Ludwig-Maximilians-Universität München
Department of Physics

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T_M1/TV: Quantum Mechanics II

Exam

19 February 2018

- Do not use any own material, except for a pen.
- Please make sure that each sheet of paper carries your name or ID (Matrikelnummer).
- Please indicate very clearly which problem any of your calculations/results belong to, and start a new page for each problem.
- Please answer the questions in English or German.
- The maximal number of points is 94.
- You will have 180 minutes to solve the exam.

I agree that my result will be published together with my ID on the lecture webpage. Signature: ______________________

Name: ______________________

ID: ______________________

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Problem 1: Short questions [15 points]

a) Which of the following wave functions are consistent? Explain your answer.

i) Two electrons are in the state

$$\frac{1}{\sqrt{2}}\left(f(r_1)g(r_2) - g(r_1)f(r_2)\right)|+\rangle \otimes |+\rangle\,.$$

ii) Three electrons are in the state

$$\left(f(r_1)g(r_2)h(r_3) - f(r_2)g(r_3)h(r_1) + f(r_3)g(r_1)h(r_2)\right)|+\rangle \otimes |+\rangle \otimes |+\rangle\,.$$

b) For a system of three identical particles, correctly (anti-)symmetrise and normalise the state

$$|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle\,,$$

where $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ are orthonormal states of the single-particle Hilbert space. Do this in the case where

i) the particles are bosons, ii) the particles are fermions.

c) Three spin-$\frac{1}{2}$ particles are in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\downarrow\rangle + i|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right)\,.$$

i) What are the eigenvalues of the density matrix corresponding to the state $|\psi\rangle$?

ii) Trace out the third particle, and write the reduced density matrix explicitly in the basis

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\,.$$

iii) Compute the entanglement entropy for the reduced density matrix

$$\rho_{\text{reduced}} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 2 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix}\,.$$

iv) Is there a state for the three-particle system such that the entanglement entropy of the first two particles is $\log(3/4)$? Why?

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Problem 2: Angular momentum [16 points]

An irreducible spherical tensor operator $\mathbf{T}^{(l)}$ of rank l has components $T_m^{(l)}$, which satisfy the following commutation relations with the generators J_z, $J_\pm$ of the angular momentum algebra:

$$[J_z, T_m^{(l)}] = m\hbar\, T_m^{(l)}\,, \quad [J_\pm, T_m^{(l)}] = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\, T_{m\pm 1}^{(l)}\,.$$

a) Consider the matrix elements

$$\langle j_1, m_1 | T_m^{(l)} | j_2, m_2 \rangle$$

of the tensor operator. Use the commutation relations from above to derive the selection rule $m_2 + m = m_1$.

b) Consider the operator

$$S = \frac{1}{\sqrt{2}}\left(J_- T_1^{(1)} - J_+ T_{-1}^{(1)}\right) - J_z T_0^{(1)}\,.$$

Show that it is a spherical scalar operator.

c) For two irreducible spherical tensor operators $\mathbf{S}^{(k)}$ and $\mathbf{T}^{(l)}$, the product $\mathbf{S}^{(k)}\mathbf{T}^{(l)}$, with components $S_m^{(k)} T_n^{(l)}$, is also a spherical tensor operator, but it is not irreducible. Rather, one can decompose it as

$$\mathbf{S}^{(k)}\mathbf{T}^{(l)} = \mathbf{U}^{(l_1)} + \mathbf{U}^{(l_2)} + \ldots\,,$$

where $\mathbf{U}^{(l_i)}$ are irreducible spherical tensor operators. Which ranks l_i can possibly appear? You only need to state the constraints; a derivation is not required.

d) State what the Wigner-Eckart theorem says, and use it to explain why for any two irreducible spherical tensor operators $\mathbf{S}^{(l)}$ and $\mathbf{T}^{(l)}$,

$$\langle j_1, m_1 | S_m^{(l)} | j_2, m_2 \rangle \, \langle j_1, n_1 | T_n^{(l)} | j_2, n_2 \rangle \; = \; \langle j_1, m_1 | T_m^{(l)} | j_2, m_2 \rangle \, \langle j_1, n_1 | S_n^{(l)} | j_2, n_2 \rangle$$

for any choice of j_1, j_2, m_1, m_2, n_1, n_2, and m, n.

