# Exercises for Conformal Field Theory (MD4)

## Problem set 5, due November 27, 2019

*If you have questions write an E-mail to: mtraube@mpp.mpg.de*

## 1 Singular Vectors and Minimal Models

Recall exercise 2 on sheet 3 where we constructed a highest weight representation (Verma-module) $V_{(h,c)}$ from a chiral primary field $\phi$ of conformal dimension $h$. We want to determine conditions on $h, c$ for which the Verma-module doesn't contain null-vectors. Let $|\psi\rangle \neq |\phi\rangle$ be a state in $V_{(h,c)}$ s.th.

$$L_n |\psi\rangle = 0, \quad \forall n \geq 1 \quad . \tag{1}$$

Such a state $|\psi\rangle$ is called *singular vector*. Relations (1) tell that the descendants of $|\psi\rangle$ span their own Verma-module inside $V_{(h,c)}$, i.e. a subrepresentation of the Virasoro-algebra. Thus the Verma-module is *reducible*. By the construction of excercise 2 on sheet 3 this also yields that we have another primary field $\psi(z)$ corresponding to the state $|\psi\rangle$ in our theory. (Of course it is also a descendant of the primary $\phi$).

A) Show that the Verma-module built on $|\psi\rangle$ is orthogonal to any state in $V_{(h,c)}$.

Thus if the Verma-module contains a singular vector we better quotient out its corresponding submodule to get rid of null fields. In the lecture you learned about the Kac-determinant (at level $N$) for a Verma-module $V_{(h,c)}$:

$$\det M^N(h,c) = A_N \prod_{1\leq m,n\leq N} (h - h_{m,n}(c))^{P(N-mn)} \quad . \tag{2}$$

Roots of the Kac-determinant correspond to singular vectors at the corresponding level in the Verma-module. It turns out, that an important class of CFTs have singular vectors, the so called *minimal models*. They occur for highest weight representations with

$$\begin{aligned} c &= 1 - 6\frac{(p-q)^2}{pq} \\ h_{r,s} &= \frac{(pr-qs)^2-(p-q)^2}{4pq} \end{aligned} \tag{3}$$

where $1 \leq r < q-1$, $1 \leq s < p$ and $p, q$ are fixed coprime positive integers. They are unitary (i.e. don't have negative norm states) for $p = m+1$, $q = m$ and $m \geq 3$, i.e.

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)} \\ h_{r,s}(m) &= \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \quad . \end{aligned} \tag{4}$$

Minimal models have the nice property that they are closed under fusion and only finitely many propagating conformal primaries are present (they are examples of *rational conformal field theories*). Since they are very computable and have a vast range of applications (see e.g. Di Francesco et. al. chapter 7.4) it is worthwhile to study them more closely. Let $V_{R,S}$ be a highest weight representation (Verma-module) with $h = h_{R,S}$ and central charge from (3).

B) Show

$$h_{r,s} = h_{q-r,p-s}, \quad h_{r,s} = h_{q+r,p+s}, \quad h_{r,s} + rs = h_{r,-s} \quad . \tag{5}$$

C) Use this and the Kac-determinant to show that $V_{R,S}$ has singular vectors at level $RS$ and $(q-R)(p-S)$. Show that the Verma-modules built on those singular vectors are again reducible. You should find that they have again two common reducible submodules.

*Hint: Use the relations* (5) *to compute the conformal weight of the singular vectors.*

The above is an iterative procedure. At every step one finds, that there are two reducible submodules which have two common reducible submodules.

D) Show that at the k-th iteration the two common reducible submodules are

$$V_{kq+R,(-1)^k S+(1-(1)^k)\frac{p}{2}}, \quad V_{R,kp+(-1)^k S+(1-(-1)^k)\frac{p}{2}} \quad . \tag{6}$$

*Hint:You already verified this for $k=1$, hence you can use induction.*

Thus the honest state space of the minimal model is

$$\begin{aligned} M_{R,S} &\equiv V_{R,S}/(V_{q+R,p-S} \oplus V_{R,2p-S}) \\ &= V_{R,S} - (V_{q+R,p-S} + V_{R,2p-S}) + \cdots + (-1)^k \left( V_{kq+R,(-1)^k S+(1-(1)^k)\frac{p}{2}} + V_{R,kp+(-1)^k S+(1-(-1)^k)\frac{p}{2}} \right) \cdots \quad . \end{aligned} \tag{7}$$

E) Compute the character $\chi(M_{R,S})(q) = tr_{M_{R,S}} q^{L_0 - \frac{c}{24}}$.

