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Introduction to Compact Riemann Surfaces

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The theory of Riemann surfaces is a classical field of mathematics where geometry and analysis play equally important roles. The purpose of these notes is to present some basic facts of this theory to make this book more self contained. In particular we will deal with classical descriptions of Riemann surfaces, Abelian differentials, periods on Riemann surfaces, meromorphic functions, theta functions, and uniformization techniques.

Motivated by the concrete point of view on Riemann surfaces of this book we choose essentially an analytic presentation. Concrete analytic tools and constructions available on Riemann surfaces and their applications to the theory are explained in detail. Most of them are proven or accompanied with sketches of proofs. For the same reason, difficult non-constructive proofs of some classical existence results in the theory of Riemann surfaces (such as the existence of conformal coordinates, of holomorphic and Abelian differentials, of meromorphic sections of holomorphic line bundles) are omitted. The language of the geometric approach is explained in the section on holomorphic line bundles.

This chapter is based on the notes of a graduate course given at the Technische Universität Berlin. There exists a huge literature on Riemann surfaces including many excellent classical monographs. Our list [FK92, Jos06, Bos, Bea78, AS60, Gu66, Lew64, Spr81] for further reading is by no means complete.

1.1 Definition of a Riemann surface and basic examples

Let $\mathcal{R}$ be a two-dimensional real manifold, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $\mathcal{R}$, i.e., $\cup_{\alpha\in A}U_\alpha = \mathcal{R}$. A *local parameter (local coordinate, coordinate chart)* is a pair (U_α, z_α) of U_α with a homeomorphism $z_\alpha : U_\alpha \to V_\alpha$ to an open subset $V_\alpha \subset \mathbb{C}$. Two coordinate charts (U_α, z_α) and (U_β, z_β) are called *compatible* if the mapping

$$f_{\beta,\alpha} = z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta) , \qquad (1.1)$$

which is called a *transition function* is holomorphic. The local parameter (U_α, z_α) will be often identified with the mapping z_a if its domain is clear or irrelevant.

If all the local parameters $\{U_\alpha, z_\alpha\}_{\alpha \in A}$ are compatible, they form a *complex atlas* $\mathcal{A}$ of $\mathcal{R}$. Two complex atlases $\mathcal{A} = \{U_\alpha, z_\alpha\}$ and $\tilde{\mathcal{A}} = \{\tilde{U}_\beta, \tilde{z}_\beta\}$ are compatible if $\mathcal{A} \cup \tilde{\mathcal{A}}$ is a complex atlas. An equivalence class Σ of complex atlases is called a *complex structure*. It can be identified with a maximal atlas $\mathcal{A}^*$, which consists of all coordinate charts, compatible with an atlas $\mathcal{A} \subset \Sigma$.

Definition 1. *A Riemann surface is a connected one-dimensional complex analytic manifold, that is, a connected two-dimensional real manifold $\mathcal{R}$ with a complex structure Σ on it.*

When it is clear which complex structure is considered, we use the notation $\mathcal{R}$ for the Riemann surface.

If $\{U, z\}$ is a coordinate on $\mathcal{R}$ then for every open set $V \subset U$ and every function $f : \mathbb{C} \to \mathbb{C}$, which is holomorphic and bijective on $z(V)$, $\{V, f \circ z\}$ is also a local parameter on $\mathcal{R}$.

The coordinate charts establish homeomorphisms of domains in $\mathcal{R}$ with domains in $\mathbb{C}$. This means that locally the Riemann surface is just a domain in $\mathbb{C}$. But for any point $P \in \mathcal{R}$ there are many possible choices of these homeomorphisms. Therefore one can associate to $\mathcal{R}$ only the notions from the theory of analytic functions in $\mathbb{C}$ that are invariant with respect to biholomorphic maps, i.e. those that one can define without choosing a specific local parameter. For example, one can talk about the angle between two smooth curves γ and $\tilde{\gamma}$ on $\mathcal{R}$ intersecting at some point $P \in \mathcal{R}$. This angle is equal to the one between the curves $z(\gamma)$ and $z(\tilde{\gamma})$ that lie in $\mathbb{C}$ and intersect at the point $z(P)$, where z is some local parameter at P. This definition is invariant with respect to the choice of z.

If $(\mathcal{R}, \Sigma)$ is a Riemann surface, then the manifold $\mathcal{R}$ is oriented.

The simplest examples of Riemann surfaces are any domain (connected open subset) $U \subset \mathbb{C}$ in the complex plane, the whole complex plane $\mathbb{C}$, and the extended complex plane (or *Riemann sphere*) $\hat{\mathbb{C}} = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The complex structures on U and $\mathbb{C}$ are defined by single coordinate charts (U, id) and $(\mathbb{C}, id)$. The extended complex plane is the simplest compact Riemann surface. To define the complex structure on it we use two charts $(U_1, z_2), (U_2, z_2)$ with

$$U_1 = \mathbb{C} , \qquad z_1 = z ,$$
$$U_2 = (\mathbb{C} \backslash \{0\}) \cup \{\infty\} , \qquad z_2 = 1/z .$$

The transition functions

$$f_{1,2} = z_1 \circ z_2^{-1} , \quad f_{2,1} = z_2 \circ z_1^{-1} \ : \mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{0\}$$

are holomorphic

$$f_{1,2}(z) = f_{2,1}(z) = 1/z \ .$$

To a large extent the beauty of the theory of Riemann surfaces is due to the fact that Riemann surfaces can be described in many completely different ways. Interrelations between these descriptions make up an essential part of the theory. The basic examples of Riemann surfaces we are going to discuss now are exactly these foundations the whole theory is based on.

1.1.1 Non-singular algebraic curves

Definition 2. *An algebraic curve C is a subset in $\mathbb{C}^2$*

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 \mid \mathcal{P}(\mu, \lambda) = 0\} \ , \tag{1.2}$$

where $\mathcal{P}$ is an irreducible polynomial in λ and μ

$$\mathcal{P}(\mu, \lambda) = \sum_{i=0}^{N} \sum_{j=0}^{M} p_{ij} \mu^i \lambda^j \ .$$

The curve C is called non-singular if

$$\mathrm{grad}_{\mathbb{C}} \mathcal{P}_{|\mathcal{P}=0} = \left(\frac{\partial \mathcal{P}}{\partial \mu}, \frac{\partial \mathcal{P}}{\partial \lambda} \right)_{|\mathcal{P}(\mu,\lambda)=0} \tag{1.3}$$

is nowhere zero on $\mathbb{C}$.

The complex structure on C is defined as follows: the variable λ is taken as local parameter in the neighborhoods of the points where $\partial\mathcal{P}/\partial\mu \neq 0$, and the variable μ is taken as local parameter near the points where $\partial\mathcal{P}/\partial\lambda \neq 0$. The holomorphic compatibility of the introduced local parameters results from the complex version of the implicit function theorem.

The Riemann surface C can be made a compact Riemann surface

$$\hat{C} = C \cup \{\infty^{(1)}\} \cup \ldots \cup \{\infty^{N}\}$$

by adjoining points $\infty^{(1)}, \ldots, \infty^{(N)}$ at infinity ($\lambda \to \infty, \mu \to \infty$), and introducing admissible local parameters at these points, see Fig. 1.1.

Definition 3. *Let $\mathcal{R}$ be a Riemann surface such that there exists an open subset*

$$U_\infty = U_\infty^{(1)} \cup \ldots \cup U_\infty^{(N)} \subset \mathcal{R}$$

such that $\mathcal{R} \backslash U_\infty$ is compact and $U_\infty^{(n)}$ are homeomorphic to punctured discs

$$z_n : U_\infty^{(n)} \to D\backslash\{0\} = \{z \in \mathbb{C} \mid 0 < |z| < 1\} \ ,$$

where the homeomorphisms z_n are holomorphically compatible with the complex structure of $\mathcal{R}$. Then $\mathcal{R}$ is called a compact Riemann surface with punctures.

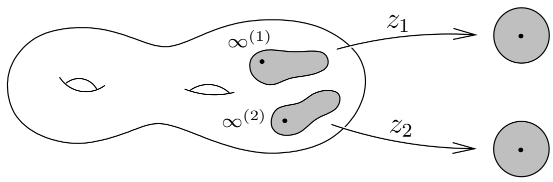

Fig. 1.1. A compact Riemann surface with punctures.

Let us extend the homeomorphisms z_n

$$z_n : \hat{U}_\infty^{(n)} = U_\infty^{(n)} \cup \{\infty^{(n)}\} \to D = \{z \mid |z| < 1\} , \tag{1.4}$$

by setting $z_n(\infty^{(n)}) = 0, \quad n = 1, \ldots, N$. A complex atlas for a new Riemann surface

$$\hat{\mathcal{R}} = \mathcal{R} \cup \{\infty^{(1)}\} \cup \ldots \cup \{\infty^{(N)}\}$$

is defined as a union of a complex atlas $\mathcal{A}$ of $\mathcal{R}$ with the coordinate charts (1.4) compatible with $\mathcal{A}$ due to Definition 3. The Riemann surface $\hat{\mathcal{R}}$ is called the *compactification* of the punctured Riemann surface $\mathcal{R}$.

Hyperelliptic curves.

