exercise 49 (central tutorial)

For solving problems in perturbation theory and initial value problems the Green's function plays an important role. It is defined as the solution to the equation

$$\hat{H}_x G(\underline{x}, \underline{y}) = \delta(\underline{x} - \underline{y}) \tag{1}$$

where $\hat{H}_x$ is a linear operator acting on x. We want to calculate the Green's function of a massive particle.

(i) The Hamiltonian of the free particle is given by $\hat{H}_0 = \frac{\hat{p}^2}{2m}$. Choose $z \in \mathbb{C}$ such that $\hat{H}_0 - z$ has an inverse defined as $\left\langle \underline{x} \middle| (\hat{H}_0 - z)(\hat{H}_0 - z)^{-1} \middle| \underline{x}' \right\rangle = \delta(\underline{x} - \underline{x}')$. Prove that $(\hat{H}_0 - z)^{-1}$ satisfying

$$\left\langle \underline{p} \middle| \frac{1}{\hat{H}_0 - z} \middle| \underline{p}' \right\rangle = \delta(\underline{p} - \underline{p}')\left(\frac{\underline{p}'^2}{2m} - z\right)^{-1} \tag{2}$$

is the inverse of $\hat{H}_0 - z$.
(Use that $\langle \underline{x} \,|\, \underline{p} \rangle = (2\pi\hbar)^{-d/2} \mathrm{e}^{\frac{i}{\hbar}\underline{p}\underline{x}}$, where d is the dimension of $\underline{x}$ and $\underline{p}$.)

(ii) $(\hat{H}_0 - z)^{-1}$ is called the *resolvent* of $\hat{H}_0$. Show that for $d = 3$ one has

$$\left\langle \underline{x} \middle| \frac{1}{\hat{H}_0 - z} \middle| \underline{x}' \right\rangle = \frac{m}{2\pi\hbar^2 \left|\underline{x}' - \underline{x}\right|} \exp\left(\frac{i}{\hbar}\sqrt{2mz}\left|\underline{x}' - \underline{x}\right|\right) \tag{3}$$

(iii) For which values of m and z is (3) a Green's function of the linear operator $-\Delta + k^2$.

(iv) By taking the limit $z \to 0$ we get a Green's function for $\hat{H}_0$. However in certain cases one encounters singularies when taking this limit. One example is the one dimensional resolvent of $\hat{H}_0$. Derive the analogue of (3) for $d = 1$.

(v) By taking the limit $z \to 0$ a singularity arises. In order to avoid that define:

$$G(x,y) = \lim_{z \to 0}\left[\left\langle \underline{x} \middle| \frac{1}{\hat{H}_0 - z} \middle| \underline{x}' \right\rangle - \sum_{i=-\infty}^{+\infty} A_i(x,y)(\sqrt{z})^i\right] \tag{4}$$

Which conditions do the coefficients $A_i(x,y)$ have to fulfill such that $G(x,y)$ converges and is a Green's function of $\hat{H}_0$?

(vi) Consider the one dimensional electrostatic problem

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = f(x)$$
$$\phi(x) = 0 \text{ for } x \to -\infty \tag{5}$$

where $f(x)$ has compact support on $[0, L]$. Derive an integral expression for $\phi(x)$ which solves (5). Show that the boundary condition in (5) fixes the remaining free parameter A_0. What is the physical interpretation of this model?

Exercise 50

Using the definitions given in the lecture, calculate the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ and the total cross section σ_{tot} for the Yukawa potential:

$$V(r) = \frac{V_0 \mathrm{e}^{-r/\alpha}}{r} \tag{6}$$

Check your result by taking the limit $\alpha \to \infty$. For the differential cross section you should get the Rutherford cross section.

Exercise 51 (central tutorial)

Consider the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V} = \frac{\hat{p}^2}{2m} + \lambda\delta(x)$. The eigenstates $|\, k\rangle$ with eigenvalue $\frac{k^2}{2m}$ of this Hamiltonian are given by

$$|\, k\rangle = |\, \bar{k}\rangle - \frac{1}{\hat{H}_0 - \frac{k^2}{2m} - i\epsilon}\hat{V}\,|\, k\rangle \tag{7}$$

where $|\, \bar{k}\rangle$ are the eigenstates of the free Hamiltonian with $\langle x \,|\, \bar{k}\rangle = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{i}{\hbar}xk}$.

(i) Using the result for the resolvent in one dimension from Exercise 49 calculate $\langle x \,|\, k\rangle$.

(ii) $|\, k\rangle$ as a function in k has a simple pole. Find the position k_0 of this pole and evaluate the residue $|\, \Psi\rangle := \mathrm{Res}_{k=k_0}\{|\, k\rangle\}$ of it.

(iii) Show that for $\lambda < 0$, $|\, \Psi\rangle$ is a bound state (normalizable eigenstate) of $\hat{H}$.

(iv) Extract the transmission and reflection coefficients from the explicit expression of $|\, k\rangle$.

Exercise 52

Consider the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$ in one dimension where the potential is given by

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } x \geq 0 \end{cases} \tag{8}$$

(i) Make the following ansatz for the wave function $\psi(x)$

$$\psi(x) = \begin{cases} A\mathrm{e}^{ik_1x} + B\mathrm{e}^{-ik_1x} & \text{for } x < 0 \\ C\mathrm{e}^{ik_2x} + D\mathrm{e}^{-ik_2x} & \text{for } x \geq 0 \end{cases} \tag{9}$$

and solve the time independent Schrödinger equation to get expressions for k_1 and k_2.

(ii) By matching the boundary conditions $\lim_{x\to 0^+}\psi(x) = \lim_{x\to 0^-}\psi(x)$ and $\lim_{x\to 0^+}\psi'(x) = \lim_{x\to 0^-}\psi'(x)$ find a relation between the coefficients A, B, C and D. Why do these boundary conditions make sense?

(iii) Find the transmission and reflection coefficient for a wave coming from $-\infty$.