Ludwig-Maximilians-Universität München
Department of Physics

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T_M1/TV: Quantum Mechanics II

First Exam
18 February 2019

- Do not use any own material, except for a pen.
- Please write in blue or black. Do not use a pencil.
- Please make sure that each sheet of paper carries your name or ID (Matrikelnummer).
- Please indicate very clearly which problem any of your calculations/results belong to, and start a new page for each problem.
- Please answer the questions in English or German.
- The maximal number of points is 101.
- You will have 180 minutes to solve the exam.

I agree that my result will be published together with my ID on the lecture webpage. Signature: ____________________

Name: ____________________

ID: ____________________

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Problem 1: Short questions [23 points]

a) Consider the irreducible representation of $so(3)$ with $j = 1$. Write down a matrix representation of the operators J_z and J_+ in the basis $|j, m\rangle$, where j is the spin and m the magnetic quantum number.

b) Write down the time evolution operator $U(t)$ in terms of the Hamilton operator H for the case that the Hamilton operator does not depend on time. Show that $U(t)\psi(x)$ for a (wave) function $\psi(x)$ satisfies the Schrödinger equation.

c) Suppose your system consists of N identical copies of a two-level particle (*i.e.*, a single particle can occupy two states). What is the dimension of the Hilbert space if the particle is

i) bosonic?

ii) fermionic?

iii) neither (*i.e.*, if the particles were distinguishable)?

d) In the following, we focus exclusively on the spin degrees of freedom of the system of particles under consideration.

i) Consider a system of two distinguishable particles of spin 1/2. We write $|\uparrow\rangle$ ($|\downarrow\rangle$) for the one-particle state with S_z eigenvalue $\hbar/2$ ($-\hbar/2$). Suppose the system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\uparrow\downarrow\rangle + \frac{i}{\sqrt{3}}|\uparrow\uparrow\rangle - \frac{1}{\sqrt{3}}|\downarrow\uparrow\rangle \, .$$

Are the particles entangled? Explain your answer briefly (verbal argument sufficient; you do not need to present formulae).

Problem 1: Short questions [continued]

ii) Now consider a system of 3 distinguishable particles of spin 1. Suppose the system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \big(|1,0,-1\rangle - |1,-1,0\rangle \big) ,$$

where the nth label denotes the magnetic quantum number of the nth particle. Compute the reduced density matrix and the entanglement entropy

α) for the first particle,

β) for the third particle.

e) Consider the following equations of motion for a wave function $\phi(x^\mu)$ on Minkowski space:

$$\left(\frac{1}{c^2} \partial_t^2 \pm \vec{\nabla}^{\,2} + \frac{m^2c^2}{\hbar^2} \right) \phi(x^\mu) = 0 .$$

Here $\vec{\nabla}^{\,2} = \partial_x^2 + \partial_y^2 + \partial_z^2$. Which choice of sign makes the equation Lorentz invariant? Explain your answer briefly in words.

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Problem 2: Stark effect [16 points]

Consider a hydrogen atom in a constant electric field in z-direction. We want to use perturbation theory to study the resulting split of the energy levels. Recall that the energy eigenvalues E_n of the unperturbed hydrogen atom are n^2 degenerate. A convenient orthonormal basis is labeled by three integers $|n, \ell, m\rangle$, where $\ell = 0, \ldots, n-1$ and $m = -\ell, \ldots \ell$, $n = 1, 2, \ldots$. The angular momentum operators act as

$$\vec{L}^2 |n, \ell, m\rangle = \ell(\ell+1)|n, \ell, m\rangle, \quad L_z|n, \ell, m\rangle = m|n, \ell, m\rangle.$$

and application of the unperturbed Hamilton operator H_0 gives the energy eigenvalue E_n. The parity operator P acts on the wave functions as

$$P|n, \ell, m\rangle = (-1)^\ell |n, \ell, m\rangle$$

We now want to perturb the system by an electric field in the z-direction. This means to perturb the initial Hamilton operator H_0 by $V = -eEz$, such that the system is described by

$$H = H_0 - eEz$$

a) Use the Wigner-Eckart theorem to state selection rules on the matrix elements $\langle n', \ell', m'|z|n, \ell, m\rangle$.

b) Use parity to state further selection rules.

c) Show that the first order correction to the energy of the perturbed system vanishes in the ground state.

d) Consider now the first excited state, $n = 2$.