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Problem 3: Time-dependent perturbation [15 points]

Let a spinless particle of mass m in one dimension be governed by the Hamiltonian

$$H_0 = \frac{p^2}{2m} + V_0(x)\,.$$

Consider a perturbation $\lambda V(x,t)$ of the potential, where λ is a small real parameter.

a) Suppose $V(x,t) \to 0$ for $t \to -\infty$ sufficiently fast. Suppose that the system for $t \to -\infty$ is in the state $|m\rangle$, which is the time-independent state satisfying $H_0|m\rangle = E_m|m\rangle$. In the interaction picture $|\tilde{\psi}\rangle = e^{iH_0 t/\hbar}|\psi\rangle$, the Schrödinger equation is

$$i\hbar\partial_t|\tilde{\psi}\rangle = \lambda\tilde{V}(t)|\tilde{\psi}\rangle\,.$$

Starting from this equation, show that to first order in λ

$$|\tilde{\psi}(t)\rangle = |\tilde{\psi}_m(t)\rangle = |m\rangle + \frac{\lambda}{i\hbar}\int_{-\infty}^{t} dt'\,\tilde{V}(t')|m\rangle\,.$$

Consider $V_0 = -\frac{\hbar^2}{m}A\delta(x)$ with $A > 0$, and the perturbation $V(x,t) = W(x)\sin(\omega t)$ with $W(x) = \alpha\delta(x)$. For such a perturbation, the long-term rate for the transition of a state $|m\rangle$ to an orthogonal state $|n\rangle$ derived from Fermi's golden rule is

$$\Gamma_{m\to n} = \frac{\lambda^2\pi}{2\hbar}|\langle n|W|m\rangle|^2\,\left(\delta(E_n - E_m + \hbar\omega) + \delta(E_n - E_m - \hbar\omega)\right)\,.$$

Assuming that the system is placed into a large box $[-L, L]$, the eigenfunctions of the unperturbed Hamiltonian are

$$\psi^b(x) = \sqrt{A}\,e^{-A|x|}\,,\quad \psi_k^e(x) = \tfrac{1}{\sqrt{L}}\cos(k|x| + \varphi_k)\,,\quad \psi_k^o(x) = \tfrac{1}{\sqrt{L}}\sin(kx)\,.$$

Here, $k = \frac{n\pi}{L}$ for $n \in \mathbb{Z}\setminus\{0\}$, and $\tan\varphi_k = A/k$. ψ^b represents a bound state, while ψ^e and ψ^o are even and odd wave functions of continuum states in the $L \to \infty$ limit.
Let the system initially be in the state with wave function ψ^b.

b) Calculate the energies E_b of ψ^b, E_k^e of ψ_k^e, and E_k^o of ψ_k^o.

Problem 3: Time-dependent perturbation (continued)

c) Show that according to the formula for $\Gamma_{m\to n}$ from above, the system makes no transitions to any of the states ψ_k^o, and that the transition rate to a state ψ_k^e is given by

$$\Gamma_{b\to k} = \tfrac{\lambda^2\pi}{2\hbar}\alpha^2 \tfrac{A}{L}\cos^2\varphi_k\,\delta(E_b - E_k^e + \hbar|\omega|)\,.$$

d) The rate $\Gamma_{b\to\text{free}}$ at which the state ψ^b decays into the (quasi-)continuum is given by summing the rate $\Gamma_{b\to k}$ over all values of k. In the limit $L\to\infty$, the sum becomes an integral; k becomes a continuous integration variable. Identify the measure dk and calculate $\Gamma_{b\to\text{free}}$ in this limit.
Hint: $\cos^2(\arctan(x)) = (1+x^2)^{-1}$.

