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Problem Set 1: Free Scalar Field & Green's Functions

Exercise 1. Classical theory of a scalar field

We consider the theory of a single real-valued scalar field $\phi(x)$ governed by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\eta^{\nu\mu}\partial_\nu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 \tag{1}$$

with Minkowski metric $\eta^{\nu\mu} = \eta_{\nu\mu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

(a) Derive the equations of motion (Klein–Gordon equation).

(b) Find the momentum conjugate to $\phi(x)$, denoted by $\Pi(x)$.

(c) Use $\Pi(x)$ to calculate the Hamiltonian density $\mathcal{H}$.

Recall Noether's theorem: Assume that a continuous infinitesimal transformation on the fields, $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$, leaves the Lagrangian invariant up to a total derivative $\alpha\partial_\mu\mathcal{J}^\mu$. Then, there is a conserved current

$$\partial_\mu j^\mu(x) = 0, \quad \text{where } j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu. \tag{2}$$

(d) Determine $\partial_\mu\mathcal{J}^\mu$ and show identity (2).
Hint: Consider the variation of the action for a more general infinitesimal transformation $\phi(x) \to \tilde{\phi}(x) = \phi(x) + \alpha(x)\Delta\phi(x)$ with x-dependent α.

(e) What is the conserved current under an infinitesimal transformation $x^\mu \to x^\mu - a^\mu$? Derive and interpret the associated conserved charges, which fulfill $\partial_0 \int d^3x\ j^0(x) = 0$.

Exercise 2. Basic relations of Green's functions

We consider various Green's functions, or two-point correlators, namely:

$$G^{R/A}_{AB}(t,t') = \mp i\,\Theta(\pm(t-t'))\,\langle[A(t),B(t')]_\zeta\rangle, \tag{3}$$

$$G^{>}_{AB}(t,t') = -i\langle A(t)B(t')\rangle, \tag{4}$$

$$G^{<}_{AB}(t,t') = -i\zeta\langle B(t')A(t)\rangle, \tag{5}$$

$$G^{C}_{AB}(t,t') = -i\langle\mathcal{T}_t A(t)B(t')\rangle. \tag{6}$$

Here, A and B are operators, $\mathcal{T}_t$ denotes the time-ordering operator, and $[A(t), B(t')]_\zeta = A(t)B(t') - \zeta B(t')A(t)$ with $\zeta = \pm 1$ for bosons/fermions, respectively. The superscripts R, A, $>$, $<$, and C refer to the retarded, advanced, greater, lesser, and causal Green's functions, respectively.
Show the following relations:

(a)
$$G^R_{AB}(t,t')^* = G^A_{B^\dagger A^\dagger}(t',t),$$

(b)
$$(G^R_{AB} - G^A_{AB})(t,t') = (G^>_{AB} - G^<_{AB})(t,t'),$$

(c)
$$G^<_{AB}(t,t') = \zeta G^>_{BA}(t',t),$$

(d)
$$G^<_{AB}(t,t')^* = -G^<_{B^\dagger A^\dagger}(t',t),$$

(e)
$$G^>_{AB}(t,t')^* = -G^>_{B^\dagger A^\dagger}(t',t),$$

(f)
$$G^C_{AB}(t,t') = \Theta(t-t')G^>_{AB}(t,t') + \Theta(t'-t)G^<_{AB}(t,t').$$

Exercise 3. Analytic properties of Green's functions

We want to understand how to analytically continue the Green's functions into the complex time plane (for definitions of the Green's functions see Ex. 2, (3)–(6)). In this exercise, we assume that the Hamiltonian does not explicitely depend on time and is bounded from below by zero. As usually, $\beta = 1/T$ denotes inverse temperature.

(a) Show: If $\mathrm{Re}(i(t-t')) \geq 0$ and $\mathrm{Re}(\beta - i(t-t')) \geq 0$, then $G^>_{AB}(t-t')$ can also be evaluated for complex "times" $t-t'$.

(b) Show: If $\mathrm{Re}(-i(t-t')) \geq 0$ and $\mathrm{Re}(\beta + i(t-t')) \geq 0$, then $G^<_{AB}(t-t')$ can also be evaluated for complex "times" $t-t'$.

This means that we can analytically continue $G^>$ into the lower part of the complex plane (staying above $\mathrm{Im}(t) = -\beta$) and $G^<$ into the upper part (staying below $\mathrm{Im}(t) = \beta$).

(c) Show:
$$G^>_{AB}(t-t') = \zeta G^<_{AB}(t-t'+i\beta).$$

What does this mean for G^C_{AB}?

Discussion of the problem set on Oct. 22, 2019.