(*Hint: Use the generic Virasoro-character from sheet 3 and linearity of the trace.*

F) Explain why the appearance of singular vectors puts severe constraints on the OPEs between chiral primaries of minimal models.

*(You don't have to compute anything here. Have a look at conformal Ward identities and lay out steps for a procedure to constrain OPEs using singular vectors.)*

# 2 Zero modes of the Ramond sector

As said in the last exercise sheet every free boson corresponds to a dimension in string theory. Now in superstring theory every free boson comes hand in hand with a free fermion. Therefore also the amount of free fermions corresponds to the dimensions. Let us therefore consider an even amount of free fermions $\psi^i$ with $i = 1, 2, \ldots 2n$.[1] The mode expansion of a fermion on $\mathbb{C}$ is

$$\psi^i(z) = \sum_r \psi^i_r z^{-r-\frac{1}{2}} \,. \tag{8}$$

with $r \in \mathbb{Z}$ for the Ramond (R) sector while $r \in \mathbb{Z} + \frac{1}{2}$ for the Neveu-Schwarz (NS) sector.

A) A string propagating through time is a cylinder. To go back to the string picture we apply the conformal transformation $w = x_0 + ix_1 = f(z) = \log z$ to map $\mathbb{C}$ back to the cylinder. Show that

$$\psi^i_{cyl}(w) = \sum_r \psi^i_r e^{rw} \tag{9}$$

What happens to the boundary conditions of the fermion under this coordinate transformation?

One sees that (only) in the R sector there is a zero-mode $\psi_0$. Recalling $[L_0, \psi^i_0] = 0$ it is clear that $\psi^i_0$ does not alter the $L_0$ eigenvalue. This means that for any state $|h\rangle$ with $\psi^i_0|h\rangle \neq 0$ we find a second state $\psi^i_0|h\rangle$ with the same energy. This degeneracy is actually a degeneracy of the vacuum as we can always shift the $\psi^i_0$ to the right using $\{\psi^i_r, \psi^j_s\} = \delta_{r,-s}\delta^{ij}$ without getting extra terms. One immediately sees that the $\psi_0$ obey the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \tag{10}$$

after an appropriate rescaling $\gamma^i = \sqrt{2}\psi^i_0$. Therefore the degenerate vacuum of the Ramond sector is actually a spinor. This motivates us to investigate the representation theory of the Clifford algebra. Having a free theory we can specialize to two dimensions thus $i = 1, 2$ without loss of generality and see representations in higher (even) dimensions as tensorproduct of the two dimensional theory.

[1] Superstring theory needs ten dimensions to cancel the conformal anomaly. Lightcone gauge can be used to eliminate two of them leaving eight effective euclidean dimensions, thus $2n = D - 2 = 8$

B) Representation theory is most easily done using raising and lowering operators. Build appropriate $\gamma^+, \gamma^-$ out of $\gamma^1, \gamma^2$ and verify that they obey the usual (anti)-commutation relations of raising and lowering operators.

C) Use the raising and lowering operators to construct the vector space of a highest weight representation (thus the module of the representation). What is the dimension of the vectorspace? What is the dimension in higher (even) dimension?

D) One can build a chirality operator out of the $\gamma$ matrices[2]

$$\Gamma = i^n \gamma^1 \dots \gamma^{2n} \,. \tag{11}$$

Show that

$$\{\Gamma, \gamma^i\} = 0 \qquad , \qquad \Gamma^2 = 1 \tag{12}$$

What does this imply for the vector space you constructed above?

*Note: The degeneracy of the Ramond groundstate will reappear later when looking at $N = 2$ superconformal theories.*

[2] Of course in four dimensions this is the well known $\gamma^5$