   (i) Show that to first order in perturbation theory the two states $|2, 1, \pm 1\rangle$ remain unchanged and there is no correction to the energy.

   (ii) Compute the matrix element $\langle 2, 0, 0|z|2, 1, 0\rangle$. Use that these states can be represented by the wave functions

   $$|2, 0, 0\rangle = \frac{1}{\sqrt{2\pi a}} \frac{1}{2a} \left(1 - \frac{r}{2a}\right) e^{-\frac{r}{2a}}, \quad |2, 1, 0\rangle = \frac{1}{\sqrt{2\pi a}} \frac{1}{4a^2} r e^{-\frac{r}{2a}} \cos\theta\,.$$

   To evaluate the integrals, use

   $$\int_0^\infty x^n e^{-x} = n!\,.$$

   (iii) Now compute the first order corrections to the energy.

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Problem 3: Tensor operators [15 points]

An irreducible spherical tensor operator $\mathbf{T}^{(l)}$ of rank l has spherical components $T^{(l)}_m$, which satisfy the following commutation relations with the generators J_z, $J_\pm$ of the angular momentum algebra:

$$[J_z, T^{(l)}_m] = m\hbar\, T^{(l)}_m \,, \quad [J_\pm, T^{(l)}_m] = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, T^{(l)}_{m\pm 1} \,.$$

a) Suppose you study the matrix elements of a spherical component $T^{(3)}_n$ of a tensor operator of rank 3 in the basis of states $|j, m\rangle$, where j labels the angular momentum and m the magnetic quantum number. Which matrix elements can be non-vanishing?

b) Consider two commuting vector operators V and W with Cartesian components V_x, V_y, V_z and W_x, W_y, W_z. They satisfy the following commutation relations with the angular momentum operator:

$$[J_i, V_j] = i\hbar\epsilon_{ijk} V_k \,, \qquad i, j, k \in \{x, y, z\} \,,$$

and likewise for W. What are the spherical components of the two operators?

c) Consider the set of operators $V_i W_j$, with $i, j \in \{x, y, z\}$. Which irreducible tensor operators can you form? Explain.

d) In part b) you should have obtained that (among other operators) you can form a rank 2 tensor operator. Spell out its spherical components $T^{(2)}_2, T^{(2)}_1, T^{(2)}_0$ in terms of the spherical components V_m, W_m ($m = 0, \pm 1$).

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Problem 4: Time dependent perturbation theory [18 points]

In the first part of this problem, you develop time-dependent perturbation theory. In the second part, you apply it to a specific example.

a) Consider a time-independent Hamilton operator H_0 with a time-dependent perturbation $V(t)$:

$$H(t) = H_0 + \lambda V(t)$$

Here, the operator $V(t)$ is constant in time, except that it is turned on at $t = 0$ and turned off at some later time T. The constant λ is a dimensionless small parameter.

For $t < 0$ let the system be in the eigenstate $|m\rangle$ of H_0. Let $|\tilde{\psi}_m(t)\rangle_I$ denote the state of the system and $\lambda\tilde{V}_I(x,t)$ the perturbation at time t in the interaction picture.

(i) Show that

$$i\hbar\partial_t|\tilde{\psi}_m(t)\rangle_I = \lambda\tilde{V}_I(x,t)|\tilde{\psi}_m(t)\rangle_I\,.$$

(ii) For $t \in [0,T]$, derive the expression for $|\tilde{\psi}_m(t)\rangle_I$ in terms of unperturbed states to first order in λ.

