Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 5, discussed November 18 - November 22, 2019

Exercise 31

(i) Calculate $\hat{q}\hat{p}^2\hat{q} - \hat{p}\hat{q}^2\hat{p}$.

(ii) Write the Hamiltonian $\hat{H} = \hat{q}\hat{p}^2\hat{q}$ in canonical form, which corresponds to moving $\hat{p}$ operators to the left of $\hat{q}$.

Exercise 32 (Central tutorial)

Consider the action

$$S = \int_{t_i}^{t_f} dt \; (p\dot{q} - H(p,q)), \tag{1}$$

and treat $q(t)$ and $p(t)$ as independent variables.

(i) Derive the Hamilton equations of motions from the variation of the action. What are the required boundary conditions on the variation δq and δp?

(ii) Show that if the Hamiltonian is not explicitly time dependent, i.e. $H(p,q,t) = H(p,q)$, then $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$.

Exercise 33 (Central tutorial)

(i) Generalise the Hamiltonian equations of motion to systems with N degrees of freedom p_i and q_i, $i = 1, \ldots, N$.

We now move to the field theory case where we have $N \to \infty$. The generalised coordinates become $q_i(t) \to \phi_{\boldsymbol{x}}(t) = \phi(t, \boldsymbol{x})$ and $p_i(t) \to \pi_{\boldsymbol{x}}(t) = \pi(t, \boldsymbol{x})$, where $\phi(t, \boldsymbol{x})$ is the scalar field and $\pi(t, \boldsymbol{x})$ is the canonical momentum.

(ii) Consider the action of a massless and free scalar field $\phi = \phi(t, \boldsymbol{x})$ in 4 dimensions,

$$S[\phi, \partial_\mu \phi] = \frac{1}{2} \int d^4x \; \partial_\mu \phi \partial^\mu \phi = \frac{1}{2} \int dt \; d^3x \left[(\partial_t \phi)^2 - (\nabla \phi)^2 \right], \tag{2}$$

where the summation over a Greek index runs over all dimensions $\mu = 1, \ldots, 4$, and $\partial_0 = \partial_t$, $\partial_1 = \partial_x$, etc.. (By convention summation over Roman indices runs over spatial dimensions $i = 1, 2, 3$ only, i.e. $(\nabla\phi)^2 = \partial_i \phi \partial^i \phi$.)

a) Use the Lagrangian density $\mathcal{L}$, where

$$S = \int dt \; L = \int d^4x \; \mathcal{L}$$

to find the canonical momenta $\pi(t, \boldsymbol{x})$.

b) Find the Hamiltonian $H(\phi, \pi)$.

c) Derive the Lagrange equation of motion.

d) Derive Hamilton's equations of motion.

Exercise 34

Prove that:

$$[\hat{p}, \hat{p}^n \hat{q}^m] = -i\hbar \frac{\partial}{\partial \hat{q}} (\hat{p}^n \hat{q}^m) \tag{3}$$

Exercise 35

Does the hermiticity of $\hat{H}$ follow from the unitarity of the time evolution operator?

Exercise 36

Let's consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{4}$$

(i) Show that the equation

$$\frac{\partial \hat{\rho}_u}{\partial \beta} = -\hat{H}\hat{\rho}_u \tag{5}$$

can be written in the integral form:

$$\hat{\rho}_u(\beta) = \hat{\rho}_0(\beta) - \int_0^\beta d\beta' \hat{\rho}_0(\beta - \beta')\hat{V}\hat{\rho}_u(\beta') \qquad \text{with} \qquad \hat{\rho}_0(\beta) = e^{-\beta \hat{H}_0} \tag{6}$$

(ii) Now let $\hat{V}$ be a small perturbation, $\hat{V} \ll \hat{H}_0$. Find the first order in perturbation theory for $\rho(\boldsymbol{x}, \boldsymbol{x}'; \beta)$ with $\hat{V} = V(\hat{x})$.

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`