Exercises for Conformal Field Theory (MD4)

Problem set 6, due December 4, 2019

If you have questions write an E-mail to: mtraube@mpp.mpg.de

1 $\frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{k+1}}$ cosets and minimal models

In this exercise we show that the unitary minimal models from the last sheet are in fact coset CFTs of the form

$$\frac{\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{k+1}} \quad . \tag{1}$$

A first hint stems from looking at the central charge of the coset CFT, which reads

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)} \quad . \tag{2}$$

Recall from the last exercise sheet that unitary minimal models have central charge

$$c = 1 - \frac{6}{m(m+1)} \tag{3}$$

hence we expect to find unitary minimal models with $m = k+2$. In the (p,q) notation from the last sheet these are $(k+3, k+2)$ minimal models. The equivalence can be shown by showing that the integrable highest weight representations (IHWR) are in fact exactly the unitary minimal models. Recall from the lecture that for $\hat{\mathfrak{su}}(2)_k$ these are labeled by $0 \leq l \leq k$, hence there are $k+1$ such representations. Thus IHWR of the direct sum $\hat{\mathfrak{su}}(2)_k \oplus \hat{\mathfrak{su}}(2)_1$ are tensor products $(l)_k \otimes (\epsilon)_1$. To derive the branching rules we have to decompose these tensor product representations in terms of IHWR $(s)_{k+1}$ times some multiplicity factor which will correspond to the representation of the coset. On the level of characters this reads

$$\chi^k_{(l)} \chi^1_{(\epsilon)} = \sum_{0 \leq s \leq k+1} \chi^{k+1}_{(s)} \chi_{(l,\epsilon,s)} \tag{4}$$

where $\chi_{(l,\epsilon,s)}$ is then a character of a coset representation. The goal is to show that $\chi_{(l,\epsilon,s)}$ are exactly the characters for $(k+3, k+2)$ minimal models, i.e. the characters for all possible $V_{R,S}$ from the last sheet. There is a formula for the characters of $\hat{\mathfrak{su}}(2)_k$ characters in terms of generalized Θ-functions

$$\chi^k_{(l)}(\tau, z) = \frac{\Theta_{l+1,k+2}(\tau,z) - \Theta_{-l-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z) - \Theta_{-1,2}(\tau,z)} \tag{5}$$

with

$$\Theta_{l,k}(\tau,z) = q^{\frac{l^2}{4k}} \sum_{n\in\mathbb{Z}} q^{kn^2+ln} \omega^{kn+\frac{l}{2}}, \quad \omega \equiv e^{-2\pi i z} \tag{6}$$

Note that the denominator is independent of l, k in the characters. You already computed the character for $(0)_1$ of $\hat{\mathfrak{su}}(2)_1$ on sheet 4. In general the characters for $\hat{\mathfrak{su}}(2)_1$ read

$$\chi^1_{(\epsilon)} = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{n^2+n\epsilon+\frac{\epsilon^2}{4}} \omega^{n+\frac{\epsilon}{2}} \quad . \tag{7}$$

Hence we rewrite the lhs of (4) to get rid of the denomintaor in (5). This yields

$$D_{l+1,k+2} \chi^1_{(\epsilon)} = \sum_{0\leq s\leq k+1} D_{s+1,k+3} \chi_{(l,\epsilon,s)} \tag{8}$$

with $D_{l+1,k+2} = \Theta_{l+1,k+2}(\tau,z) - \Theta_{-l-1,k+2}(\tau,z)$. In the following we massage the lhs of (8) long enough to read off the rhs.

A) This was quite some information. Take your time and read the above carefully. Make sure you understand the formulas appearing. (You don't have to worry how to derive them, after all they drop out of the Kac-character formula.) If you have questions write an email!

B) Explain why the above characters extract more information about the theories than the generic Virasoro-character.

In the proof we need some identites for $D_{l,k}$.

C) Show

$$D_{l,k} = q^{\frac{l^2}{4k}} \sum_{m\in\mathbb{Z}} q^{ml+km^2} \left(\omega^{km+\frac{l}{2}} - \omega^{-km-\frac{l}{2}}\right) \tag{9}$$

$$\begin{aligned} D_{l+2tk,k} &= D_{l,k}, \quad t\in\mathbb{Z} \\ D_{l,k} &= -D_{-l,k} \\ D_{k,k} &= 0 \\ D_{l+1,k+1} &= -D_{2(k+1)-l-1,k+1} \quad . \end{aligned} \tag{10}$$

D.1) Denote the summation indices $m, \tilde{m}$ for the sums in $D_{l+1,k+2}$ and $\chi^1_{(\epsilon)}$ on the lhs of (8). Introduce a new summation index

$$s = l + \epsilon + 2(\tilde{m} - m) \tag{11}$$

and write (8) in the form

$$D_{l+1,k+2}\chi^1_{(\epsilon)} = \frac{1}{\eta(q)} \sum_{\substack{s\in\mathbb{Z} \\ s+l+\epsilon=0\, mod\, 2}} q^{A_{l,s}} D_{s+1,k+3} \quad . \tag{12}$$

D.2) Next we use the identities (10). Introduce $s = s_0 + 2(k+3)n$, $n \in \mathbb{Z}$, $0 \le s_0 \le 2k+5$ and bring (12) in the form

$$\begin{aligned} &\sum_{\substack{0\le s_0\le k+1 \\ s_0+l+\epsilon=0\, mod\, 2}} D_{s_0+1,k+3} \left[\sum_{n\in\mathbb{Z}} \frac{q^{B_{l,s_0}(n)} - q^{B_{l,-s_0-2}(n)}}{\eta(q)}\right] \\ \equiv &\sum_{\substack{0\le s_0\le k+1 \\ s_0+l+\epsilon=0\, mod\, 2}} D_{s_0+1,k+3}\, \chi_{(l,\epsilon,s_0)} \end{aligned} \tag{13}$$

You should find

$$B_{l,s_0}(n) = \frac{\left((l+1)^2(k+3) - (s_0+1)(k+2) + 2n(k+2)(k+3)\right)^2}{4(k+2)(k+3)} \quad . \tag{14}$$

D.3) Choose $R(l), S(s_0)$ s.th.

$$\chi_{(l,\epsilon,s_0)}(q) = \chi_{R,S}(q) \tag{15}$$

where $\chi_{R,S}(q)$ is the character for a unitary $(p,q) = (k+3, k+2)$ minimal model in the $h_{R,S}$ highest weight representation (HWR) from the last sheet.

E) Infer from the ranges of l, s_0 that every possible value for R, S in the $(k+3, k+2)$ minimal model is reached.

Comment 1. You may worry that (13) is not quite (4). But since $\epsilon \in \{0,1\}$, for fixed values of l, s_0 the extra condition $l + s_0 + \epsilon = 0$, $mod\, 2$ uniquely determines ϵ, thus the relation doesn't matter.