Let us consider the important special case of hyperelliptic curves

$$\mu^2 = \prod_{j=1}^{N} (\lambda - \lambda_j) , \quad N \geq 3 , \quad \lambda_j \in \mathbb{C} . \tag{1.5}$$

When $N = 3$ or 4 the curve (1.5) is called elliptic. The curve is non-singular if all the points λ_j are different

$$\lambda_j \neq \lambda_i , \quad i, j = 1, \ldots, N .$$

In this case the choice of local parameters can be additionally specified. Namely, in the neighborhood of the points (μ_0, λ_0) with $\lambda_0 \neq \lambda_j \quad \forall j$, the local parameter is the homeomorphism

$$(\mu, \lambda) \to \lambda . \tag{1.6}$$

In the neighborhood of each point $(0, \lambda_j)$ it is defined by the homeomorphism

$$(\mu, \lambda) \to \sqrt{\lambda - \lambda_j} . \tag{1.7}$$

For odd $N = 2g + 1$, the curve (1.5) has one puncture ∞

$$P \to \infty \Longleftrightarrow \lambda \to \infty ,$$

and a local parameter in its neighborhood is given by the homeomorphism

$$z_\infty : (\mu, \lambda) \to \frac{1}{\sqrt{\lambda}} \, . \tag{1.8}$$

For even $N = 2g + 2$ there are two punctures $\infty^\pm$ distinguished by the condition

$$P \to \infty^\pm \Longleftrightarrow \frac{\mu}{\lambda^{g+1}} \to \pm 1 \, , \qquad \lambda \to \infty \, ,$$

and the local parameters in the neighborhood of both points are given by the homeomorphism

$$z_{\infty^\pm} : (\mu, \lambda) \to \lambda^{-1} \, . \tag{1.9}$$

Theorem 1. *The local parameters (1.6, 1.7, 1.8, 1.9) describe a compact Riemann surface*

$$\hat{C} = C \cup \{\infty\} \qquad if\ N\ is\ odd \, ,$$
$$\hat{C} = C \cup \{\infty^\pm\} \qquad if\ N\ is\ even \, ,$$

of the hyperelliptic curve (1.5).

One prefers to consider compact Riemann surfaces and thus the compactification $\hat{C}$ is called *the Riemann surface of the curve* C.

It turns out that all compact Riemann surfaces can be described as compactifications of algebraic curves (see for example [Jos06]).

1.1.2 Quotients under group actions

Definition 4. *Let* Δ *be a domain in* $\mathbb{C}$*. A group* $G : \Delta \to \Delta$ *of holomorphic transformations acts discontinuously on* Δ *if for any* $P \in \Delta$ *there exists a neighborhood* $V \ni P$ *such that*

$$gV \cap V = \emptyset \, , \qquad \forall g \in G \, , \qquad g \neq I \, . \tag{1.10}$$

The quotient space Δ/G is defined by the equivalence relation

$$P \sim P' \Leftrightarrow \exists g \in G : P' = gP \, .$$

By the natural projection $\pi : \Delta \to \Delta/G$ every point is mapped to its equivalence class. Every point $P \in \Delta$ has a neighborhood V satisfying (1.10). Then $U = \pi(V)$ is open and $\pi_{|V} : V \to U$ is a homeomorphism. Its inversion $z : U \to V \subset \Delta \subset \mathbb{C}$ is a local parameter. One can cover Δ/G by domains of this type. The transition functions are the corresponding group elements g; therefore they are holomorphic.

Theorem 2. *Δ/G is a Riemann surface.*

Tori

Let us consider the case $\Delta = \mathbb{C}$ and the group G generated by two translations

$$z \to z + w\ , \qquad z \to z + w'\ ,$$

where $w, w' \in \mathbb{C}$ are two non-parallel vectors, $\mathrm{Im}\ w'/w \neq 0$, see Fig. 1.2. The group G is commutative and consists of the elements

$$g_{n,m}(z) = z + nw + mw'\ , \qquad n, m \in \mathbb{Z}\ . \tag{1.11}$$

The factor $\mathbb{C}/G$ has a nice geometrical realization as the parallelogram

$$T = \{z \in \mathbb{C} \mid z = aw + bw',\ a, b \in [0, 1)\}\ .$$

There are no G-equivalent points in T and on the other hand every point in $\mathbb{C}$ is equivalent to some point in T. Since the edges of the parallelogram T are G-equivalent $z \sim z + w,\ z \sim z + w'$, $\mathcal{R}$ is a compact Riemann surface, which is topologically a torus. We discuss this case in more detail in Sect. 1.5.5.

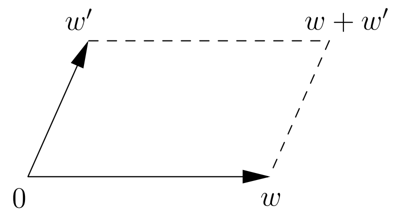

Fig. 1.2. A complex torus

The uniformization theorem (see for example [Jos06]) claims that all compact Riemann surfaces can be obtained as quotients Δ/G.

1.1.3 Polyhedral surfaces as Riemann surfaces

One can build a Riemann surface gluing together pieces of the complex plane $\mathbb{C}$.

Consider a finite set of disjoint polygons F_i and identify isometrically pairs of edges in such a way that the result is a compact oriented polyhedral surface $\mathcal{P}$. A polyhedron in 3-dimensional Euclidean space is an example of such a surface.

Theorem 3. *The polyhedral surface $\mathcal{P}$ is a Riemann surface.*

In order to define a complex structure on a polyhedral surface let us distinguish three kinds of points (see Fig. 1.3):

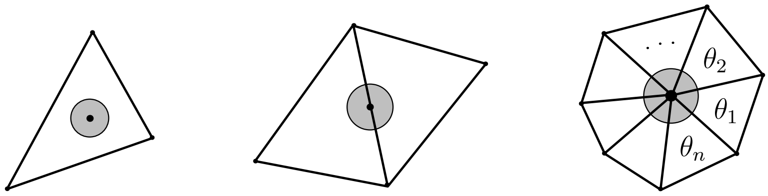

Fig. 1.3. Three kinds of points on a polyhedral surface

1. inner points of triangles,
2. inner points of edges,
3. vertices.

One can map isometrically the corresponding polygon F_i (or pairs of neighboring polygons) into $\mathbb{C}$. This provides local parameters at the points of the first and the second kind. Let P be a vertex and $F_i, \ldots, F_n$ the sequence of successive polygons with this vertex (see the point (iii) above). Denote by θ_i the angle of F_i at P. Then define

$$\gamma = \frac{2\pi}{\sum_{i=1}^{n} \theta_i} .$$

Consider a suitably small ball neighborhood of P, which is the union $U^r = \cup_i F_i^r$, where $F_i^r = \{Q \in F_i \mid \mid Q - P \mid < r\}$. Each F_i^r is a sector with angle θ_i at P. We map it as above into $\mathbb{C}$ with P mapped to the origin and then apply $z \mapsto z^\gamma$, which produces a sector with the angle $\gamma\theta_i$. The mappings corresponding to different polygons F_i can be adjusted to provide a homeomorphism of U^r onto a disc in $\mathbb{C}$. All transition functions of the constructed charts are holomorphic since they are compositions of maps of the form $z \mapsto az + b$ and $z \mapsto z^\gamma$ (away from the origin).

It turns out that any compact Riemann surface can be recovered from some polyhedral surface [Bos].

1.1.4 Complex structure generated by the metric

There is a smooth version of the previous construction. Let $(\mathcal{R}, g)$ be a two-real dimensional orientable differential manifold with Riemannian metric g.

Definition 5. *Two metrics g and $\tilde{g}$ are called conformally equivalent if they differ by a function on $\mathcal{R}$*

$$g \sim \tilde{g} \Leftrightarrow g = f\tilde{g} , \qquad f : \mathcal{R} \to \mathbb{R}_+ . \tag{1.12}$$

The transformation (1.12) preserves angles. This relation defines classes of conformally equivalent metrics.

Let $(x, y) : U \subset \mathcal{R} \to \mathbb{R}^2$ be a local coordinate. In terms of the complex variable $z = x + \mathrm{i}y$ the metric can be written as

$$g = A\mathrm{d}z^2 + 2B\mathrm{d}z\mathrm{d}\bar{z} + \bar{A}\mathrm{d}\bar{z}^2, \qquad A \in \mathbb{C}, B \in \mathbb{R}, B > |A| \,. \tag{1.13}$$

Note that the complex coordinate z is *not* compatible with the complex structure we will define on $\mathcal{R}$ with the help of g.

Definition 6. *A coordinate $w : U \to \mathbb{C}$ is called conformal if the metric in this coordinate is of the form*

$$g = \mathrm{e}^{\phi}\mathrm{d}w, \mathrm{d}\bar{w} \,, \tag{1.14}$$

i.e., it is conformally equivalent to the standard metric $\mathrm{d}w\mathrm{d}\bar{w}$ *of* $\mathbb{R}^2 = \mathbb{C}$.

If $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is an immersed surface in $\mathbb{R}^3$ then the first fundamental form $< dF, dF >$ induces a metric on U. When the standard coordinate (x, y) of $\mathbb{R}^2 \supset U$ is conformal, the parameter lines

$$F(x, \Delta m) \,, \qquad F(\Delta n, y) \,, \qquad x, y \in \mathbb{R} \,, \qquad n, m \in \mathbb{Z} \,, \qquad \Delta \to 0$$

comprise an infinitesimal square net on the surface.

It is easy to show that every compact Riemann surface admits a conformal Riemannian metric. Indeed, each point $P \in \mathcal{R}$ possesses a local parameter $z_P : U_P \to D_P \subset \mathbb{C}$, where D_P is a small open disc. Since $\mathcal{R}$ is compact there exists a finite covering $\cup_{i=1}^{n} U_{P_i} = \mathcal{R}$. For each i choose a smooth function $m_i : D_{P_i} \to \mathbb{R}$ with

$$m_i > 0 \quad \text{on } D_i \,, \qquad m_i = 0 \quad \text{on } \mathbb{C} \setminus D_i \,.$$

$m_i(z_{P_i})\mathrm{d}z_{P_i}\mathrm{d}\bar{z}_{P_i}$ is a conformal metric on U_{P_i}. The sum of these metrics over $i = 1, \ldots, n$ yields a conformal metric on $\mathcal{R}$.

Moreover, any metric can be brought to conformal form (1.14) due to the following fundamental theorem.

