Prof. Dr. J. von Delft, Dr. O. Yevtushenko, F. Kugler Nov. 15, 2019

Problem Set 5: Driven harmonic oscillator and basic formulae in the Keldysh formalism

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Exercise 1. Driven harmonic oscillator

In the lecture, we derived the fundamental relation between expectation values of operators and fields,

$$\langle T_c[a^i(t)a^{\dagger j}(t')]\rangle = \langle \phi^i(t)\bar{\phi}^j(t')\rangle, \tag{1}$$

where T_c denotes contour time ordering and $i, j \in \{\pm\}$ are contour indices.

(a) Use Eq. (1) to establish the relation between operator and field representation of the retarded and Keldysh Green's functions:

$$G^R(t,t') = -i\theta(t-t')\langle[a(t), a^\dagger(t')]\rangle = -i\langle\phi_c(t)\bar{\phi}_q(t')\rangle, \tag{2}$$
$$G^K(t,t') = -i\langle\{a(t), a^\dagger(t')\}\rangle = -i\langle\phi_c(t)\bar{\phi}_c(t')\rangle. \tag{3}$$

What is the corresponding relation for $G^A(t,t')$?

Consider now a many-body system of a single bosonic state:

$$H_0 = \omega_0 a^\dagger a \quad \Rightarrow \quad S[\bar{\phi}, \phi] = \int_C \mathrm{d}t\, \bar{\phi}[i\partial_t - \omega_0]\phi. \tag{4}$$

The connection from model (4) to the quantum harmonic oscillator can be made via the transformation

$$\phi = \frac{1}{\sqrt{2\omega_0}}(\omega_0 X + iP), \quad \bar{\phi} = \frac{1}{\sqrt{2\omega_0}}(\omega_0 X - iP), \tag{5}$$

where X and P are real fields.

(b) Starting from the action (4), show that the corresponding action for the real X and P fields is given by (boundary terms are neglected in the continuous form)

$$S[X,P] = \int_C \mathrm{d}t\, \big[P\dot{X} - \frac{1}{2}P^2 - \frac{\omega_0^2}{2}X^2\big], \tag{6}$$

which is the Hamiltonian representation of the path integral for the harmonic oscillator.

(c) Integrate out the P-fields to obtain the Lagrangian representation. Show that after transforming to Keldysh space via

$$X^c = \frac{X^+ + X^-}{\sqrt{2}}, \quad X^q = \frac{X^+ - X^-}{\sqrt{2}}, \tag{7}$$

one obtains the action

$$S[X^c, X^q] = \int_{-\infty}^{\infty} \mathrm{d}t \left[- X^q \ddot{X}^c - \omega_0^2 X^c X^q \right]. \tag{8}$$

(d) Integrate out X^q and show that, for the harmonic oscillator, X^c obeys classical mechanics: $\ddot{X}^c = -\omega_0^2 X^c$.
Remark: For general systems, the validity of this relation is controlled by the strength of the fluctuations in X^q. This is the origin of the terms "classical" and "quantum" component for X^c and X^q, respectively.

(e) Let's now add a term $-\frac{\omega_1^2}{2}(X - y(t))^2$ to the action (6). Here $y(t)$ is a given external function of time (vanishing at $\pm\infty$). How can such a term arise physically? Show that the corresponding action for the ϕ, $\bar{\phi}$ fields is given by

$$S[\bar{\phi}, \phi] = \int_C \mathrm{d}t \left[\bar{\phi}(i\partial_t - \omega_2)\phi + \frac{V(t)}{\sqrt{2}}(\bar{\phi} + \phi) \right], \tag{9}$$

where $\omega_2 = \sqrt{\omega_0^2 + \omega_1^2}$, and determine the function $V(t)$.

(f) Assume now that $V(t)$ has different contributions on the forward and backward branch. Transform Eq. (9) into Keldysh space (convention: $V^{c/q} = (V^+ \pm V^-)/2$) and compute $Z[V^c, V^q] = \int \mathcal{D}[\bar{\phi}, \phi]\, e^{-(S+S_V)}$. Check that $Z[V^c, 0] = 1$.

(g) Compute the expectation value $\langle X(t) \rangle$ and show (using the explicit form of the bare Green's functions) that it is essentially given by the real part of the Fourier transform of $V_c(t)$.
Hint: Use the generating functional via $\delta Z[V_c, V_q]/\delta V_q(t)$.

