Prof. Dr. Viatcheslav Mukhanov Winter term 2019/2020

Exercises on Quantum Mechanics II (TM1/TV)

Problem set 12, discussed January 20 - January 24, 2020

Exercise 68 - Central Tutorial

In this exercise we want to discuss quantum teleportation. For this, suppose Alice holds an unknown spin-$\frac{1}{2}$ state $|\,\phi\rangle = a\,|+\rangle + b\,|-\rangle$, which she would like to transmit to Bob. However, they can only communicate their measurement results classically, i.e. for example via e-mail. Suppose that Alice and Bob in addition are each in possession of one member of a pair in an entangled spin-$\frac{1}{2}$ system, which is in the singlet state $|\,\Psi\rangle$.

(i) Suppose that Alice measures the two spin particles she holds in the so-called *Bell basis*, which consists of the four states

$$\begin{aligned} \left|\,\Phi^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}\left[|++\rangle \pm |--\rangle\right], \\ \left|\,\Psi^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}\left[|+-\rangle \pm |-+\rangle\right], \end{aligned} \tag{1}$$

where, for example $|+-\rangle := |+\rangle\otimes|-\rangle$. Write the complete state $|\,\phi\rangle\otimes|\,\Psi\rangle$ of the three particles by using the Bell basis for Alice's particles. Determine the probabilities of the outcome of Alice's measurement.

(ii) After Alice has done her measurement, what is the state of Bob's particle? What can Alice and Bob do in order to turn Bob's particle into the state $|\,\phi\rangle$?

Solution

(i) We use the convention that in the complete system, the first two factors of the tensor product of three spin-$\frac{1}{2}$ Hilbert spaces correspond to Alice's particles. The complete state of the three particles is then written as

$$\begin{aligned} |\,\phi\rangle\otimes|\,\Psi\rangle &= \frac{1}{\sqrt{2}}\left[a\,|+\rangle + b\,|-\rangle\right]\otimes\left[|+-\rangle - |-+\rangle\right] \\ &= \frac{1}{\sqrt{2}}\left[a\,|\underline{++}-\rangle - a\,|\underline{+-}+\rangle + b\,|\underline{-+}-\rangle - b\,|\underline{--}+\rangle\right], \end{aligned} \tag{2}$$

where, in the last line, the underlined entries correspond to Alice's particles, which we now want to express in the Bell basis. We invert the transformation given on the exercise sheet,

$$\begin{aligned} |++\rangle &= \frac{1}{\sqrt{2}}\left[\left|\,\Phi^{+}\right\rangle + \left|\,\Phi^{-}\right\rangle\right], \qquad |--\rangle = \frac{1}{\sqrt{2}}\left[\left|\,\Phi^{+}\right\rangle - \left|\,\Phi^{-}\right\rangle\right], \\ |+-\rangle &= \frac{1}{\sqrt{2}}\left[\left|\,\Psi^{+}\right\rangle + \left|\,\Psi^{-}\right\rangle\right], \qquad |-+\rangle = \frac{1}{\sqrt{2}}\left[\left|\,\Psi^{+}\right\rangle - \left|\,\Psi^{-}\right\rangle\right], \end{aligned} \tag{3}$$

and substitute these expression for the underlined parts in eq. 2:

$$
\begin{aligned}
|\,\phi\rangle \otimes |\,\Psi\rangle = \frac{1}{2}\Big[& a\,\big|\,\Phi^+\big\rangle \otimes |\,-\rangle + a\,\big|\,\Phi^-\big\rangle \otimes |\,-\rangle - a\,\big|\,\Psi^+\big\rangle \otimes |\,+\rangle - a\,\big|\,\Psi^-\big\rangle \otimes |\,+\rangle \\
& + b\,\big|\,\Psi^+\big\rangle \otimes |\,-\rangle - b\,\big|\,\Psi^-\big\rangle \otimes |\,-\rangle - b\,\big|\,\Phi^+\big\rangle \otimes |\,+\rangle + b\,\big|\,\Phi^-\big\rangle \otimes |\,+\rangle \Big] \\
= \frac{1}{2}\Big[& \big|\,\Phi^+\big\rangle \otimes (a\,|\,-\rangle - b\,|\,+\rangle) + \big|\,\Phi^-\big\rangle \otimes (a\,|\,-\rangle + b\,|\,+\rangle) \\
& + \big|\,\Psi^+\big\rangle \otimes (-a\,|\,+\rangle + b\,|\,-\rangle) + \big|\,\Psi^-\big\rangle \otimes (-a\,|\,+\rangle - b\,|\,-\rangle) \Big].
\end{aligned} \tag{4}
$$

