Exercises for Conformal Field Theory (MD4)

Problem set 4, due November 20, 2019

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1 Vertex operators

Let us consider the free boson CFT from the lecture. Next to the current $j(z) = i\partial X(z, \overline{z})$ there is another conformal primary

$$V_\alpha(z, \overline{z}) = :e^{i\alpha X(z,\overline{z})}: \tag{1}$$

called vertex. In the following we will verify that $V(z, \overline{z})$ is indeed a conformal primary and compute its eigenvalues under the current $j(z)$. This is most easily done using Wick's theorem. In case you are unfamiliar with that we collected all necessary information at the end of the sheet.

A) Let us compute the momentum of the vertex operator. As we have a free theory $p = \pi$ and due to $\pi = j_0$ this amounts to computing the commutator $[j_0, V]$. To do so recall the integration of the mode expansion of the current

$$X(z, \overline{z}) = x_0 + -i\, j_0 \log z + i \sum_{n\neq 0} \frac{j_n}{n} z^{-n} + \text{antiholomorphic} \tag{2}$$

from the lecture and use $[j_m, j_n] = m\delta_{m,-n}$ as well as the Heisenberg commutation relation.

B) Given a conformal primary $\phi(z, \overline{z})$ with momentum p_ϕ

$$[j_m, \phi_n] = p_\phi\ \phi_{m+n} \tag{3}$$

derive the singular part of the OPE $j(z)\phi(w, \overline{w})$. State the result you expect for the OPE $j(z)V_\alpha(w, \overline{w})$.

C) Verify the guess for $j(z)V_\alpha(w, \overline{w})$ from the last exercise using Wick's theorem. As you can see, the momentum can be read of from the OPE.

D) Show that $V_\alpha(w, \overline{w})$ is a conformal primary by computing the OPE $T(z)V_\alpha(w, \overline{w})$ using Wick's theorem. What is the conformal dimension?

E) The Lagrangian of the free boson $\mathcal{L} = \frac{1}{2}\partial X\, \overline{\partial} X$ is invariant under shifts $X(z, \overline{z}) \to X(z, \overline{z}) + a$ for constant a. Use this symmetry to deduce for which values of α and β the correlator of two vertex operators

$$\langle V_\alpha(z, \overline{z})V_\beta(w, \overline{w})\rangle \tag{4}$$

is non-zero. Interpret this result in terms of the symmetry and state the result of the above correlator.

F) By the operator-state correspondence there is a state

$$|\alpha\rangle := \lim_{|z|\to 0} V_\alpha(z, \overline{z})|0\rangle \tag{5}$$

Furthermore the primary $j(z) = i\partial X(z, \overline{z})$ gives rise to

$$\lim_{|z|\to 0} i\partial X(z, \overline{z})|0\rangle = j_{-1}|0\rangle \tag{6}$$

As such a general state in the free boson CFT is the combination of both

$$\lim_{|z|\to 0} :(i\partial X(z, \overline{z}))^{n_1}(i\partial^2 X(z, \overline{z}))^{n_2} \ldots\ \ V_\alpha(z, \overline{z}):|0\rangle = j_{-1}^{n_1} j_{-2}^{n_2} \ldots |\alpha\rangle\ , \tag{7}$$

where we omitted the antiholomorphic piece. State its conformal weight and its momentum.

G) We specialize to momenta $\alpha = \sqrt{2}m$ with $m \in \mathbb{Z}$. Denote the Hilbert space spanned by the above vectors with $V_{\mathbb{Z}}$. Compute the character

$$\chi(\tau, z) = Tr_{V_{\mathbb{Z}}} \left(q^{L_0 - \frac{c}{24}} x^{j_0} \right) , \quad x = e^{-2\pi i z} \quad . \tag{8}$$

(Note: This is also the character for the $\hat{\mathfrak{su}}(2)_1$ vacuum representation.)