Theorem 4. *Conformal equivalence classes of metrics on an orientable two-manifold $\mathcal{R}$ are in one to one correspondence with the complex structures on $\mathcal{R}$.*

Let us show how one finds conformal coordinates. The metric (1.13) can be written as follows (we suppose $A \neq 0$)

$$g = s(\mathrm{d}z + \mu\mathrm{d}\bar{z})(\mathrm{d}\bar{z} + \bar{\mu}\mathrm{d}z) \,, \qquad s > 0 \,, \tag{1.15}$$

where

$$\mu = \frac{\bar{A}}{2B}(1 + |\mu|^2) \,, \qquad s = \frac{2B}{1 + |\mu|^2} \,.$$

Here $|\mu|$ is a solution of the quadratic equation

$$|\mu| + \frac{1}{|\mu|} = \frac{2B}{|A|} \,,$$

which can be chosen $|\mu| < 1$. Comparing (1.15) and (1.14) we get

$$\mathrm{d}w = \lambda(\mathrm{d}z + \mu \mathrm{d}\bar{z})$$

or

$$\mathrm{d}w = \lambda(\mathrm{d}\bar{z} + \bar{\mu}\mathrm{d}z).$$

In the first case the map $w(z, \bar{z})$ satisfies the equation

$$w_{\bar{z}} = \mu w_z \tag{1.16}$$

and preserves the orientation $w :\supset \mathbb{C}U \to V \subset \mathbb{C}$ since $|\mu| < 1$. In the second case $w : U \to V$ inverses the orientation.

Equation (1.16) is called the *Beltrami equation* and $\mu(z, \bar{z})$ is called the *Beltrami coefficient.*

By analytic methods (see for example [Spi79]) one can prove that for any Beltrami coefficient μ there exists a local solution to the Beltrami equation in the corresponding functional class. This allows us to introduce local conformal coordinates.

Proposition 1. *Let $\mathcal{R}$ be a two-dimensional orientable manifold with a metric g and a positively oriented atlas $((x_\alpha, y_\alpha) : U_\alpha \to \mathbb{R}^2)_{\alpha \in A}$ on $\mathcal{R}$. Let $(x, y) : U \subset \mathcal{R} \to \mathbb{R}^2$ be one of these coordinate charts around a point $P \in U$, let $z = x + \mathrm{i}y$ and $\mu(z, \bar{z})$ be the Beltrami coefficient and let $w_\beta(z, \bar{z})$ be a solution to the Beltrami equation (1.16) in a neighborhood $V_\beta \subset V = z(U)$ with $P \in U_\beta = z^{-1}(V_\beta)$. Then the coordinate w_β is conformal and the atlas $(w_\beta : U_\beta \to \mathbb{C})_{\beta \in B}$ defines a complex structure on $\mathcal{R}$.*

Only the holomorphicity of the transition function may require a comment. Let $w : U \to \mathbb{C}, \tilde{w} : \tilde{U} \to \mathbb{C}$ be two local parameters with a non-empty intersection $U \cap \tilde{U} \neq \emptyset$. Both coordinates are conformal

$$g = \mathrm{e}^{\phi} \mathrm{d}w \, \mathrm{d}\bar{w} = \mathrm{e}^{\tilde{\phi}} \mathrm{d}\tilde{w} \, \mathrm{d}\bar{\tilde{w}},$$

which happens in one of the two cases

$$\frac{\partial \tilde{w}}{\partial \bar{w}} = 0 \quad \text{or} \quad \frac{\partial \tilde{w}}{\partial w} = 0 \tag{1.17}$$

only. The transition function $\tilde{w}(w)$ is holomorphic and not antiholomorphic since the map $w \to \tilde{w}$ preserves orientation.

Repeating these arguments one observes that conformally equivalent metrics generate the same complex structure, and Theorem 4 follows.

1.2 Holomorphic mappings

Definition 7. *A mapping*

$$f : M \to N$$

between Riemann surfaces is called holomorphic if for every local parameter (U, z) *on* M *and every local parameter* (V, w) *on* N *with* $U \cap f^{-1}(V) \neq \emptyset$*, the mapping*

$$w \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \to w(V)$$

is holomorphic.

A holomorphic mapping to $\mathbb{C}$ is called a *holomorphic function*, a holomorphic mapping to $\hat{\mathbb{C}}$ is called a *meromorphic function*.

The following lemma characterizes the local behavior of holomorphic mappings.

Lemma 1. *Let* $f : M \to N$ *be a holomorphic mapping. Then for any* $a \in M$ *there exist* $k \in \mathbb{N}$ *and local parameters* $(U, z), (V, w)$ *such that* $a \in U, f(a) \in V$ *and* $F = w \circ f \circ z^{-1} : z(U) \to w(V)$ *equals*

$$F(z) = z^k. \tag{1.18}$$

Corollary 1. *Let* $f : M \to N$ *be a non-constant holomorphic mapping, then* f *is open, i.e., the image of an open set is open.*

If M is compact then $f(M)$ is compact as a continuous image of a compact set and open due to the previous claim. This implies that in this case the corresponding non-constant holomorphic mapping is surjective and its image $N = f(M)$ compact.

We see that there exist no non-constant holomorphic mappings $f : M \to \mathbb{C}$, which is the issue of the classical Liouville theorem

Theorem 5. *On a compact Riemann surface there exists no non-constant holomorphic function.*

Non-constant holomorphic mappings of Riemann surfaces $f : M \to N$ are *discrete*: for any point $P \in N$ the set $S_P = f^{-1}(P)$ is discrete, i.e. for any point $a \in M$ there is a neighborhood $V \subset M$ intersecting with S_P in at most one point, $|V \cap S_P| \leq 1$. Non-discreteness of S for a holomorphic mapping would imply the existence of a limiting point in S_P and finally $f = \text{const}, f : M \to P \in N$. Non-constant holomorphic mappings of Riemann surfaces are also called *holomorphic coverings*.

Definition 8. *Let* $f : M \to N$ *be a holomorphic covering. A point* $P \in M$ *is called a* branch point *of* f *if it has no neighborhood* $V \ni P$ *such that* $f\big|_V$ *is injective. A covering without branch points is called* unramified *(*ramified*covering!ramified or* branched covering *in the opposite case).*

Note that various definitions of a covering are used in the literature (see for example [Ber57, Jos06, Bea78]). In particular, often the term “covering” is used for unramified coverings of our definition. Ramified coverings are important in the theory of Riemann surfaces.

The number $k \in \mathbb{N}$ in Lemma 1 can be described in topological terms. There exist neighborhoods $U \ni a, V \ni f(a)$ such that for any $Q \in V \setminus \{f(a)\}$ the set $f^{-1}(Q) \cap U$ consists of k points. One says that f has the *multiplicity* k at a. Lemma 1 allows us to characterize the branch points of a holomorphic covering $f : M \to N$ as the points with the multiplicity $k > 1$. Equivalently, P is a branch point of the covering $f : M \to N$ if

$$\left. \frac{\partial (w \circ f \circ z^{-1})}{\partial z} \right|_{z(P)} = 0 \,, \tag{1.19}$$

where z and w are local parameters at P and $f(P)$ respectively. Due to the chain rule this condition is independent of the choice of the local parameters. The number $b_f(P) = k - 1$ is called the *branch number* of f at $P \in M$. The next lemma follows immediately from Lemma 1.

Lemma 2. *Let $f : M \to N$ be a holomorphic covering. Then the set of branch points*

$$B = \{P \in M \mid b_f(P) > 0\}$$

is discrete. If M is compact, then B is finite.

The projection $A = f(B)$ of the set of branch points is also finite. The number m of preimages for any point in $N \setminus A$ is the same since any two points $Q_1, Q_2 \in N \setminus A$ can be connected by a curve $l \subset N \setminus A$. Combined with the topological characterization of the branch numbers this fact implies the following theorem.

Theorem 6. *Let $f : M \to N$ be a non-constant holomorphic mapping between two compact Riemann surfaces. Then there is a number $m \in \mathbb{N}$ such that f takes every value $Q \in N$ precisely m times, counting multiplicities. That is, for all $Q \in N$*

$$\sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m \,. \tag{1.20}$$

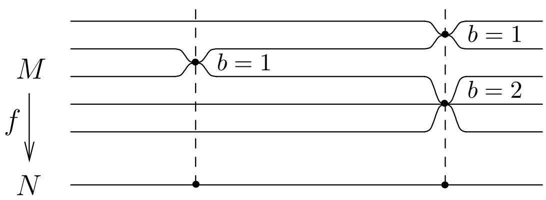

Fig. 1.4. Covering

Definition 9. *The number m above is called the* degree *of f, and the covering $f : M \to N$ is called* m-sheeted.

Applying Theorem 6 to holomorphic mappings $f : \mathcal{R} \to \hat{\mathbb{C}}$ we get

Corollary 2. *A non-constant meromorphic function on a compact Riemann surface assumes every value m times, where m is the number of its poles (counting multiplicities).*

1.2.1 Algebraic curves as coverings

Let C be a non-singular algebraic curve (1.2) and $\hat{C}$ its compactification. The map

$$(\mu, \lambda) \to \lambda \tag{1.21}$$

is a holomorphic covering $\hat{C} \to \hat{\mathbb{C}}$. If N is the degree of the polynomial $\mathcal{P}(\mu, \lambda)$ in μ

$$\mathcal{P}(\mu, \lambda) = \mu^N p_N(\lambda) + \mu^{N-1} p_{N-1}(\lambda) + \ldots + p_0(\lambda) ,$$

where all $p_i(\lambda)$ are polynomials, then $\lambda : \hat{C} \to \hat{\mathbb{C}}$ is an N-sheeted covering, see Fig. 1.4.

The points with $\partial\mathcal{P}/\partial\mu = 0$ are the branch points of the covering $\lambda : C \to \mathbb{C}$. At these points $\partial\mathcal{P}/\partial\lambda \neq 0$, and μ is a local parameter. The derivative of λ with respect to the local parameter vanishes

$$\frac{\partial\lambda}{\partial\mu} = -\frac{\partial\mathcal{P}/\partial\mu}{\partial\mathcal{P}/\partial\lambda} = 0 ,$$

which characterizes (1.19) the branch points of the covering (1.21). In the same way the map $(\mu, \lambda) \mapsto \mu$ is a holomorphic covering of the μ-plane. The branch points of this covering are the points with $\partial\mathcal{P}/\partial\lambda = 0$.