The probability for Alice to measure, e.g. $|\,\Phi^+\rangle$ is

$$
P_{\Phi^+} = \left|\big\langle \Phi^+ \,\big|\, [|\,\phi\rangle \otimes |\,\Psi\rangle] \right|^2 = \frac{1}{4}\,\big\langle \Phi^+ \,|\, \Phi^+ \big\rangle\,(|a|^2 + |b|^2) = \frac{1}{4}, \tag{5}
$$

for a normalised state $|\,\phi\rangle$. The probabilities for Alice are hence equal to 1/4 in all four cases.

(ii) The state of Bob's particle depends on the outcome of Alice's measurement. In each case the state is given by a unitary transformation U of the original unknown state $|\,\phi\rangle = a\,|\,+\rangle + b\,|\,-\rangle$. If we represent the states $|\,+\rangle \to (1,0)^T$ and $|\,-\rangle \to (0,1)^T$, we have,

$$
\begin{aligned}
&\text{for } \big|\,\Phi^+\big\rangle: \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{for } \big|\,\Phi^-\big\rangle: \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\text{for } \big|\,\Psi^+\big\rangle: \quad U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{for } \big|\,\Phi^-\big\rangle: \quad U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{6}
$$

In order to turn Bob's particle into the state $|\,\phi\rangle$, Alice (classically) communicates the result of her measurement to Bob, and Bob just applies the corresponding rotation to undo the effect of U to his state .

Exercise 69

Consider a spin-$\frac{1}{2}$ atom in a Stern-Gerlach experiment. From the Schrödinger equation, derive differential equations describing the behaviour of the atom when passing through a magnetic field in the z direction, active in a region $a < x < b$.
Hint: Separate the total wave function into

$$
\psi(x,z,t) = \psi_x(x,t)\psi_z(z,t), \tag{7}
$$

and consider how the action of the potential in the Hamiltonian depends on the spin.

Solution We start by substituting the separated wave function into a general classical Hamiltonian,

$$
\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\Delta\psi + \hat{V}\psi \\
i\hbar\frac{1}{\psi_x}\frac{\partial\psi_x}{\partial t} + i\hbar\frac{1}{\psi_z}\frac{\partial\psi_z}{\partial t} &= -\frac{\hbar^2}{2m}\frac{1}{\psi_x}\frac{\partial^2\psi_x}{\partial x^2} - \frac{\hbar^2}{2m}\frac{1}{\psi_z}\frac{\partial^2\psi_z}{\partial z^2} + \frac{1}{\psi_z\psi_x}\hat{V}(\psi_x\psi_z).
\end{aligned} \tag{8}
$$

The potential in the Hamiltonian has the form $\hat{V} = \boldsymbol{\mu B}.\hat{\boldsymbol{\sigma}}$. As stated in the question, we have $\boldsymbol{B} = B_z\boldsymbol{e_z}$. For a very localized magnetic field, we can Taylor expand the field in z,

$$
B_z(z) = \left[B_0 + \frac{\partial B_z}{\partial z}\Big|_{z=0} z + \mathcal{O}(z^2) \right] \theta(b-x)\theta(x-a), \tag{9}
$$

where the θ-functions impose that the magnetic field only acts in the region $a < x < b$. The first term in the expansion represents only a constant energy shift and can therefore be ignored. Now splitting ψ_z into spin-up and spin-down component,

$$
\psi_z(z,t) = \alpha\psi_z^{\uparrow}(z,t)\,|\uparrow\rangle + \beta\psi_z^{\downarrow}(z,t)\,|\downarrow\rangle\,, \tag{10}
$$

