Exercises for Conformal Field Theory (MD4)

Problem set 1, due October 30, 2019

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1 The Conformal Group

The conformal group is built out of operators that generate conformal transformations. Here we will verify that these operators indeed correspond to translations, rotations, dilations and special conformal transformations.

A) Show that the momentum operator $P_\mu = -i\partial_\mu$ generates translations. In particular, verify that

$$\exp\big(ia^\mu P_\mu\big)\, f\big(x^\nu\big) = f\big(x^\nu + a^\nu\big) \tag{1}$$

for some differentiable function $f(x^\nu)$ and some constants a^μ. Note that this operation describes a translation $x^\nu \to x^\nu + a^\nu$.

B) For the dilation operator $D = -ix^\mu\partial_\mu$ verify that

$$\exp\big(iaD\big)\, x^\nu = e^a\, x^\nu \tag{2}$$

for some constant a. Note that this operation is a dilation $x^\nu \to e^a\, x^\nu$.

C) Show that the operator $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ generates rotations. Specialize to 3 dimensions and verify that

$$\exp\big(i\alpha L_{12}\big)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{3}$$

with α constant. Note that the matrix on the right-hand side is a rotation matrix.

D) Show that the operator $K_\mu = -i\,(\,2x_\mu x^\nu\partial_\nu - x^\nu x_\nu\partial_\mu)$ generates special conformal transformations. In particular, up to first order in the constants b^μ, verify that

$$\exp\big(ib^\mu K_\mu\big)\, x^\rho = x^\rho + 2\,\big(b^\mu x_\mu\big)\, x^\rho - \big(x^\mu x_\mu\big)\, b^\rho + \mathcal{O}\big(b^2\big)\ . \tag{4}$$

The situation for all orders in b^μ will be investigated in the next problem.

2 Special Conformal Transformation

In this problem we will show that the formula for *finite* special conformal transformations

$$x'^\mu = \frac{x^\mu - (x^\nu x_\nu)b^\mu}{1 - 2\,(x^\nu b_\nu) + (b^\nu b_\nu)(x^\rho x_\rho)} \tag{5}$$

is indeed the correct expression corresponding to the *infinitesimal* version

$$x'^\mu = x^\mu + 2\,\big(x^\nu b_\nu\big)x^\mu - \big(x^\nu x_\nu\big)\, b^\mu\ . \tag{6}$$

We will do this in two steps:

A) First, show that the finite special conformal transformations (5) lead to the scale factor

$$\Lambda(x) = \Big(1 - 2\,(x^\nu b_\nu) + (b^\nu b_\nu)(x^\rho x_\rho)\Big)^{-2} \tag{7}$$

and so they are indeed conformal transformations.

B) Second, expand (5) for $b^\mu \ll 1$ and show that up to first order in b^μ it is equal to the infinitesimal version (6).

3 Complex Analysis

During the first part of the lecture, some results from complex analysis will be important. The following is a recollection of important facts and some sample exercises.

Definition 1. Let $U \subset \mathbb{C}$ be an open subset. A smooth complex valued function $f(z) = u(z) + iv(z)$ is *holomorphic in* U if its real and imaginary part satisfy the *Cauchy-Riemann-equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{8}$$

, where $z = x + iy$, at every point in U.

Definition 2. A complex valued function f has *pole at a point* $z_0 \in U$ if $0 = \frac{1}{f(z_0)}$. The function f is *meromorphic on* U if there is a sequence of points $\{z_0, z_1, z_2, \ldots\} \subset U$ without accumulation point, s.th. f has poles on $\{z_0, z_1, z_2, \ldots\}$ and is holomorphic on $U \backslash \{z_0, z_1, z_2, \ldots\}$. A meromorphic function has a *pole of order* N *at* $z_0 \in U$ if N is the smallst positive integer s.th. $(z - z_0)^N f(z)$ is holomorphic and non-vanishing in a small neighborhood of the point.

The condition on not having an accumulation point of poles ensures, that for a meromorphic function you can find small paths around a pole s.th. no other pole is inside the contour.

Fact 1. *A meromorphic function* f *on* U *can be written as* $f(z) = \frac{g(z)}{h(z)}$, *where* $g(z), h(z)$ *are holomorphic functions on* U *and* $h(z) \neq 0 \, \forall z \in U$.

These facts may seem abstract, for all practical pourposes we need the following fact.

Fact 2. *Let* $U \subset \mathbb{C}$ *be open. Let* $0 \leq r < R \in \mathbb{R}$ *be constants and* z_0 *be a point in* U *and* f *is a holomorphic function on the annulus* $A_{z_0}(r, R) = \{z \in \mathbb{C} \,|\, r < |z - z_0| < R\} \subset U$. *Then* f *can be expanded in a Laurent-series on* $A(r, R)$, *i.e.*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \forall\, z \in A(r, R). \tag{9}$$

In addition the Laurent-series is absolutely convergent in $A_{z_0}(r, R)$.

Absolut convergence is a technical remark. You can use it to exchange integral and summation whenever necessary. Note that if the inner radius r can be taken zero, and the Laurent-series has only finitely many terms of negative order, then f has a pole at z_0, i.e. f is meromorphic on the disk $D_{z_0}(R)$. We will almost exclusively deal with this case. The next lemma is one of the fundamental results in complex analysis.

Fact 3. *Let* $U \subset \mathbb{C}$ *be open,* $z_0 \in U$ *a point and* f *a holomorphic function on* U. *Then for any continuous path* $\mathcal{C}$, *circling* z_0 *counterclockwise once, it holds*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)} dz \quad . \tag{10}$$

This is the famous Cauchy-integral-formula.

But even more is true!

Fact 4. *In the situation of fact 3 it holds*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} dz \quad . \tag{11}$$

Fact 5. *In the situation of fact 3 it holds*

$$0 = \oint_{\mathcal{C}} f(z) dz \quad . \tag{12}$$

In the following exercises you can always assume that U is a disk of very large radius.

A) In the situation of fact 3, calculate

$$\oint_{|z|>|z_0|} \frac{1}{(z - z_0)^n} dz, \quad n \in \mathbb{Z} \tag{13}$$

where the integration contour denotes an arbitrary path of type as in fact 3.

B) In the situation of fact 3 assume that $0 \in \mathbb{C}$ is not encircled by the contour. Calculate

$$\oint_{\mathcal{C}(z_0)} \frac{z^m}{(z-z_0)^n} dz\,, \qquad m,n \in \mathbb{Z} \quad . \tag{14}$$

C) Assume f is holomorphic on $A_{z_0}(r,R)$. Find an expression for the Laurent-coefficients.

D) Let f be a meromorphic function with poles at points $z_0,\ldots,z_k$ with $|z_i| < |z_k|$ for $i \neq k$. Show

$$\oint_{|z|>|z_k|} f(z) = \sum_{j=0}^{k} \oint_{\mathcal{C}(z_j)} f(z) \tag{15}$$

where $\mathcal{C}(z_i)$ is a small contour around z_i. *(Hint: You may want to draw a picture!)*