Hyperelliptic curves

Before we consider the hyperelliptic case let us recall a conventional description of the Riemann surface of the function $\mu = \sqrt{\lambda}$. One takes two copies of the complex plane $\mathbb{C}$ with cuts $[0, \infty]$ and glues them together crosswise along this cut (see Fig. 1.5). The image in Fig. 1.5 visualizes the points of the curve

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 \mid \mu^2 = \lambda\} ,$$

and the point $\lambda = 0$ gives an idea of a branch point.

The compactification $\hat{C}$ of the hyperelliptic curve

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 \mid \mu^2 = \prod_{i=1}^{N}(\lambda - \lambda_i)\} \tag{1.22}$$

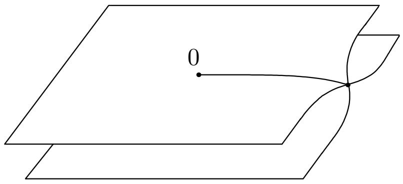

Fig. 1.5. Riemann surface of $\sqrt{\lambda}$

is a two sheeted covering of the extended complex plane $\lambda : \hat{C} \to \hat{\mathbb{C}}$. The branch points of this covering are

$$\begin{aligned} &(0, \lambda_i),\ i = 1, \dots, N \text{ and } \infty \text{ for } N = 2g+1\ , \\ &(0, \lambda_i),\ i = 1, \dots, N \qquad\qquad \text{for } N = 2g+2\ , \end{aligned}$$

with the branch numbers $b_\lambda = 1$ at these points. Only the branching at $\lambda = \infty$ possibly needs some clarification. The local parameter at $\infty \in \hat{\mathbb{C}}$ is $1/\lambda$, whereas the local parameter at the point $\infty \in \hat{C}$ of the curve $\hat{C}$ with $N = 2g+1$ is $1/\sqrt{\lambda}$ due to (1.8). In these coordinates the covering mapping reads as (compare with (1.18))

$$\frac{1}{\lambda} = \left(\frac{1}{\sqrt{\lambda}}\right)^2 ,$$

which shows that $b_\lambda(\infty) = 1$.

One can imagine the Riemann surface $\hat{C}$ with $N = 2g+2$ as two Riemann spheres with the cuts

$$[\lambda_1, \lambda_2], [\lambda_3, \lambda_4], \dots, [\lambda_{2g+1}, \lambda_{2g+2}]$$

glued together crosswise along the cuts. Fig. 1.6 presents a topological image of this Riemann surface. The image in Fig. 1.7 shows the Riemann surface "from above" or "the first" sheet on the covering $\lambda : C \to \mathbb{C}$.

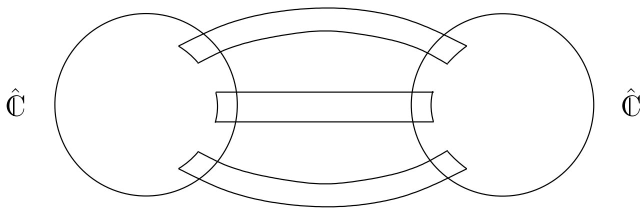

Fig. 1.6. Topological image of a hyperelliptic surface

Hyperelliptic curves possess a holomorphic involution

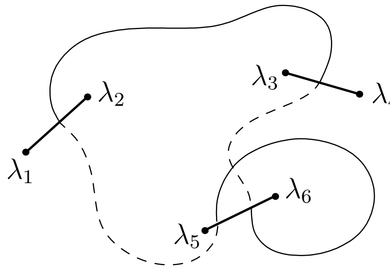

Fig. 1.7. Hyperelliptic surface C as a two-sheeted cover. The parts of the curves on C that lie on the second sheet are indicated by dotted lines.

$$h : (\mu, \lambda) \to (-\mu, \lambda) , \tag{1.23}$$

which interchanges the sheets of the covering $\lambda : \hat{C} \to \hat{\mathbb{C}}$. It is called the *hyperelliptic involution*. The branch points of the covering are the fixed points of h.

The cuts in Fig. 1.7 are conventional and belong to the image shown in Fig. 1.7 and not to the hyperelliptic Riemann surface itself, which is determined by its branch points alone. In particular, the two images shown in Fig. 1.8 correspond to the same Riemann surface and to the same covering $(\mu, \lambda) \to \lambda$.

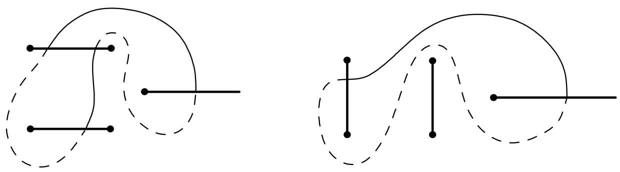

Fig. 1.8. Two equivalent images of the same hyperelliptic Riemann surface

1.2.2 Symmetric Riemann surfaces as coverings

The construction of Sect. 1.1.2 can be also applied to Riemann surfaces.

Theorem 7. *Let $\mathcal{R}$ be a (compact) Riemann surface and let G be a finite group of holomorphic automorphisms*[1] *of order $|G|$. Then $\mathcal{R}/G$ is a Riemann surface with the complex structure determined by the condition that the canonical projection*

$$\pi : \mathcal{R} \to \mathcal{R}/G$$

[1] This group is always finite if the genus ≥ 2.

is holomorphic. This is an $|G|$-sheeted covering, ramified at the fixed points of G.

The canonical projection π defines an $|G|$-sheeted covering. Denote by

$$G_{P_0} = \{g \in G \mid gP_0 = P_0\}$$

the *stabilizer* of P_0. It is always possible to choose a neighborhood U of P_0, which is invariant with respect to all elements of G_{P_0} and such that $U \cap gU = \emptyset$ for all $g \in G \setminus G_{P_0}$. Let us normalize the local parameter z on U by $z(P_0) = 0$. The local parameter w in $\pi(U)$, which is $|G_{P_0}|$-sheetedly covered by U is defined as the product of the values of the local parameter z at all equivalent points lying in U. In terms of the local parameter z all the elements of the stabilizer are represented by the functions $\tilde{g} = z \circ g \circ z^{-1} : z(U) \to z(U)$, which vanish at $z = 0$. Since $\tilde{g}(z)$ are also invertible they can be represented as $\tilde{g}(z) = zh_g(z)$ with $h_g(0) \neq 0$. Finally the $w - z$ coordinate charts representation of π

$$w \circ \pi \circ z^{-1} : z \to z^{|G_{P_0}|} \prod_{g \in G_{P_0}} h_g(z)$$

shows that the branch number of P_0 is $|G_{P_0}|$.

The compact Riemann surface $\hat{C}$ of the hyperelliptic curve

$$\mu^2 = \prod_{n=1}^{2N} (\lambda^2 - \lambda_n^2) \, , \qquad \lambda_i^2 \neq \lambda_j^2, \ \lambda_k \neq 0 \tag{1.24}$$

has the following group of holomorphic automorphisms

$$\begin{aligned} h &: (\mu, \lambda) \to (-\mu, \lambda) \\ i_1 &: (\mu, \lambda) \to (\mu, -\lambda) \\ i_2 = hi_1 &: (\mu, \lambda) \to (-\mu, -\lambda) \, . \end{aligned}$$

The hyperelliptic involution h interchanges the sheets of the covering $\lambda : \hat{C} \to \hat{\mathbb{C}}$, therefore the factor $\hat{C}/h$ is the Riemann sphere. The covering

$$\hat{C} \to \hat{C}/h = \hat{\mathbb{C}}$$

is ramified at all the points $\lambda = \pm\lambda_n$.

The involution i_1 has four fixed points on $\hat{C}$: two points with $\lambda = 0$ and two points with $\lambda = \infty$. The covering

$$\hat{C} \to \hat{C}_1 = \hat{C}/i_1 \tag{1.25}$$

is ramified at these points. The mapping (1.25) is given by

$$(\mu, \lambda) \to (\mu, \Lambda) \, , \qquad \Lambda = \lambda^2 \, ,$$

and $\hat{C}_1$ is the Riemann surface of the curve

$$\mu^2 = \prod_{n=1}^{2N} (\Lambda - \lambda_n^2) \ .$$

The involution i_2 has no fixed points. The covering

$$\hat{C} \to \hat{C}_2 = \hat{C}/i_2 \tag{1.26}$$

is unramified. The mapping (1.26) is given by

$$(\mu, \lambda) \to (M, \Lambda) \ , \qquad M = \mu\lambda, \ \Lambda = \lambda^2 \ ,$$

and $\hat{C}_2$ is the Riemann surface of the curve

$$M^2 = \Lambda \prod_{n=1}^{2N} (\Lambda - \lambda_n^2) \ .$$

1.3 Topology of Riemann surfaces

1.3.1 Spheres with handles

We have seen in Sect. 1.1 that any Riemann surface is a two-dimensional orientable real manifold. In this section we present basic facts about the topology of these manifolds focusing on the compact case. We start with an intuitive fundamental classification theorem.

Theorem 8. (and Definition) *Every compact Riemann surface is homeomorphic to a sphere with handles (i.e., a topological manifold homeomorphic to a sphere with handles in Euclidean 3-space). The number $g \in \mathbb{N}$ of handles is called the* genus *of $\mathcal{R}$. Two manifolds with different genera are not homeomorphic.*

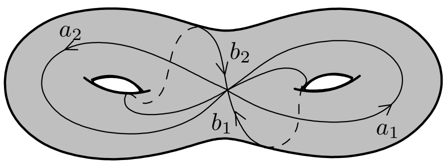

Fig. 1.9. A sphere with 2 handles

The genus of the compactification $\hat{C}$ of the hyperelliptic curve (1.22) with $N = 2g + 1$ or $N = 2g + 2$ is equal to g.

For many purposes it is convenient to use planar images of spheres with handles.

