Determinants and Graßmann numbers

Determinants of operators can formally be written as (path) integrals over some auxiliary variables. In order for this to be possible, these auxiliary variables have to be anti-commuting rather than ordinary commuting numbers. Two anti-commuting numbers (or Graßmann numbers) θ and η satisfy

$$\theta\eta = -\eta\theta \tag{1.1}$$

and hence $\theta^2 = 0$. Because of this, the most general function of one Graßmann variable θ is

$$f(\theta) = A + B\theta \tag{1.2}$$

with $A, B \in \mathbb{C}$. Integrals over Graßmann variables ("Berezin integrals") are defined by

$$\int d\theta (A + B\theta) := B \ . \tag{1.3}$$

Defining the derivative

$$\frac{d}{d\theta}\theta = 1 \ , \qquad \frac{d}{d\theta}A = 0 \ , \qquad (A \in \mathbb{C}) \ , \tag{1.4}$$

the Berezin integral of a total derivative is zero and the Berezin integral is translation invariant, i.e.,

$$\begin{aligned}
\int d\theta \frac{d}{d\theta} f(\theta) &= 0 \ , && (1.5)\\
\int d\theta f(\theta + a) &= \int d\theta f(\theta) \ , \qquad \text{for } a \in \mathbb{C} \ . && (1.6)
\end{aligned}$$

Proof: Let $f(\theta) = A + B\theta$ with $A, B \in \mathbb{C}$. Then

$$\begin{aligned}
\int d\theta \frac{d}{d\theta} f(\theta) &= \int d\theta \frac{d}{d\theta}(A + B\theta) && (1.7)\\
&= \int d\theta \left(\frac{d}{d\theta}A + (\frac{d}{d\theta}B)\theta + B\frac{d}{d\theta}\theta \right) && (1.8)\\
&= \int d\theta B = 0 && (1.9)
\end{aligned}$$

Moreover, $f(\theta + a) = A + Ba + B\theta$. Thus,

$$\int d\theta f(\theta + a) = \int d\theta (A + Ba + B\theta) = B = \int d\theta f(\theta) \ . \tag{1.10}$$

The properties (1.5) and (1.6) mimic similar properties of ordinary integrals of the type $\int_{-\infty}^{\infty} dx f(x)$, which is the motivation for the unusual definition (1.3). Note that, for Graßmann variables, integration and differentiation are equivalent operations.

If one has several linearly independent Graßmann variables θ_i ($i = 1, \ldots, n$), where

$$\forall_{i,j} : \quad \theta_i\theta_j = -\theta_j\theta_i \ , \tag{1.11}$$

one defines

$$\int d\theta_1 \dots d\theta_n f(\theta_i) = c \ , \tag{1.12}$$

where c is the coefficient in front of the $\theta_n\theta_{n-1}\dots\theta_1$-term in $f(\theta_i)$ (note the order):

$$f = \dots + c\theta_n\theta_{n-1}\dots\theta_1 \ . \tag{1.13}$$

Now say we have n Graßmann variables ψ_i and n other independent Graßmann variables χ_i. Then one can easily show that

$$\int \Big(\prod_{m=1}^{n} d\psi_m d\chi_m\Big) e^{\sum_{k=1}^{n}\chi_k\lambda_k\psi_k} = \prod_{m=1}^{n}\lambda_m \ , \tag{1.14}$$

where $\lambda_m \in \mathbb{C}$ are ordinary c-numbers and the exponential is defined via its power series expansion.
Proof:

$$\begin{aligned}
\int \Big(\prod_{m=1}^{n} d\psi_m d\chi_m\Big) e^{\sum_{k=1}^{n}\chi_k\lambda_k\psi_k} &= \int d\psi_1 d\chi_1\dots d\psi_n d\chi_n \left[1+\dots+\frac{1}{n!}\Big(\sum_{k=1}^{n}\chi_k\lambda_k\psi_k\Big)^n\right] \\
&= \int d\psi_1 d\chi_1\dots d\psi_n d\chi_n \frac{1}{n!}\Big(\sum_{k=1}^{n}\chi_k\lambda_k\psi_k\Big)^n \\
&= \int d\psi_1 d\chi_1\dots d\psi_n d\chi_n (\chi_n\lambda_n\psi_n)\dots(\chi_1\lambda_1\psi_1) \\
&= \prod_{m=1}^{n}\lambda_m \ .
\end{aligned} \tag{1.15}$$

Formula (1.14) implies

$$\int \Big(\prod_{m=1}^{n} d\psi_m d\chi_m\Big) e^{\sum_{k,l=1}^{n}\chi_k\Lambda_{kl}\psi_l} = \det\Lambda \ , \tag{1.16}$$

for a symmetric $n\times n$ matrix Λ with eigenvalues λ_m.
Proof: As Λ is symmetric, it can be diagonalized via an orthogonal matrix, i.e.

