Exercises for Conformal Field Theory (MD4)

Christmas sheet part 2

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Here is the rest of the construction split into several exercises. There will be typed solutions to these exercises. A textbook account is in Ben-Zvi/ Frenkel: Vertex Algebras and Algebraic Curves.
So let's continue. Next we split the state space into two pieces. In the following capital latin letters run over $\mathfrak{n}_+$ (the positive roots), small latin letters run over $\mathfrak{n}_- \oplus \mathfrak{h}$, greek letters deonte an arbitrary element in $\mathfrak{sl}(N)$. We define the following vector spaces

$$\begin{aligned} C_0^\bullet(\mathfrak{sl}(N))_k &= \mathrm{span}_\mathbb{C}\left\{\bar{J}^{a_1}_{n_1}\dots\bar{J}^{a_l}_{n_r} c^{A_1}_{m_1}\dots c^{A_s}_{m_s}\,|0\rangle\right\} \\ \tilde{C}^\bullet(\mathfrak{sl}(N))_k &= \mathrm{span}_\mathbb{C}\left\{\bar{J}^{A_1}_{n_1}\dots\bar{J}^{A_s}_{n_s} b^{A_1}_{m_1}\dots b^{A_t}_{m_t}\,|0\rangle\right\} \quad . \end{aligned} \tag{1}$$

C) Show that the differential maps

$$d : C_0^\bullet(\mathfrak{sl}(N))_k \to C_0^\bullet(\mathfrak{sl}(N))_k, \quad d : \tilde{C}^\bullet(\mathfrak{sl}(N))_k \to \tilde{C}^\bullet(\mathfrak{sl}(N))_k \quad . \tag{2}$$

D) Show that $C_0^\bullet(\mathfrak{su}(N))_k$ and $\tilde{C}^\bullet(\mathfrak{su}(N))_k$ are closed under commutation relations. Conclude that one can split the CFT into two parts

$$C^\bullet(\mathfrak{sl}(N))_k \cong C_0^\bullet(\mathfrak{sl}(N))_k \otimes \tilde{C}^\bullet(\mathfrak{sl}(N))_k \tag{3}$$

Thus the cohomology of the complex will be the product of the cohomologies of its factors (as chiral CFTs). We start with computing the cohomology of $\tilde{C}^\bullet(\mathfrak{sl}(N))_k$.

E) From the action of the differential show

$$[d, \bar{J}^A_n] = 0, \quad \left\{d, b^A_n\right\} = \bar{J}^A_n + \sum_{i=1}^{r} \delta^{A,i} \quad . \tag{4}$$

Thus the differential on $\tilde{C}^\bullet(\mathfrak{su}(N))_k$ preserves the modes of the factors. Therefore we factor $\tilde{C}^\bullet(\mathfrak{su}(N))_k = \bigotimes_A \bigotimes_n V^A_n$ with

$$V^A_n \equiv \mathrm{span}_\mathbb{C}\left\{\left(\bar{J}^A_n\right)^k (b^A_n)^\epsilon\,|0\rangle \,\middle|\, n\in\mathbb{N}, \epsilon = 0,1\right\} \tag{5}$$

and compute only the cohomology of V^A_n.

F) Show

$$H^i(V^A_n, d) = \begin{cases} \mathbb{C},\ if\, i = 0 \\ 0,\ else \end{cases} \tag{6}$$

(Hint: Consider the case where A is a simple root separately.)

So we computed

$$H^i(\tilde{C}^\bullet(\mathfrak{su}(N))_k, d) = \begin{cases} \mathbb{C},\ i = 0 \\ 0,\ else \end{cases} \tag{7}$$

This tells us that the cohomology of the total complex is determined by the cohomology of $C_0^\bullet(\mathfrak{su}(N))_k$. The computation of this cohomology can be done using a spectral sequence argument.

G) Infer from your computation in *A*) that it holds

$$d_0^2 = 0, \quad \chi^2 = 0, \quad d_0\chi + \chi d_0 = 0. \tag{8}$$

So far we haven't considered the conformal dimensions of our fields. For the construction to work we have to deform the usual Sugawara + (b, c) energy-momentum tensor (see e.g. exercise 15.6 in di Francesco for the construction). The upshot is that using the deformed emt the level of our fields reads

$$\begin{aligned} lv(J^\alpha_n) &= -n - \sum a_i,\ where\ \alpha = \sum a_i\alpha_i, \\ lv(b^A_n) &= -n - \sum A_i, \\ lv(c^A_n) &= -n + \sum A_i,\ where\ A = \sum A_i\alpha_i \end{aligned} \tag{9}$$

Note that fields for different generators of the Lie-algebra have different conformal weights.

H) Show that $Q(z)$ is conformal dimension 1. This implies that the differential d has level 0. Hence its cohomology inherits the level grading.