Proposition 2. *Let Π_g be a sphere $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ with $2g$ holes bounded by the non-intersecting curves*

$$\gamma_1, \gamma_1', \ldots, \gamma_g, \gamma_g' \,. \tag{1.27}$$

and identify the curves $\gamma_i \approx \gamma_i'$, $i = 1, \ldots, g$ in such a way that the orientations of these curves with respect to Π_g are opposite (see Fig. 1.10). Then Π_g is homeomorphic to a sphere with g handles.

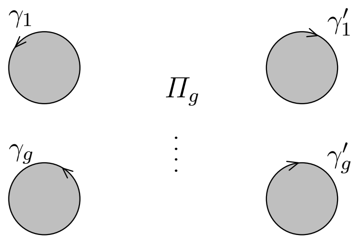

Fig. 1.10. A planar image of a sphere with g handles

To check this claim one should cut up all the handles of a sphere with g handles.

A normalized simply-connected image of a sphere with g handles is described in the following proposition.

Proposition 3. *Let F_g be a 4g-gon with the edges*

$$a_1, b_1, a_1', b_1', \ldots, a_g, b_g, a_g', b_g' \,, \tag{1.28}$$

listed in the order of traversing the boundary of F_g and the boundary curves

$$a_i \approx a_i', b_i \approx b_i',\ i = 1, \ldots, g$$

are identified in such a way that the orientations of the edges a_i and a_i' as well as b_i and b_i' with respect to F_g are opposite (see Fig. 1.11). Then F_g is homeomorphic to a sphere with g handles. The sphere without handles ($g = 0$) is homeomorphic to a 2-gon with the edges identified.

This claim is visualized in Figs. 1.12, 1.13. One choice of closed curves $a_1, b_1, \ldots, a_g, b_g$ on a sphere with handles is shown in Fig. 1.9.

Let us consider *a triangulation* $\mathcal{T}$ of $\mathcal{R}$, i.e., a set $\{T_i\}$ of topological triangles, which are glued along their edges (the identification of vertices or edges of individual triangles is not excluded), and which comprise $\mathcal{R}$. More generally, one can consider cell decompositions of $\mathcal{R}$ into topological polygons $\{T_i\}$. Obviously, compact Riemann surfaces possess finite triangulations.

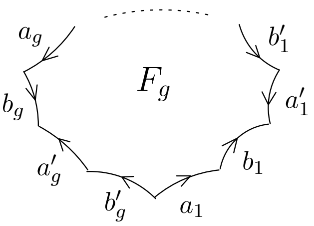

Fig. 1.11. Simply-connected image of a sphere with g handles

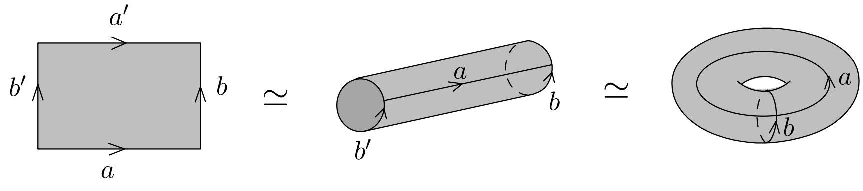

Fig. 1.12. Gluing a torus

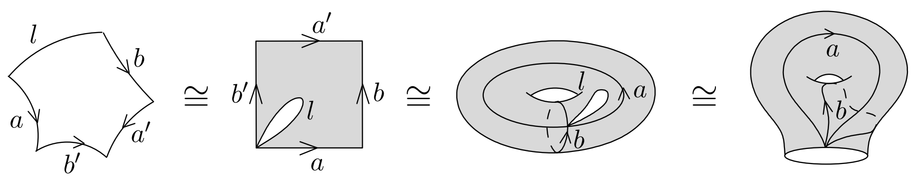

Fig. 1.13. Gluing a handle

Definition 10. *Let $\mathcal{T}$ be a triangulation of a compact two-real dimensional manifold $\mathcal{R}$ and F be the number of triangles, E the number of edges, V the number of vertices of $\mathcal{T}$. The number*

$$\chi = F - E + V \tag{1.29}$$

is called the Euler characteristic *of $\mathcal{R}$.*

Proposition 4. *The Euler characteristic $\chi(\mathcal{R})$ of a compact Riemann surface $\mathcal{R}$ is independent of the triangulation of $\mathcal{R}$.*

A differential geometric proof of this fact is based on the Gauss–Bonnet theorem (see for example [Spi79]). Introduce a conformal metric $\mathrm{e}^u \,\mathrm{d}z\,\mathrm{d}\bar{z}$ on a Riemann surface (see Sect. 1.1.4). The Gauss–Bonnet theorem provides us with the following formula for the Euler characteristic

$$\chi(\mathcal{R}) = \frac{1}{2\pi} \int_{\mathcal{R}} K \ , \tag{1.30}$$

where

$$K = -2u_{z\bar{z}}\mathrm{e}^{-u}$$

is the curvature of the metric. The right hand side in (1.30) is independent of the triangulation, the left hand side is independent of the metric we introduced on $\mathcal{R}$. This proves that the Euler characteristic is a topological invariant of $\mathcal{R}$.

A triangulation of the simply-connected model F_g of Proposition 3 gives a formula for χ in terms of the genus.

Corollary 3. *The Euler characteristic $\chi(\mathcal{R})$ of a compact Riemann surface $\mathcal{R}$ of genus g is equal to*

$$\chi(\mathcal{R}) = 2 - 2g \,. \tag{1.31}$$

Theorem 9 (Riemann-Hurwitz). *Let $f : \hat{\mathcal{R}} \to \mathcal{R}$ be an N-sheeted covering of the compact Riemann surface $\mathcal{R}$ of genus g. Then the genus $\hat{g}$ of $\hat{\mathcal{R}}$ is equal to*

$$\hat{g} = N(g-1) + 1 + \frac{b}{2} \,, \tag{1.32}$$

where

$$b = \sum_{P \in \hat{\mathcal{R}}} b_f(P) \tag{1.33}$$

is the total branching number.

This formula is equivalent to the corresponding identity for the Euler characteristic

$$\chi(\hat{\mathcal{R}}) = N\chi(\mathcal{R}) - b \,.$$

The latter follows easily if one chooses a triangulation of $\mathcal{R}$ so that the set of its vertices contains the projection to $\mathcal{R}$ of all branch points of the covering.

1.3.2 Fundamental group

Let γ be a closed curve with initial and terminal point P, i.e., a continuous map $\gamma : [0,1] \to \mathcal{R}$ with $\gamma(0) = \gamma(1) = P$.

Definition 11. *Two closed curves γ^1, γ^2 on $\mathcal{R}$ with the initial and terminal point P are called homotopic if one can be continuously deformed to the another, i.e., if there exists a continuous map $\gamma : [0,1] \times [0,1] \to \mathcal{R}$ such that $\gamma(t,0) = \gamma^1(t),\ \gamma(t,1) = \gamma^2(t),\ \gamma(0,\lambda) = \gamma(1,\lambda) = P$. The set of homotopic curves forms a* homotopy *class, which we denote by $\Gamma = [\gamma]$.*

There is a natural composition of such curves:

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} ,$$

which is well-defined also for the corresponding homotopic classes

$$\Gamma_1 \cdot \Gamma_2 = [\gamma_1 \cdot \gamma_2] \, .$$

The set of homotopy classes of curves forms a group $\pi_1(\mathcal{R}, P)$ with the multiplication defined above. The curves that can be contracted to a point correspond to the identity element of the group. It is easy to see that the groups $\pi_1(\mathcal{R}, P)$ and $\pi_1(\mathcal{R}, Q)$ based at different points are isomorphic as groups, and one can omit the second argument in the notation. The elements of this group are freely homotopic closed curves (i.e. cycles without reference to the base point P).

Definition 12. *The group* $\pi_1(\mathcal{R})$ *is called the* fundamental group *of* $\mathcal{R}$*.*

Examples
1. Sphere with N holes

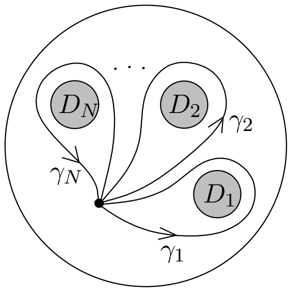

Fig. 1.14. Fundamental group of a sphere with N holes

The fundamental group is generated by the homotopy classes of the closed curves $\gamma_1, \dots, \gamma_N$ each going around one of the holes (Fig. 1.14). The curve $\gamma_1\gamma_2\dots\gamma_N$ can be contracted to a point, which implies the relation

$$\Gamma_1\Gamma_2\dots\Gamma_N = 1 \tag{1.34}$$

in $\pi_1(S^2 \setminus \{\bigcup_{n=1}^N D_n\})$.

2. Compact Riemann surface of genus g

It is convenient to consider the $4g$-gon model F_g (Fig. 1.15). The curves $a_1, b_1, \dots, a_g, b_g$ are closed on $\mathcal{R}$. Their homotopy classes, which we denote by $A_1, B_1, \dots, A_g, B_g$ generate $\pi_1(\mathcal{R})$. The contractible boundary of F_g implies the only relation in the fundamental group:

$$A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1} = 1 \, . \tag{1.35}$$

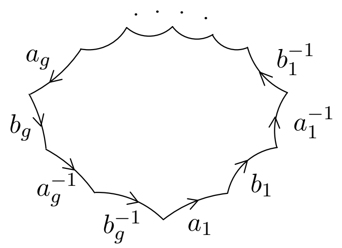

Fig. 1.15. Fundamental group of a compact surface of genus g

1.3.3 First homology group

Formal sums of points $\sum n_i P_i$, oriented curves γ_i,

$$\gamma = \sum n_i \gamma_i \in C_1$$

and oriented domains D_i,

$$D = \sum n_i D_i \in C_2$$

with integer coefficients $n_i \in \mathbb{Z}$ form abelian groups C_0, C_1 and C_2 respectively. The elements of these groups are called *0-chains*, *1-chains* and *2-chains*, respectively.

