Prof. Dr. J. von Delft, Dr. O. Yevtushenko, F. Kugler Nov. 20, 2019

Problem Set 6: Spin representations

Exercise 1. Fermionic representation of $S = 1/2$ magnets

We consider a spin system on a lattice, with a spin-1/2 representation on each lattice site. Hence, the spin operators on sites r, r' fulfill

$$[S_a(r), S_b(r')] = i\epsilon_{abc} S_c(r) \delta_{r,r'}, \tag{1}$$

$$\sum_{a \in \{x,y,z\}} S_a(r) S_a(r) = 3/4. \tag{2}$$

(a) Consider a *Schwinger*-fermion representation of the spin operators, defined by

$$S_a(r) = \frac{1}{2} f^\dagger_\alpha(r) \sigma^a_{\alpha\beta} f_\beta(r), \quad \alpha, \beta \in \{1, 2\}, \tag{3}$$

where f_α are fermionic operators, $\{f_\alpha(r), f^\dagger_\beta(r'\} = \delta_{\alpha\beta}\delta_{r,r'}$, and σ^a are Pauli matrices. Check that $[S_x(r), S_y(r)] = iS_z(r)$ holds in this representation. Is there a gauge redundancy in this mapping?

(b) Check that (2) does not hold for all states in the fermionic on-site Hilbert space. Formulate a constraint on the Hilbert space such that (2) is fulfilled.

(c) Consider now a *Majorana*-fermion representation, defined by

$$S_a(r) = -\frac{i}{2} \epsilon_{abc} \gamma_b(r) \gamma_c(r). \tag{4}$$

Here, $\gamma_a, a \in \{x, y, z\}$, are "real" (i.e., self-adjoint) fermions in the sense that they satisfy

$$\{\gamma_a(r), \gamma_b(r')\} = \delta_{ab}\delta_{r,r'}, \qquad \gamma^\dagger(r) = \gamma(r). \tag{5}$$

Check that $[S_x(r), S_y(r)] = iS_z(r)$ also holds in this representation. Does (2) hold as well? Is there a gauge redundancy in this mapping?

(d) For the construction of a Majorana path integral, we want to characterize the local Hilbert space in terms of Majorana operators γ_i. In order to do this, it is convenient to introduce a further fourth "Majorana" operator γ_4 that completes the operator algebra (5). Express now γ_i as real or imaginary parts of two complex fermions c, d.

(e) Based on (d), argue that the Majorana Lagrangian is of the form

$$\mathcal{L} = \frac{1}{2} \sum_r \gamma_{a,r} \partial_\tau \gamma_{a,r} + H[\gamma]. \tag{6}$$

(f) To construct $H[\gamma]$, we need the Hamiltonian to be normal ordered. We want to show that a quadratic Hamiltonian in the Majorana basis is quite often trivial. Let us assume the following rather general Hamiltonian

$$H_r = J \sum_{i,j\in\{x,y\}} \gamma_i(r)\gamma_j(r). \tag{7}$$

and bring it into a normal ordered form by using the Dirac fermions c, d at first. Can you think of a possibility to construct a quadratic Hamiltonian in Majorana representation that is nontrivial?

(g) As a minimal example, we consider a single spin in a magnetic field h:

$$H = -hS_z.$$

Write down the action for the Majorana fermions γ_x and γ_y, and transform it to the Matsubara-frequency representation. Compute the magnetization $M = \langle S_z \rangle$ as a Majorana-fermion correlator. Finally, compute the susceptibility $\chi = \mathrm{d}M/\mathrm{d}h|_{h=0}$ and demonstrate the Curie law for high temperatures.

You can check your results by solving this problem with standard (quantum-mechanical) techniques.

Discussion of the problem set on Nov. 26, 2019.