$$\Lambda_{kl} = \sum_{s=1}^{n} M_{sk}M_{sl}\lambda_s \tag{1.17}$$

with λ_s the eigenvalues of Λ. The matrix M is orthogonal, i.e. $M^{-1} = M^T$ and $\mathrm{Det}M = \pm 1$. Now one has

$$\int \Big(\prod_{m=1}^{n} d\psi_m d\chi_m\Big) e^{\sum_{k,l=1}^{n}\chi_k\Lambda_{kl}\psi_l} = \int \Big(\prod_{m=1}^{n} d\psi_m d\chi_m\Big) e^{\sum_{k,l,s=1}^{n}\chi_k M_{sk}M_{sl}\lambda_s\psi_l} \ . \tag{1.18}$$

Set

$$\sum_{k=1}^{n} M_{sk}\chi_k = \phi_s \ , \quad \sum_{l=1}^{n} M_{sl}\psi_l = \theta_s \ , \tag{1.19}$$

or equivalently ($M^{-1} = M^T$)

$$\chi_k = \sum_{s=1}^{n} M_{sk}\phi_s \ , \quad \psi_l = \sum_{s=1}^{n} M_{sl}\theta_s \ . \tag{1.20}$$

Then one has

$$
\begin{aligned}
& \int \Big(\prod_{m=1}^{n} d\psi_m d\chi_m \Big) e^{\sum_{k,l=1}^{n} \chi_k \Lambda_{kl} \psi_l} \\
= \; & \int \Big(\prod_{m=1}^{n} \Big(\sum_{k,l=1}^{n} M_{km} M_{lm} \int d\theta_k d\phi_l \Big) e^{\sum_{s=1}^{n} \phi_s \lambda_s \theta_s} \\
= \; & \sum_{k_i, l_i = 1}^{n} (M_{k_1 1} M_{l_1 1}) \ldots (M_{k_n n} M_{l_n n}) \int (d\theta_{k_1} d\phi_{l_1}) \ldots (d\theta_{k_n} d\phi_{l_n}) e^{\sum_{s=1}^{n} \phi_s \lambda_s \theta_s} \\
= \; & \sum_{k_i, l_i = 1}^{n} (M_{k_1 1} M_{l_1 1}) \ldots (M_{k_n n} M_{l_n n}) \epsilon_{k_1 \ldots k_n} \epsilon_{l_1 \ldots l_n} \int (d\theta_1 d\phi_1) \ldots (d\theta_n d\phi_n) e^{\sum_{s=1}^{n} \phi_s \lambda_s \theta_s} \\
= \; & (\mathrm{Det} M)^2 \prod_{s=1}^{n} \lambda_s = \prod_{s=1}^{n} \lambda_s = \mathrm{Det} \Lambda \; . \qquad (1.21)
\end{aligned}
$$

Formula (1.14) should be compared with the result of the integral over commuting numbers α_m, β_m $(m = 1, \ldots, n)$, with $\lambda_m \in \mathbb{R}$, for which one can show

$$
\int_{-\infty}^{\infty} \Big(\prod_{m=1}^{n} d\alpha_m d\beta_m \Big) e^{2\pi i \sum_{k=1}^{n} \alpha_k \lambda_k \beta_k} = \prod_{m=1}^{n} \frac{1}{\lambda_m} \; . \qquad (1.22)
$$

This follows simply from

$$
\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta e^{i\lambda\alpha\beta} = 2\pi \int_{-\infty}^{\infty} d\alpha \delta(\lambda\alpha) = \frac{2\pi}{\lambda} \; . \qquad (1.23)
$$

The fact that one can invert the result of a Gaussian integral by replacing the commuting variables by Graßmann valued variables, carries over to path integrals. This is commonly used in QFT where fermionic path integrals are used to express the determinant of a differential operator. For instance, formula (1.16) can be generalized to the context of a path integral over Graßman valued fields $\psi(x)$, $\chi(x)$, resulting in

$$
\int \mathcal{D}\psi \mathcal{D}\chi \; e^{\int d^D x \chi \Delta \psi} = \det \Delta \; , \qquad (1.24)
$$

where Δ is some self-adjoint differential operator. This can be seen as follows. The fields $\psi(x)$ and $\chi(x)$ can be expanded in (c-number valued) eigenfunctions $\Psi_i(x)$ of Δ with Graßmann valued coefficients ψ_i and χ_i, i.e.

$$
\begin{aligned}
\psi(x) = \sum_i \psi_i \Psi_i(x) \quad &, \qquad \chi(x) = \sum_i \chi_i \Psi_i(x) \; , \\
\Delta \Psi_i(x) \quad &= \quad \lambda_i \Psi_i(x) \; . \qquad (1.25)
\end{aligned}
$$

The eigenfunctions can be chosen in an orthonormal way, i.e.

$$
\int d^D x \Psi_i(x) \Psi_j(x) = \delta_{ij} \; , \qquad (1.26)
$$

and the measure can be defined as $\mathcal{D}\psi\mathcal{D}\chi = \prod_i d\psi_i d\chi_i$.

Analogously to (1.24) one has for commuting fields ϕ_1 and ϕ_2

$$
\int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \; e^{2\pi i \int d^D x \phi_1 \Delta \phi_2} = (\det \Delta)^{-1} \; . \qquad (1.27)
$$