The boundary operator ∂ maps the corresponding elements to their oriented boundaries, defining the group homomorphisms $\partial : C_1 \to C_0, \partial : C_2 \to C_1$.

C_1 contains two important subgroups, of cycles and of boundaries. A closed oriented curve γ is called a *cycle* (i.e. $\partial\gamma = 0$), and $\gamma = \partial D$ is called a *boundary*. We denote these subgroups by

$$Z = \{\gamma \in C_1 \mid \ \partial\gamma = 0\} \ , \qquad B = \partial C_2 \ .$$

Because $\partial^2 = 0$, every boundary is a cycle and we have $B \subset Z \subset C_1$. Two elements of C_1 are called *homologous* if their difference is a boundary:

$$\gamma_1 \sim \gamma_2, \ \gamma_1, \gamma_2 \in C_1 \Leftrightarrow \gamma_1 - \gamma_2 \in B, \ \text{i.e.} \ \exists D \in C_2 : \delta D = \gamma_1 - \gamma_2 \ .$$

Definition 13. *The factor group*

$$H_1(\mathcal{R}, \mathbb{Z}) = Z/B$$

is called the first homology *group of* $\mathcal{R}$*.*

Freely homotopic closed curves are homologous. However, the converse is false in general, as one can see from the example in Fig. 1.16.

The first homology group is the fundamental group "made commutative". More precisely

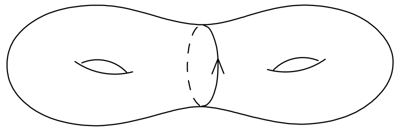

Fig. 1.16. A cycle homologous to zero but not homotopic to a point.

$$H_1(\mathcal{R}, \mathbb{Z}) = \frac{\pi(\mathcal{R})}{[\pi(\mathcal{R}), \pi(\mathcal{R})]} ,$$

where the denominator is the commutator subgroup, i.e., the subgroup of $\pi(\mathcal{R})$ generated by all elements of the form $ABA^{-1}B^{-1}$, $A, B \in \pi(\mathcal{R})$.

To introduce intersection numbers of elements of the first homology group it is convenient to represent them by smooth cycles. Every element of $H_1(\mathcal{R}, \mathbb{Z})$ can be represented by a C^∞-cycle without self-intersections. Moreover, given two elements of $H_1(\mathcal{R}, \mathbb{Z})$ one can represent them by smooth cycles intersecting transversally in a finite number of points.

Let γ_1 and γ_2 be two curves intersecting transversally at the point P. One associates to this point a number $(\gamma_1 \circ \gamma_2)_P = \pm 1$, where the sign is determined by the orientation of the basis $\gamma_1'(P), \gamma_2'(P)$ as it is shown in Fig. 1.17.

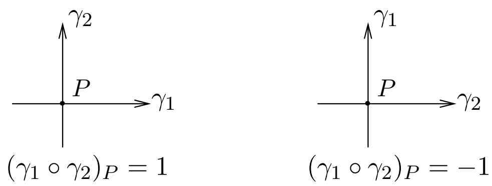

Fig. 1.17. Intersection number at a point.

Definition 14. *Let γ_1, γ_2 be two smooth cycles intersecting transversally in finitely many points. The* intersection number *of γ_1 and γ_2 is defined by*

$$\gamma_1 \circ \gamma_2 = \sum_{P \in \gamma_1 \cap \gamma_2} (\gamma_1 \circ \gamma_2)_P . \tag{1.36}$$

Lemma 3. *The intersection number of any boundary with any cycle vanishes.*

Since (1.36) is bilinear it is enough to check the statement for a boundary of a domain $\beta = \delta D$ and a simple cycle γ. This follows from the fact that the cycle γ enters D as many times as it leaves D (see Fig. 1.18).

To define the intersection number for homology classes represent $\gamma, \gamma' \in H_1(\mathcal{R}, \mathbb{Z})$ by C^∞-cycles

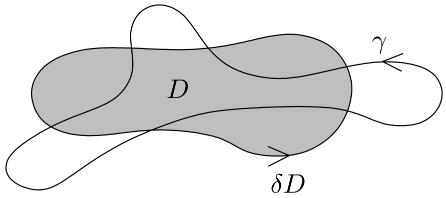

Fig. 1.18. $\gamma \circ \delta D = 0$.

$$\gamma = \sum_i n_i \gamma_i, \qquad \gamma' = \sum_j m_j \gamma'_j,$$

where γ_i, γ'_j are smooth curves intersecting transversally. Define $\gamma \circ \gamma' = \sum_{ij} n_i m_j \gamma_i \circ \gamma'_j$. Due to Lemma 3 the intersection number is well defined for homology classes.

Theorem 10. *The intersection number is a bilinear skew-symmetric map*

$$\circ : H_1(\mathcal{R}, \mathbb{Z}) \times H_1(\mathcal{R}, \mathbb{Z}) \to \mathbb{Z} \ .$$

Examples

1. The homology group of a sphere with N holes

The homology group is generated by the loops $\gamma_1, \ldots, \gamma_{N-1}$ (see Fig. 1.14). For the homology class of the loop γ_N one has

$$\gamma_N = - \sum_{i=1}^{N-1} \gamma_i \ ,$$

since $\sum_{i=1}^{N} \gamma_i$ is a boundary.

2. Homology group of a compact Riemann surface of genus g

Since the homotopy group is generated by the cycles $a_1, b_1, \ldots, a_g, b_g$ shown in Fig. 1.15 this is also true for the homology group. The intersection numbers of these cycles are as follows:

$$a_i \circ b_j = \delta_{ij} \ , \qquad a_i \circ a_j = b_i \circ b_j = 0 \ . \tag{1.37}$$

The cycles $a_1, b_1, \ldots, a_g, b_g$ constitute a basis of the homology group. Their intersection numbers imply the linear independence.

Definition 15. *A homology basis $a_1, b_1, \ldots, a_g, b_g$ of a compact Riemann surface of genus g with the intersection numbers (1.37) is called* canonical basis of cycles.

A canonical basis of cycles is by no means unique. Let (a, b) be a canonical basis of cycles. We represent it by a $2g$-dimensional vector

$$\begin{pmatrix} a \\ b \end{pmatrix} , \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_g \end{pmatrix} , \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix} .$$

Any other basis $(\tilde{a}, \tilde{b})$ of $H_1(\mathcal{R}, \mathbb{Z})$ is then given by the transformation

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix} , \qquad A \in GL(2g, \mathbb{Z}) . \tag{1.38}$$

Substituting the right hand side of equation (1.38) in

$$J = \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \circ (\tilde{a}, \tilde{b}) , \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

we obtain that the basis $(\tilde{a}, \tilde{b})$ is canonical if and only if A is symplectic, $A \in Sp(g, \mathbb{Z})$, i.e.,

$$J = AJA^T . \tag{1.39}$$

Two examples of canonical cycle bases are presented in Figs. 1.19, 1.20. The curves b_i in Fig. 1.19 connect identified points of the boundary curves and are therefore closed. In Fig. 1.20 the parts of the cycles lying on the "lower" sheet of the covering are marked by dotted lines.

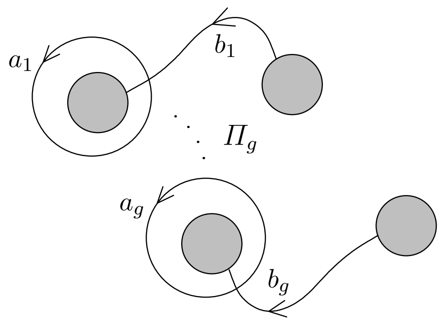

Fig. 1.19. A canonical cycle basis for the planar model Π_g of a compact Riemann surface.

1.4 Abelian differentials

Differentials on a Riemann surface are much easier to handle than functions, and they are the basic tool to investigate and construct functions.

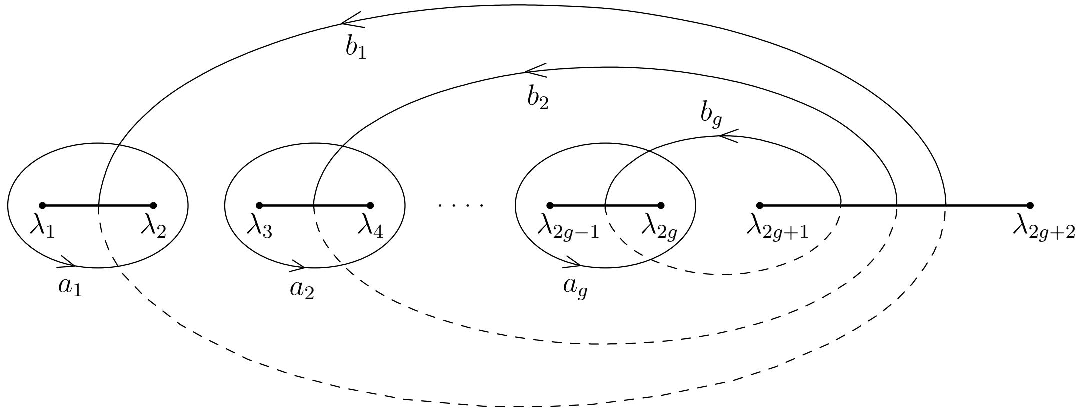

Fig. 1.20. A canonical cycle basis of a hyperelliptic Riemann surface.

1.4.1 Differential forms and integration formulas

If smooth complex valued functions $f(z,\bar{z})$, $p(z,\bar{z})$, $q(z,\bar{z})$, $s(z,\bar{z})$ are assigned to each local coordinate on $\mathcal{R}$ such that

$$\begin{aligned} f &= f(z,\bar{z}) , \\ \omega &= p(z,\bar{z})\mathrm{d}z + q(z,\bar{z})\mathrm{d}\bar{z} , \\ S &= s(z,\bar{z})\mathrm{d}z \wedge \mathrm{d}\bar{z} \end{aligned} \tag{1.40}$$

are invariant under coordinate changes (1.1), one says that the function (0-form) f, the differential (1-form) ω and the 2-form S are defined on $\mathcal{R}$.