Equation (8) ask for a spectral sequence. We only have to introduce an artificial bidegree s.th. $\text{bideg}(d_0) = (0, 1)$ and $\text{bideg}(\chi) = (1, 0)$.

I) Show that the following choice will do the job

$$\begin{aligned} \text{bigdeg}(J^\alpha_n) &= (-\sum a_i, \sum a_i) \\ \text{bideg}(c^A_n) &= (\sum A_i, -\sum A_i + 1) \\ \text{bideg}(b^A_n) &= (-\sum A_i, \sum A_i - 1)\,. \end{aligned} \tag{10}$$

Hint: The vector corresponding to the field $Q(z)$ under operator state correspondence is: $Q = \sum_{A\in\Delta_+} J^A_{-1} \otimes c^A_0 \, |0\rangle - \frac{1}{2}\sum_{A,B,C\in\Delta_+} f^{AB}_C \, 1 \otimes c^A_0 c^B_0 b^C_{-1} \, |0\rangle + \sum_{i=1}^l 1 \otimes c^{\alpha_i}_0 \, |0\rangle$

After all this we can finally start computing the cohomology of the second complex. On the first page of the spectral sequence is the cohomology of χ.

J) Recall that in the Cartan-Weyl basis the Killing form on generators corresponding to simple roots reads $\kappa(J^{\alpha_i}, J^{-\alpha_j}) = \frac{2}{(\alpha_i,\alpha_i)}\delta_{ij} = N_i\delta_{ij}$. Use this to show

$$\sum_{i=1}^{l}\sum_B f_i^{a,B} c^B(z) = \sum_B \kappa(\sum_{i=1}^{l}\frac{1}{N_i}J^{-\alpha_i}, J^a], J^B)c^B(z) \tag{11}$$

In the following we denote $W_- = \sum_{i=1}^l \frac{1}{N_i} J^{-\alpha_i}$.

For the next steps we need some facts about semi-simple Lie-algebras. An element X_- in a semi-simple Lie-algebra $\mathfrak{g}$ is called *nilpotent* if $\text{ad}^N_{X_-}(Y) = 0$, $\forall Y \in \mathfrak{g}$ for some $N > 0$. Any nilpotent element X_- can be uniquely completed to an $\mathfrak{sl}(2)$-subalgebra (Y_+, H_0, X_-) inside $\mathfrak{g}$. We can then decompose $\mathfrak{g}$ in terms of representations of this $\mathfrak{sl}(2)$ subalgebra.

K) Show that W_- is a nilpotent element in $\mathfrak{sl}(N)$.

We call the W_- subalgebra (W_+, W_0, W_-) and decompse $\mathfrak{sl}(N)$ in terms of (W_+, W_0, W_-) representations. Being an $\mathfrak{sl}(2)$ representation, $\mathfrak{g}$ decomposes as a direct sum of irreducible (lowest weight) representations. In our case we get $\mathfrak{sl}(N) = \bigoplus_{i=1}^{N-1}(\mathbf{2i+1})$. In addition we have that the lowest weight vector W^i_- in $\mathbf{2i+1}$ is a sum of generators J^α such that $\sum a_j = i$, where $\alpha = \sum a_j\alpha_j$.

L) To the lowest elements W^i_- we can associate a field $\overline{W}^i_-(z) = \sum \overline{W}^i_{-,n} z^{-n-1}$. Show

$$\left[\chi, \overline{W}^i_-(z)\right] = 0 \quad . \tag{12}$$

As you might have guessed from the notation these fields will constitute the cohomology. We want to mimic the trick we used in the computation in F) and split the complex $C^\bullet_0$ into two subcomplexes. We denote $\mathcal{W}$ for the subspace spanned by applying the fields $\overline{W}^i_-(z)$ to the vacuum.

M) Show that there is a splitting of complexes

$$C^\bullet_0(\mathfrak{sl}(N)) \cong \mathcal{W} \otimes B^\bullet_0 \tag{13}$$

wrt the differential χ.

Hint: Extend $\left\{W^i_-\right\}$ to a basis $(\left\{W^i_-\right\}, \{I^{a_j}\})$ of $\mathfrak{sl}(N)$.

We are almost done. What is left is to show that the cohomology of $B_0^\bullet$ is $\mathbb{C}$ in ghost degree 0 and zero else. Since then the cohomology of the complex is $\mathcal{W}$ in degree zero and the spectral sequence collapses at the first page. This means that the cohomology of $(C_0^\bullet(\mathfrak{sl}(N)), d)$ is $\mathcal{W}$! from the definition of the shifted conformal weights we get that the conformal weight of a generator $\overline{W}_-^i(z)$ is i and we are done.

N) Show

$$\left[\chi, \overline{I}_n^A\right] = b_{n+1}^A \quad . \tag{14}$$