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Problem 4: Elastic scattering [14 Points]

We consider elastic scattering of a quantum mechanical particle in the Yukawa potential

$$V(r) = \frac{V_0\, e^{-\mu r}}{r}\,, \qquad \mu > 0\,.$$

Here r is the radial distance in $\mathbb{R}^3$, and $V_0 \neq 0$ is a real constant.

a) Is the Yukawa potential short-ranged? Explain your answer.

b) For a general potential $V(\vec{r})$, the scattering amplitude in the first Born approximation is

$$f(\vec{k}, \vec{k}') = -\frac{m}{2\pi\hbar^2} \int d^3r\, V(\vec{r}) e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\,,$$

where $\vec{k}$ is the wave vector of the incoming plane wave, and $\vec{k}'$ the wave vector in the scattered direction. Show that if $V(\vec{r}) = V(r)$ is spherically symmetric, the scattering amplitude can be written as

$$f(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty dr\, r\, V(r)\, \sin(qr)\,,$$

where $q = k\sqrt{2(1-\cos\theta)} = 2k \sin\frac{\theta}{2}$.

c) Show that the differential cross section for the Yukawa potential is

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{1}{|2k^2(1-\cos\theta) + \mu^2|^2}\,.$$

d) Calculate the total cross section for the Yukawa potential.

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Problem 5: A second quantized fermionic system [20 points]

Consider a system of identical fermions, where the single-particle Hilbert space $\mathcal{H}$ has dimension 2. Let $|\phi_1\rangle$, $|\phi_2\rangle$ be an orthonormal eigenbasis of $\mathcal{H}$.
The second quantised Hamiltonian of the system is

$$H = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta(a_1^\dagger a_2^\dagger + a_2 a_1)\,.$$

a) Compute the action of H on all states in the occupation number basis of the Fock space.

b) Suppose there are two fermions in the system. Write down the state of the system in first quantised formalism.

c) Consider the transformation of operators $(a_i, a_i^\dagger) \mapsto (b_i, b_i^\dagger)$ given by

$$a_i = \sum_j \left(\alpha_{ij} b_j + \beta_{ij} b_j^\dagger\right)\,,$$

where the complex coefficients α_{ij}, β_{ij} form matrices α, β. Show that requiring that the transformation preserves canonical anticommutation relations leads to the condition

$$\alpha\alpha^\dagger + \beta\beta^\dagger = 1\,, \qquad \alpha\beta^T + \beta\alpha^T = 0\,.$$

d) Compute the expectation values of the operators $a_1 a_2 a_i^\dagger a_j a_2^\dagger a_1^\dagger$ $(i, j = 1, 2)$ in the Fock space vacuum.

e) Consider the $a_i, a_i^\dagger$ in the Heisenberg picture. Derive the explicit differential equations for the a_i, $a_i^\dagger$ from the Heisenberg equations of motion.

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Problem 6: Klein-Gordon equation [14 points]

Consider the Klein-Gordon equation describing a spinless particle in the presence of an electromagnetic field described by a vector potential A_μ

$$(D_\mu D^\mu + \frac{m^2c^2}{\hbar^2})\phi(x) = 0 \ ,$$

where D_μ is the covariant derivative operator

$$D_\mu = \partial_\mu + \frac{ie}{\hbar c}A_\mu \ .$$

a) Show that the operator $\partial_\mu \partial^\mu$ transforms trivially under Lorentz transformations. Your starting point should be the transformation rules for x^μ.

b) Under a gauge transformation, A_μ transforms as

$$A_\mu \mapsto A'_\mu = A_\mu - \partial_\mu \lambda \, .$$

Find a gauge transformed wave function $\phi'(x)$ that satisfies the Klein-Gordon equation with potential A'_μ, provided ϕ satisfies it with potential A_μ.

c) Consider

$$\tilde{j}^\mu = \frac{i\hbar}{2m}(\phi^\dagger D^\mu \phi + \alpha \phi \bar{D}^\mu \phi^\dagger) \ ,$$

where $\alpha \in \mathbb{R}$.

i) How does $\tilde{j}^\mu$ transform under Lorentz transformations?

ii) Compute all values of α for which $\tilde{j}^\mu$ is gauge invariant.

iii) Compute all values of α for which $\tilde{j}^\mu$ is a conserved current.