The 1-form ω is called a form of type (1,0) (resp. a form of type (0,1)) if it may locally be written $\omega = p\,\mathrm{d}z$ (resp. $\omega = q\,\mathrm{d}\bar{z}$). The space of differentials is obviously a direct sum of the subspaces of (1,0) and (0,1) forms.

The exterior product of two 1-forms ω_1 and ω_2 is the 2-form

$$\omega_1 \wedge \omega_2 = (p_1 q_2 - p_2 q_1)\mathrm{d}z \wedge \mathrm{d}\bar{z} .$$

The differential operator d, which transforms k-forms into $(k+1)$-forms is defined by

$$\begin{aligned} \mathrm{d}f &= f_z \mathrm{d}z + f_{\bar{z}}\mathrm{d}\bar{z} , \\ \mathrm{d}\omega &= (q_z - p_{\bar{z}})\mathrm{d}z \wedge \mathrm{d}\bar{z} , \\ \mathrm{d}S &= 0 . \end{aligned} \tag{1.41}$$

Definition 16. *A differential* $\mathrm{d}f$ *is called* exact. *A differential* ω *with* $\mathrm{d}\omega = 0$ *is called* closed.

Using (1.41), one can also easily check that

$$d^2 = 0$$

whenever d^2 is defined and

$$d(f\omega) = df \wedge \omega + f d\omega \tag{1.42}$$

for any function f and 1-form ω. This implies in particular that any exact form is closed.

One can integrate differentials over 1-chains (i.e., smooth oriented curves and their formal sums), $\int_\gamma \omega$, and 2-forms over 2-chains (formal sums of oriented domains): $\int_D S$.

The most important integration formula is

Theorem 11 (Stokes's theorem). *Let D be a 2-chain with a piecewise smooth boundary ∂D. Then Stokes's formula*

$$\int_D d\omega = \int_{\partial D} \omega \tag{1.43}$$

holds for any differential ω.

The difference of two homologic curves $\gamma - \tilde{\gamma}$ is a boundary for some D, which implies

Corollary 4. *A differential ω is closed, $d\omega = 0$, if and only if for any two homological paths γ and $\tilde{\gamma}$*

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$$

holds.

Corollary 5. *Let ω be a closed differential, F_g be a simply connected model of Riemann surface of genus g (see Sect. 1.3) and P_0 be some point in F_g. Then the function*

$$f(P) = \int_{P_0}^{P} \omega\,, \qquad P \in F_g\,,$$

where the integration path lies in F_g is well-defined on F_g.

Let $\gamma_1, \ldots, \gamma_n$ be a homology basis of $\mathcal{R}$ and ω a closed differential. Periods of ω are defined by

$$\Lambda_i = \int_{\gamma_i} \omega\,.$$

Any closed curve γ on $\mathcal{R}$ is homological to $\sum n_i \gamma_i$ with some $n_i \in \mathbb{Z}$, which implies

$$\int_\gamma \omega = \sum n_i \Lambda_i \ ,$$

i.e., Λ_i generate the lattice of periods of ω. In particular, if $\mathcal{R}$ is a Riemann surface of genus g with canonical homology basis $a_1, b_1, \dots, a_g, b_g$, we denote the corresponding periods by

$$A_i = \int_{a_i} \omega \ , \qquad B_i = \int_{b_i} \omega \ .$$

Theorem 12 (Riemann's bilinear relations). *Let $\mathcal{R}$ be a Riemann surface of genus g with a canonical basis $a_i, b_i,\ i = 1, \dots, g$ and let ω and ω' be two closed differentials on $\mathcal{R}$ with periods $A_i, B_i, A_i', B_i',\ i = 1, \dots, g$. Then*

$$\int_{\mathcal{R}} \omega \wedge \omega' = \sum_{j=1}^{g} (A_j B_j' - A_j' B_j) \ . \tag{1.44}$$

The Riemann surface $\mathcal{R}$ cut along all the cycles $a_i, b_i,\ i = 1, \dots, g$ of the fundamental group is the simply connected domain F_g with the boundary (see Figs. 1.11, 1.15)

$$\partial F_g = \sum_{i=1}^{g} a_i + a_i^{-1} + b_i + b_i^{-1} \ . \tag{1.45}$$

Stokes's theorem with $D = F_g$ implies

$$\int_{\mathcal{R}} \omega \wedge \omega' = \int_{\partial F_g} \omega'(P) \int_{P_0}^{P} \omega \ ,$$

where P_0 is some point in F_g and the integration path $[P_0, P]$ lies in F_g.

The curves a_j and a_j^{-1} of the boundary of F_g are identical on $\mathcal{R}$ but have opposite orientation. For the points P_j and P_j' lying on a_j and a_j^{-1} respectively and coinciding on $\mathcal{R}$ we have (see Fig. 1.21)

$$\omega'(P_j) = \omega'(P_j') \ , \quad \int_{P_0}^{P_j} \omega - \int_{P_0}^{P_j'} \omega = \int_{P_j'}^{P_j} \omega = -B_j \ . \tag{1.46}$$

In the same way for the points $Q_j \in b_j$ and $Q_j' \in b_j^{-1}$ coinciding on $\mathcal{R}$ one gets

$$\omega'(Q_j) = \omega'(Q_j') \ , \quad \int_{P_0}^{Q_j} \omega - \int_{P_0}^{Q_j'} \omega = \int_{Q_j'}^{Q_j} \omega = A_j \ . \tag{1.47}$$

Substituting, we obtain

$$\int_{\partial F_g} \omega'(P) \int_{P_0}^{P} \omega = \sum_{j=1}^{g} \Big(- B_j \int_{a_j} \omega' + A_j \int_{b_j} \omega' \Big) = \sum_{j=1}^{g} (A_j B_j' - A_j' B_j) \ .$$

Finally, to complete the proof of Riemann's bilinear identity, one checks directly that the right hand side of (1.44) is invariant under the transformation (1.38, 1.39). Therefore the claim holds for an arbitrary canonical basis of $H_1(\mathcal{R}, \mathbb{C})$.

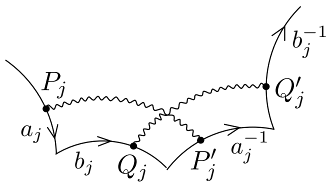

Fig. 1.21. Illustrating the proof of the Riemann bilinear relation.

1.4.2 Abelian differentials of the first, second and third kind

Definition 17. *A differential ω on a Riemann surface $\mathcal{R}$ is called holomorphic (or an Abelian differential of the first kind) if in any local chart it is represented as*

$$\omega = h(z)\mathrm{d}z$$

where $h(z)$ is holomorphic. The differential $\bar{\omega}$ is called anti-holomorphic.

Holomorphic and anti-holomorphic differentials are closed.

Holomorphic differentials form a complex vector space $H^1(\mathcal{R}, \mathbb{C})$. It is not difficult to show that the dimension of this space is at most g. Indeed, Riemann's bilinear identity with $\omega' = \bar{\omega}$ implies that the periods A_j, B_j of a holomorphic differential ω satisfy

$$\mathrm{Im} \sum_{j=1}^{g} A_j \bar{B}_j < 0 \ . \tag{1.48}$$

Thus, if all a-periods of the holomorphic differential ω are zero then $\omega \equiv 0$. If $\omega_1, \ldots, \omega_{g+1}$ are holomorphic, then there exists a linear combination of them with all zero a-periods, i.e., the differentials are linearly dependent.

Theorem 13. *The dimension of the space of holomorphic differentials of a compact Riemann surface is equal to its genus*

$$\dim\ H^1(\mathcal{R}, \mathbb{C}) = g(\mathcal{R}) \ .$$

The existence part of this theorem is more difficult and can be proved by analytic methods [FK92]. However, when the Riemann surface $\mathcal{R}$ is concretely described, one can usually present the basis $\omega_1, \dots, \omega_g$ of holomorphic differentials explicitly.

On a hyperelliptic curve one can check the holomorphicity using the corresponding local coordinates described in Sect. 1.1.1.

Theorem 14. *The differentials*

$$\omega_j = \frac{\lambda^{j-1} \mathrm{d}\lambda}{\mu}, \qquad j = 1, \dots, g \tag{1.49}$$

form a basis of holomorphic differentials of the hyperelliptic Riemann surface

$$\mu^2 = \prod_{i=1}^{N} (\lambda - \lambda_i) \qquad \lambda_i \neq \lambda_j \ , \tag{1.50}$$

where $N = 2g + 2$ or $N = 2g + 1$.

Another example is the holomorphic differential

$$\omega = \mathrm{d}z$$

on the torus $\mathbb{C}/G$ of Sect. 1.3. Here z is the coordinate of $\mathbb{C}$.

Since a differential with all zero a-periods vanishes identically, the matrix of a-periods $A_{ij} = \int_{a_i} \omega_j$ of any basis $\omega_j, \ j = 1, \dots, g$ of $H^1(\mathcal{R}, \mathbb{C})$ is invertible. The basis can be normalized.

Definition 18. *Let $a_j, b_j, \ j = 1, \dots, g$ be a canonical basis of $H_1(\mathcal{R}, \mathbb{Z})$. The dual basis of holomorphic differentials $\omega_k, \ k = 1, \dots, g$, normalized by*

$$\int_{a_j} \omega_k = 2\pi \mathrm{i} \delta_{jk}$$

is called canonical basis of differentials.

We consider also differentials with singularities.

Definition 19. *A differential Ω is called meromorphic or* Abelian differential *if in any local chart $z : U \to \mathbb{C}$ it is of the form*

$$\Omega = g(z) \mathrm{d}z \ ,$$

where $g(z)$ is meromorphic. The integral $\int_{P_0}^{P} \Omega$ of a meromorphic differential is called the Abelian integral.

