Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 6, discussed November 25 - November 29, 2019

Exercise 37

Consider the transition amplitude for a free particle in the path integral representation,

$$K(x_f, x_i; T) = \int\limits_{\begin{cases} x(T)=x_f \\ x(0)=x_i \end{cases}} \mathcal{D}x \, \exp\left(\frac{i}{\hbar} S[x(t)]\right) \tag{1}$$

Why does $\mathcal{D}x = \mathcal{D}y$ hold for a transformation of $x(t)$ according to $x(t) = x_{class.}(t) + y(t)$, where $y(t)$ is arbitrary?

Exercise 38 (Central Tutorial)

In the lecture it was shown that, by using the transformation from Exercise 37, the transition amplitude for a free particle can be approximated as

$$K(x_f, x_i; T) = F(T) \, \exp\left(\frac{i}{\hbar} S[x_{class}]\right) \,, \tag{2}$$

where $F(T)$ is an undetermined function of $T = t_f - t_i$. Check by direct calculation that

$$F(T) = \int_{y(t_i)=0}^{y(t_f)=0} \mathcal{D}y \, \exp\left(\frac{i}{\hbar}\frac{m}{2}\int_0^T \dot{y}^2\right) = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}} \tag{3}$$

Hint: The following integral might be useful:

$$\int dx \, \exp\left[-\frac{ax^2}{2} + ibx\right] = \sqrt{\frac{2\pi}{a}} \, \exp\left[-\frac{b^2}{2a}\right] \tag{4}$$

Exercise 39 (Central Tutorial)

Consider the path integral with Hamiltonian $\hat{H} = \hat{q}\hat{p}^2\hat{q}$. Derive the measure for the Lagrangian path integral. *Hints*: 1) Follow the derivation of the path integral for $\hat{H} = \hat{p}^2 + V(\hat{q})$. The integration measure is explicitly dependent on $q(t)$. 2) The integral (4) might be useful.

Exercise 40

Show that for analytic operators $\hat{A}(\hat{p}, \hat{q})$ the following holds:

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A}, \hat{H}] \tag{5}$$

Remark: Generally we have $\frac{d\hat{A}}{dt} \neq \frac{\partial\hat{A}}{\partial\hat{q}}\dot{\hat{q}} + \frac{\partial\hat{A}}{\partial\hat{p}}\dot{\hat{p}}$.

Exercise 41

Consider a Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \hat{V}(t), \tag{6}$$

where $\hat{H}_0$ describes the free part of the system and $\hat{V}(t)$ describes interactions. In the interaction picture we consider the time evolution of operators according to the free part of the Hamiltonian, $\hat{U}_0(t) = \exp\left(-\frac{i}{\hbar}\hat{H}_0 t\right)$, satisfying $\frac{\partial \hat{U}_0}{\partial t} = -\frac{i}{\hbar}\hat{H}_0\hat{U}_0$.

(i) How is a state $|\,\phi(t)\rangle_I$ defined in the interaction picture?

(ii) Show that $|\,\phi(t)\rangle_I$ satisfies

$$i\hbar\frac{\partial}{\partial t}\,|\,\phi(t)\rangle_I = \hat{V}_I(t)\,|\,\phi(t)\rangle_I\,, \tag{7}$$

where $\hat{V}_I(t) = \hat{U}_0^\dagger \hat{V}(t)\hat{U}_0$.

(iii) Take $|\,\phi(t)\rangle_I = \hat{U}_V(t)\,|\,\phi_0\rangle$, and making use of your result of (ii), find the differential equation, satisfied by $\hat{U}_V$. Solve this equation iteratively and use it to express $|\,\phi(t)\rangle_I$ only in terms of $\hat{V}(t)$ and $|\,\phi_0\rangle$.

Exercise 42

Evaluate directly the matrix elements of the evolution operator

$$\mathcal{K}(x_f, x_i; t_f, t_i) = \langle x_f\,|\,e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)}\,|\,x_i\rangle$$

where $\hat{H} = \frac{\hat{p}^2}{2m}$ is the Hamiltonian of the free particle. How would the result change for a particle moving in d space dimensions?