(iii) We are interested in the transition probabilities $P_{m\to n}(t)$, $t > T$, between eigenstates $|m\rangle$ and $|n\rangle$ of H_0 under this perturbation. Compute $P_{m\to n}(t)$ for $m \neq n$ in first order perturbation theory.
Result:

$$P_{m\to n}(t) = 4\lambda^2|\langle n|V|m\rangle|^2\frac{\sin^2[(E_m - E_n)T/2\hbar]}{(E_m - E_n)^2}$$

Problem 4: Time dependent perturbation theory [continued]

b) Consider now the example of an infinite square well in one dimension. For the unperturbed problem, the energy eigenvalues are

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

for $n = 1, 2, \ldots$, and the eigenfunctions are

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

At time $t = 0$ the system is perturbed such that the potential becomes

$$V(x) = \begin{cases} V_0\,, & 0 \le x \le \frac{a}{2} \\ 0\,, & \frac{a}{2} < x \le a \\ \infty\,, & \text{otherwise} \end{cases}$$

where $V_0 \ll E_1$. After time T, the perturbation is turned off.

(i) Sketch the potential V as a function of x.

(ii) The system is initially ($t < 0$) in the ground state. Find the probability that it is in the first excited state for $t > T$.

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Problem 5: First Born approximation [9 points]

A beam of particles (mass m) is elastically scattered by a potential V given by

$$V = \begin{cases} V_0\,, & |x| \le L, |y| \le L, |z| \le L \\ 0\,, & \text{otherwise} \end{cases}$$

The incident particles propagate in z-direction with wave-vector $\vec{k} = k\vec{e}_z$. Compute the differential cross section $d\sigma/d\Omega$ as a function of the scattering angle θ and the corresponding azimuthal angle ϕ, in first Born approximation.

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Problem 6: Second Quantisation [20 points]

Consider a system of bosonic particles in 1 dimension. Suppose the Hamilton operator is $\hat{H} = \sum_{k=0}^{\infty} E_k \hat{a}_k^\dagger \hat{a}_k$, where the operator $\hat{a}_k^\dagger$ creates a single particle in a state of definite energy E_k, for $k \in \mathbb{N}_0$, and $[\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}$. The single particle Hilbert space is the direct sum of the 1-dimensional eigenspaces associated to E_k.

a) Compute the matrix element $\langle\Omega|\, \hat{a}_k\, \hat{a}_l\, \hat{a}_m^\dagger\, \hat{a}_n^\dagger\, |\Omega\rangle$, where $|\Omega\rangle$ is the vacuum state of the Fock space.

b) The operator $\hat{a}_k$ in the Heisenberg picture is denoted $\hat{a}_k(t)$. Starting from the Heisenberg equation of motion, show that $\hat{a}_k(t) = \exp(-\frac{iE_k t}{\hbar})\, \hat{a}_k$.

c) Write the single-particle operator $\hat{x}$ in second-quantised form.

d) Write the operator $\hat{x}$ in terms of the field operator $\hat{\psi}(x)$.

e) Let Z denote the sequence of complex numbers z_0, z_1, z_2, ..., such that $0 < \sum_{k=0}^{\infty} |z_k|^2 < \infty$. Consider the state

$$|Z\rangle = \mathcal{N}(Z) \prod_{k=1}^{\infty} e^{z_k\, \hat{a}_k^\dagger}\, |\Omega\rangle \,,$$

where $\mathcal{N}(z)$ is a normalisation constant. Show that $|Z\rangle$ is an eigenstate for every $\hat{a}_k$, $k \in \mathbb{N}_0$. Compute the eigenvalues.

f) Show that the time-evolved expectation value of the field operator in the state $|Z\rangle$ solves the equation $i\hbar\partial_t f(x,t) = \hat{H} f(x,t)$, where $\hat{H}$ here takes the appropriate form acting on single particle wave functions.