# Exercises for Conformal Field Theory (MD4)

## Christmas sheet, due January 8, 2020

*If you have questions write an E-mail to: mtraube@mpp.mpg.de*

*This exercise sheet is split into two pieces. Exercises from C) on on the extra sheet are optional!*

## $\mathcal{W}$-algebras from Drinfel'd-Sokolov reduction

In the lecture you learned about $\mathcal{W}$-algebras, which are extension of the Virasoro-algebra by higher spin currents. In general it is a tough question to determine which higher spin extensions are possible, but there is a procedure to produce $\mathcal{W}(2,3,4,\ldots,N-1)$-algebras for specified central charges. This is the quantum Drinfel'd-Sokolov reduction (or BRST-reduction).

First recall the $(b,c)$ ghost system where $b(z), c(z)$ are chiral bosonic primaries but they have the wrong spin-statistics, i.e. they are anticommuting. The OPEs/ commutation relations read:

$$b(z)c(w) \sim \frac{1}{z-w} \quad \Leftrightarrow \quad \{b_m, c_n\} = \delta_{m,-n} \quad . \tag{1}$$

The conformal weights of the $(b,c)$ system can be chosen such that their difference is exactly 1[1], here we choose $(1,0)$ as their default conformal weights [2]. The second ingredient is the vacuum representation for the $\widehat{\mathfrak{sl}}(N)_k$ Kac-Moody algebra, i.e. the vector space obtained from $|0\rangle$ by applying $\left\{J^i_n\right\}_{i=1,\ldots,M}$ ($M = \dim \mathfrak{sl}(N)$) subject to

$$J^i_n\,|0\rangle = 0, \text{ for } n > -1, \qquad [J^i_n, J^k_m] = f^{ik}_j J^j_{n+m} + \delta^{ik} k n \delta_{n,-m} \tag{2}$$

We denote this representation as $V^0_k$. Recall that $\mathfrak{sl}(N)$ has a Cartan-Weyl decomposition

$$\mathfrak{sl}(N) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{3}$$

with $\dim \mathfrak{h} = N-1$ the rank of $\mathfrak{sl}(N)$. Let $\Delta_s = \{\alpha_1, \ldots, \alpha_{N-1}\}$ be the simple roots and $\Delta_+$ be the positive roots. Accordingly there are $\dim \mathfrak{n}_+ \equiv T$ positive roots. For every positive root $A$ we introduce a $(b^A, c^A)$ ghost system and the combined Fock-module $\bigwedge \mathfrak{n}_+$ generated by applications of the modes of the ghost systems subject to

$$b^A_n\,|0\rangle = 0,\; n \geq 0, \quad c^A_n\,|0\rangle = 0,\; n \geq 1, \qquad \left\{b^A_n, c^B_m\right\} = \delta^{AB}\delta_{n,-m} \quad . \tag{4}$$

The vector space we are interested in is the following one.

$$C(\mathfrak{sl}(N))_k \equiv V^0_k \otimes \bigwedge \mathfrak{n}_+ \tag{5}$$

We will suppress the tensor product in all our formulas. Let

$$Q(z) = \sum_{\alpha\in\Delta_+} J^A(z)c^A(z) - \frac{1}{2}\sum_{A,B,C\in\Delta_+} f^{AB}_C :c^A(z)c^B(z)b^C(z): + \sum_{i=1}^{N-1} c^{\alpha_i}(z) \quad . \tag{6}$$

A) Show that the OPE $Q(z)Q(w)$ is regular for $z = w$.

[1] In string theory the conformal weights are fixed to $(2,-1)$, since they stem from integrating over different conformal structures of the world sheet.

[2] Later we deform the energy momentum tensor and conformal weights get shifted!

Thus

$$d \equiv Q_{(0)} = \oint Q(z)dz \tag{7}$$

satisfies $\{d,d\} = 0$ (note that $Q(z)$ has odd statistics) and therefore $d^2 = 0$. Having a nilpotent operator on $C(\mathfrak{sl}(N))_k$ we can take its cohomology. Unfortunately we have to refine the procedure to extract $\mathcal{W}$-algebras. For this we introduce the so called *ghost number gh* for the elements in $C(\mathfrak{sl}(N))_k$. We define

$$gh(J^A) = 0, \quad gh(b^A) = -1, \quad gh(c^A) = 1 \quad . \tag{8}$$

With these definitions we have $gh(Q) = 1$ and ghost number in $C(\mathfrak{sl}(N))_k$ is completely determined by the second factor. We denote $(C^\bullet(\mathfrak{sl}(N))_k, d)$ for the vector space when ghost number taken into account. In the following exercises we are going to show

**Theorem 1.** *Let $H_k^\bullet(\mathfrak{sl}(N))$ be the cohomology of $(C^\bullet(\mathfrak{sl}(N))_k, d)$. Then*

*1) $H_k^i(\mathfrak{sl}(N)) = 0$ for $i \neq 0$.*

*2) In ghost number 0 we get $H_k^0(\mathfrak{sl}(N)) = \mathcal{W}(2,3,4,\dots,N-1)$ where the central charge depends on the level $k$.*

This takes some effort, but after all is a nice construction. First we introduce the field

$$\bar{J}^a(z) = \sum_n \bar{J}^a_n z^{-n-1} = J^a(z) + \sum_{A,B\in\Delta_+} f_C^{aB} {:}b^C(z)c^B(z){:} \tag{9}$$

and split the differential

$$d = d_0 + \chi \tag{10}$$

$$\begin{aligned} d_0 &= \oint \left[ \sum_{A\in\Delta_+} J^A(z)c^A(z) - \frac{1}{2} \sum_{A,B,C\in\Delta_+} f_C^{AB} {:}c^A(z)c^B(z)b^C(z){:} \right] dz, \\ \chi &= \oint \sum_{i=1}^{N-1} c^{\alpha_i}(z)dz \end{aligned} \tag{11}$$

B) Show

$$\begin{aligned} \left[\chi, \bar{J}^a(z)\right] &= \sum_{i=1}^{N-1} \sum_B f_i^{a,B}\, c^B(z), \\ \{\chi, c^a(z)\} &= 0, \\ \left\{\chi, b^A(z)\right\} &= \sum_{i=1}^{N-1} \delta^{iA} \\ \left[d_0, \bar{J}^a(z)\right] &= \sum_{A,B} f_B^{Aa} {:}\bar{J}^B(z)c^A(z){:} + k \sum_A \kappa(J^a, J^A)\partial_z c^A(z) - \sum_{A,B,d} f_B^{Ad} f_d^{Ba} \partial_z c^A(z) \\ \left\{d_0, c^A(z)\right\} &= -\frac{1}{2}\sum_{B,C} f_A^{BC} c^B(z)c^C(z) \\ \left\{d_0, b^A(z)\right\} &= \bar{J}^A(z) \end{aligned} \tag{12}$$

*(Hint: When using the Wick-formula keep an eye on extra minus signs.)*

We derived he action of the differential on the state space. In the exercises on christmas sheet part 2 we compute the cohmology for the differential.