and considering that the spin operator $\hat{\sigma}_z$ acts as $\hat{\sigma}_z\,|\!\uparrow\rangle = |\!\uparrow\rangle$ and $\hat{\sigma}_z\,|\!\downarrow\rangle = -\,|\!\downarrow\rangle$, we have

$$\begin{aligned}
i\hbar\frac{\partial\psi_x}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi_x}{\partial x^2} \\
i\hbar\frac{\partial\psi_z^\uparrow}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi_z^\uparrow}{\partial z^2} + \mu\frac{\partial B_z}{\partial z}\Big|_{z=0} z\theta(b-x)\theta(x-a)\psi_z^\uparrow \\
i\hbar\frac{\partial\psi_z^\downarrow}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi_z^\downarrow}{\partial z^2} - \mu\frac{\partial B_z}{\partial z}\Big|_{z=0} z\theta(b-x)\theta(x-a)\psi_z^\downarrow .
\end{aligned} \tag{11}$$

These equations were solved in the lecture, giving the usual Stern-Gerlach behaviour.

Exercise 70

Show that for a successful Stern-Gerlach experiment the following condition has to be satisfied,

$$\frac{\mu}{(\hbar)^{1/2}}\left(\frac{\partial H}{\partial z}\right)\frac{b^{3/2}m}{p_{0x}^{3/2}} > 1\ . \tag{12}$$

(The detector shall be at $x = b$.)

Solution The separation of the two beams is equal to

$$\triangle z = 2\int_0^{t'} v(t)dt = 2\int_0^{t'}\int_0^{t} a(t_1)dt_1 dt = \frac{Ft^2}{m} \tag{13}$$

where an approximately constant acceleration is assumed, $F = \mu\left(\frac{\partial H}{\partial z}\right)$ is the force and $t = \frac{b}{v_x} = \frac{bm}{p_x}$ is the time of flight. The condition for a successful Stern-Gerlach experiment is that the separation is bigger than the final uncertainty in z direction $\triangle z > \triangle z_f$. Under the assumption that the initial state minimizes the uncertainty relationship (so satisfies the equality) we obtain:

$$(\triangle z_f)^2 = (\triangle z_i)^2 + (t\triangle v_z)^2 = (\triangle z_i)^2 + \left(\frac{b\triangle p_z}{p_x}\right)^2 = (\triangle z_i)^2 + \left(\frac{b\hbar}{2\triangle z_i p_x}\right)^2 \tag{14}$$

This function is minimized by

$$\triangle z_i = \sqrt{\frac{b\hbar}{2p_x}} \qquad \triangle z_f = \sqrt{\frac{b\hbar}{p_x}} \tag{15}$$

putting everything together leads to the desired relation.

Exercise 71

Let $\hat{H}_{\text{int}} = \hat{S}\cdot\hat{X}$ be the interaction term between system S and device A (where $\hat{S}$ and $\hat{A}$ are the operators corresponding to the observables in which we are interested within the system and device respectively) and $\hat{X}$ is the canonical conjugate operator to $\hat{A}$, $[\hat{X},\hat{A}] = -i\hbar$. Prove that

$$[\hat{A}, e^{-\frac{i}{\hbar}\hat{H}\triangle t}] = e^{-\frac{i}{\hbar}\hat{H}\triangle t}\hat{S}\triangle t\ . \tag{16}$$

Solution The relation follows immediately from the fact that $[\hat{A}, f(\hat{X})] = i\hbar f'(\hat{X})$, which is straightforward using

$$[\hat{A}, \hat{X}^n] = i\hbar n\hat{X}^{n-1} \tag{17}$$

as checked in exercise 22.

General information

The *lecture* takes place on:
Monday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
Friday at 10:00 - 12:00 c.t. in B 052 (Theresienstraße 37)
The *central tutorial* takes place on Monday at 12:00 - 14:00 c.t. in B 139 (Theresienstraße 37)
The *webpage* for the lecture and exercises can be found at

`https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_19_20/T_M1_TV_-Quantum-Mechanics-II`