Exercise 2. Basic formulae from the lecture

In this exercise, we prove three basic formulae used in the lecture.

(a) The discrete correlation functions of a single bosonic mode with energy ω_0 at inverse temperature β are given by

$$\langle \phi_j^+ \bar{\phi}_{j'}^- \rangle \equiv iG^<_{jj'} = \frac{\rho h_+^{j'-1} h_-^{j-1}}{\det\left[- i\hat{G}^{-1} \right]},$$

$$\langle \phi_j^- \bar{\phi}_{j'}^+ \rangle \equiv iG^>_{jj'} = \frac{h_+^{N-j} h_-^{N-j'}}{\det\left[- i\hat{G}^{-1} \right]} = \frac{(h_+ h_-)^{N-1} h_+^{1-j} h_-^{1-j'}}{\det\left[- i\hat{G}^{-1} \right]},$$

$$\langle \phi_j^+ \bar{\phi}_{j'}^+ \rangle \equiv iG^{\mathbb{T}}_{jj'} = \frac{h_-^{j-j'}}{\det\left[- i\hat{G}^{-1} \right]} \times \begin{cases} 1, & j \geq j' \\ \rho(h_+ h_-)^{N-1}, & j < j' \end{cases},$$

$$\langle \phi_j^- \bar{\phi}_{j'}^- \rangle \equiv iG^{\tilde{\mathbb{T}}}_{jj'} = \frac{h_+^{j'-j}}{\det\left[- i\hat{G}^{-1} \right]} \times \begin{cases} \rho(h_+ h_-)^{N-1}, & j > j' \\ 1, & j \leq j' \end{cases},$$

with

$$\rho = e^{-\beta\omega_0}, \quad h_\pm = 1 \pm i\omega_0\delta t, \quad \det\big[-i\hat{G}^{-1}\big] = 1 - \rho\,(h_- h_+)^{N-1}.$$

Show that, in the continuum limit $N \to \infty$, $\delta t \sim 1/N$, the correlation functions are given by

$$\begin{aligned}
\langle \phi^+(t)\bar{\phi}^-(t')\rangle &= iG^<(t,t') = n_\mathrm{B} e^{-i\omega_0(t-t')}, \\
\langle \phi^-(t)\bar{\phi}^+(t')\rangle &= iG^>(t,t') = (n_\mathrm{B}+1) e^{-i\omega_0(t-t')}, \\
\langle \phi^+(t)\bar{\phi}^+(t')\rangle &= iG^\mathbb{T}(t,t') = \Theta(t-t')iG^>(t,t') + \Theta(t'-t)iG^<(t,t'), \\
\langle \phi^-(t)\bar{\phi}^-(t')\rangle &= iG^{\tilde{\mathbb{T}}}(t,t') = \Theta(t'-t)iG^>(t,t') + \Theta(t-t')iG^<(t,t'),
\end{aligned}$$

where $n_\mathrm{B} = \frac{\rho}{1-\rho}$.

(b) Using the discrete correlation functions, show that

$$G^\mathbb{T}(t,t') + G^{\tilde{\mathbb{T}}}(t,t') - G^>(t,t') - G^<(t,t') = \begin{cases} 0, & t \neq t' \\ 1, & t = t' \end{cases}.$$

(c) The partition function for noninteracting fermions and the polarization matrix are given by $(\alpha, \beta \in \{c, q\})$

$$Z[V] = e^{\mathrm{Tr}\ln[\hat{1} - \hat{G}V^\alpha\hat{\gamma}^\alpha]}, \quad \hat{\gamma}^c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\gamma}^q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{\Pi}^{\alpha\beta}(x,x') \equiv -\frac{i}{2}\frac{\delta^2 \ln Z[\hat{V}]}{\delta V^\beta(x')\delta V^\alpha(x)}\bigg|_{\hat{V}=0}, \quad x = (\mathbf{r}, t).$$

Use these equations to show that

$$\hat{\Pi}^{\alpha\beta}(x,x') = \frac{i}{2}\mathrm{Tr}\Big\{\hat{\gamma}^\alpha \hat{G}(x,x')\hat{\gamma}^\beta \hat{G}(x',x)\Big\}.$$

Discussion of the problem set on Nov. 19, 2019, starting at 16:00.