H) *(optional)* In string theory the free boson takes the role of the coordinate of the target space. As such we need D free bosons X^μ with $\mu = 1, \ldots, D$ instead of only one X. As you maybe heard, string theory naturally predicts gravity. Concretely the graviton appears in the state[1]

$$\lim_{|z| \to 0} \epsilon_{\mu\nu} : i\partial X^\mu(z, \bar{z}) \; i\bar{\partial} X^\nu(z, \bar{z}) V_k(z, \bar{z}) : |0\rangle = \epsilon_{\mu\nu} j^\mu_{-1} \bar{j}^\nu_{-1} |k\rangle \; . \tag{9}$$

Analogously to (1) compute the OPE

$$T(z)\epsilon_{\mu\nu} : i\partial X^\mu(w, \bar{w}) \; i\bar{\partial} X^\nu(w, \bar{w}) V_k(w, \bar{w}) : \; . \tag{10}$$

Next to the expected terms you will find a cubic term such that the above state is not a conformal primary. What needs to be imposed to get rid of this anomalous term? Interpret this relation!

Of course, the graviton should be massless. What condition onto the momentum k ensures the masslessness? What is the conformal dimension of a massless graviton?

Wick's theorem for free fields

The contraction of two operators A and B is defined to be

$$A^\bullet \, B^\bullet \equiv A \, B \; - :A \, B: . \tag{11}$$

Recalling the full OPE for a scalar X

$$X(z, \bar{z}) X(w, \bar{w}) = -\log|z - w|^2 + : X(z, \bar{z}) X(w, \bar{w}) : \tag{12}$$

one sees that the contraction picks out the singular part of the OPE. For free theories Wick's theorem reads

$$ABCDEF\ldots = :ABCDEF\ldots: \quad + \sum_{\text{singles}} :A^\bullet B^\bullet CDEF\ldots: \quad + \sum_{\text{doubles}} :A^\bullet B^\bullet C^{\bullet'} D^{\bullet'} EF\ldots: \quad + \ldots \tag{13}$$

The first sum is over all possibilities to have a single contraction while the second sum is over all possibilities to have two contractions and so forth. Let us give examples to clarify how to perform the first sum

$$\begin{aligned} X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) X(z_3, \bar{z}_3) = & : X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) X(z_3, \bar{z}_3) : + \\ & + : X^\bullet(z_1, \bar{z}_1) X^\bullet(z_2, \bar{z}_2) X(z_3, \bar{z}_3) : \\ & + : X^\bullet(z_1, \bar{z}_1) X(z_2, \bar{z}_2) X^\bullet(z_3, \bar{z}_3) : \\ & + : X(z_1, \bar{z}_1) X^\bullet(z_2, \bar{z}_2) X^\bullet(z_3, \bar{z}_3) : \\ = & : X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) X(z_3, \bar{z}_3) : + \\ & - \log|z_1 - z_2|^2 : X(z_3, \bar{z}_3) : \\ & - \log|z_1 - z_3|^2 : X(z_2, \bar{z}_2) : \\ & - \log|z_2 - z_3|^2 : X(z_1, \bar{z}_1) : \; . \end{aligned} \tag{14}$$

To shorten the next example we introduce the notation $X_i := X(z_i, \bar{z}_i)$ and $z_i - z_j := z_{ij}$. Furthermore we do not write out the single contractions as they should be clear from the last example.

$$\begin{aligned} X_1 X_2 X_3 X_4 =& : X_1 X_2 X_3 X_4 : + \sum_{\text{singles}} + : X_1^\bullet X_2^\bullet X_3^{\bullet'} X_4^{\bullet'} : + : X_1^\bullet X_2^{\bullet'} X_3^\bullet X_4^{\bullet'} : + : X_1^\bullet X_2^{\bullet'} X_3^{\bullet'} X_4^\bullet : \\ =& : X_1 X_2 X_3 X_4 : + \sum_{\text{singles}} + \log|z_{12}|^2 \log|z_{34}|^2 + \log|z_{13}|^2 \log|z_{24}|^2 + \log|z_{14}|^2 \log|z_{23}|^2 \end{aligned} \tag{15}$$

[1] The symmetric and traceless part of $\epsilon_{\mu\nu}$ is the fluctuation $h_{\mu\nu}$ of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.