Let z be a local parameter at the point P, $z(P) = 0$ and

$$\Omega = \sum_{k=N(P)}^{\infty} g_k z^k \mathrm{d}z\ , \qquad N \in \mathbb{Z} \tag{1.51}$$

be the representation of the differential Ω at P. The numbers $N(P)$ and g_{-1} do not depend on the choice of the local parameter and are characteristics of Ω alone. $N(P)$ is called the *order of the point* P. If $N(P)$ is negative $-N(P)$ is called the *order of the pole* of Ω at P. The number g_{-1} is called the *residue* of Ω at P. It can also be defined by

$$\mathrm{res}_P \Omega \equiv g_{-1} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \Omega\ , \tag{1.52}$$

where γ is a small closed simple loop going around P in the positive direction.

Let S be the set of singularities of Ω

$$S = \{P \in \mathcal{R} \mid N(P) < 0\}\ .$$

S is discrete, and if $\mathcal{R}$ is compact then S is also finite.

Computing the integral of an Abelian differential Ω along the boundary of the simply connected model F_g by residues, one obtains

$$\sum_{P_j \in S} \mathrm{res}_{P_j} \Omega = 0\ . \tag{1.53}$$

Definition 20. *A meromorphic differential with singularities is called an Abelian differential of the second kind if the residues are equal to zero at all singular points. A meromorphic differential with non-zero residues is called an Abelian differential of the third kind.*

The residue identity (1.53) motivates the following choice of basic meromorphic differentials. The differential of the second kind $\Omega_R^{(N)}$ has only one singularity. It is at the point $R \in \mathcal{R}$ and is of the form

$$\Omega_R^{(N)} = \left(\frac{1}{z^{N+1}} + O(1) \right) \mathrm{d}z\ , \tag{1.54}$$

where z is a local parameter at R with $z(R) = 0$. The Abelian differential $\Omega_R^{(N)}$ depends on the choice of the local parameter z. The Abelian differential of the third kind Ω_{RQ} has two singularities at the points R and Q with

$$\mathrm{res}_R \Omega_{RQ} = -\mathrm{res}_Q \Omega_{RQ} = 1\ ,$$

$$\Omega_{RQ} = \left(\frac{1}{z_R} + O(1) \right) \mathrm{d}z_R \qquad \text{near } R\ ,$$
$$\Omega_{RQ} = \left(-\frac{1}{z_Q} + O(1) \right) \mathrm{d}z_Q \qquad \text{near } Q\ , \tag{1.55}$$

where z_R and z_Q are local parameters at R and Q with $z_R(R) = z_Q(Q) = 0$. For the corresponding Abelian integrals this implies

$$\int^P \Omega_R^{(N)} = -\frac{1}{N z^N} + O(1) \qquad P \to R\,, \tag{1.56}$$

$$\int^P \Omega_{RQ} = \log z_R + O(1) \qquad P \to R\,,$$
$$\int^P \Omega_{RQ} = -\log z_Q + O(1) \qquad P \to Q\,. \tag{1.57}$$

The Abelian integrals of the first and second kind are single-valued on F_g. The Abelian integral of the third kind Ω_{RQ} is single-valued on $F_g \setminus [R, Q]$, where $[R, Q]$ is a cut from R to Q lying inside F_g.

One can add Abelian differentials of the first kind to $\Omega_R^{(N)}$ and Ω_{RQ} preserving the form of the singularities. By addition of a proper linear combination $\sum_{i=1}^{g} \alpha_i \omega_i$ the differential can be normalized as follows:

$$\int_{a_j} \Omega_R^{(N)} = 0\,, \qquad \int_{a_j} \Omega_{RQ} = 0 \tag{1.58}$$

for all a-cycles $j = 1, \ldots, g$.

Definition 21. *The differentials $\Omega_R^{(N)}$, Ω_{RQ} with the singularities (1.54), (1.55) and all zero a-periods (1.58) are called the normalized Abelian differentials of the second and third kind.*

Theorem 15. *Given a compact Riemann surface $\mathcal{R}$ with a canonical basis of cycles $a_1, b_1, \ldots, a_g, b_g$, points $R, Q \in \mathcal{R}$, a local parameter z at R, and $N \in \mathbb{N}$, there exist unique normalized Abelian differentials $\Omega_R^{(N)}$ and Ω_{RQ} of the second and third kind, respectively.*

The proof of the uniqueness is simple. The holomorphic difference of two normalized differentials with the same singularities has all zero a-periods and therefore vanishes identically. Like in the case of holomorphic differentials, the existence can be shown by analytic methods [FK92].

Abelian differentials of the second and third kind can be normalized by a more symmetric condition than (1.58). Namely, all the periods can be normalized to be purely imaginary

$$\mathrm{Re} \int_\gamma \Omega = 0\,, \qquad \forall \gamma \in H_1(\mathcal{R}, \mathbb{Z})\,.$$

Corollary 6. *The normalized Abelian differentials form a basis in the space of Abelian differentials on $\mathcal{R}$.*

Again, as in the case of holomorphic differentials, we present the basis of Abelian differentials of the second and third kind in the hyperelliptic case

$$\mu^2 = \prod_{k=1}^{M} (\lambda - \lambda_k) \ .$$

Denote the coordinates of the points R and Q by

$$R = (\mu_R, \lambda_R) \ , \qquad Q = (\mu_Q, \lambda_Q) \ .$$

We consider the case when both points R and Q are finite $\lambda_R \neq \infty$, $\lambda_Q \neq \infty$. The case $\lambda_R = \infty$ or $\lambda_Q = \infty$ is reduced to the case we consider by a fractional linear transformation. If R is not a branch point, then to get a proper singularity we multiply $d\lambda/\mu$ by $1/(\lambda - \lambda_R)^n$ and cancel the singularity at the point $\pi R = (-\mu_R, \lambda_R)$.

The following differentials are of the third kind with the singularities (1.55)

$$\hat{\Omega}_{RQ} = \left(\frac{\mu + \mu_R}{\lambda - \lambda_R} - \frac{\mu + \mu_Q}{\lambda - \lambda_Q} \right) \frac{d\lambda}{2\mu} \quad \text{if} \quad \mu_R \neq 0 \ , \quad \mu_Q \neq 0 \ ,$$

$$\hat{\Omega}_{RQ} = \left(\frac{\mu + \mu_R}{\mu(\lambda - \lambda_R)} - \frac{1}{\lambda - \lambda_Q} \right) \frac{d\lambda}{2} \quad \text{if} \quad \mu_R \neq 0 \ , \quad \mu_Q = 0 \ ,$$

$$\hat{\Omega}_{RQ} = \left(\frac{1}{\lambda - \lambda_R} - \frac{1}{\lambda - \lambda_Q} \right) \frac{d\lambda}{2} \quad \text{if} \quad \mu_R = \mu_Q = 0 \ .$$

If R is not a branch point, $\mu_R \neq 0$, then the differentials

$$\hat{\Omega}_R^{(N)} = \frac{\mu + \mu_R^{[N]}}{(\lambda - \lambda_R)^{N+1}} \frac{\mathrm{d}\lambda}{2\mu} \ ,$$

where $\mu_R^{[N]}$ is the Taylor series at R up to the term of order N,

$$\mu_R^{[N]} = \mu_R + \left.\frac{\partial \mu}{\partial \lambda}\right|_R (\lambda - \lambda_R) + \ldots + \frac{1}{N!} \left.\frac{\partial^N \mu}{\partial \lambda^N}\right|_R (\lambda - \lambda_R)^N \ ,$$

have singularities at R of the form

$$\big(z^{-N-1} + o(z^{-N-1})\big) dz \tag{1.59}$$

where $z = \lambda - \lambda_R$. If R is a branch point, $\mu_R = 0$, the following differentials have the singularities (1.59) with $z = \sqrt{\lambda - \lambda_R}$

$$\hat{\Omega}_R^{(N)} = \frac{\mathrm{d}\lambda}{2(\lambda - \lambda_R)^n \mu} \sqrt{\prod_{\substack{i=1 \\ i \neq R}}^{N} (\lambda_R - \lambda_i)} \quad \text{for} \quad N = 2n - 1 \ ,$$

$$\hat{\Omega}_R^{(N)} = \frac{\mathrm{d}\lambda}{2(\lambda - \lambda_R)^n} \quad \text{for} \quad N = 2n - 2 \ .$$

Taking proper linear combinations of these differentials with different values of N we obtain the singularity (1.54). The normalization (1.58) is obtained by addition of holomorphic differentials (1.46)

1.4.3 Periods of Abelian differentials. Jacobi variety

Definition 22. *Let $a_j, b_j,\ j = 1, \ldots, g,$ be a canonical homology basis of $\mathcal{R}$ and let $\omega_k,\ k = 1, \ldots, g,$ be the dual basis of $H^1(\mathcal{R}, \mathbb{C})$. The matrix*

$$B_{ij} = \int_{b_i} \omega_j \tag{1.60}$$

is called the period matrix *of $\mathcal{R}$.*

Theorem 16. *The period matrix is symmetric and its real part is negative definite,*

$$B_{ij} = B_{ji}\ , \tag{1.61}$$

$$\mathrm{Re}(B\alpha, \alpha)\ < 0\ , \qquad \forall \alpha \in \mathbb{R}^g \setminus \{0\}\ . \tag{1.62}$$

The symmetry of the period matrix follows from the Riemann bilinear identity (1.44) with $\omega = \omega_i$ and $\omega' = \omega_j$. The definiteness (1.62) is another form of (1.48).

The period matrix depends on the homology basis. Let us use the column notations

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \in Sp(g, \mathbb{Z})\ . \tag{1.63}$$

Let $\omega = (\omega_1, \ldots, \omega_g)$ be the canonical basis of holomorphic differentials dual to (a, b). Labeling columns of the matrices by differentials and rows by cycles we get

$$\int_{\tilde{a}} \omega = 2\pi \mathrm{i} \mathsf{A} + \mathsf{B} B, \qquad \int_{\tilde{b}} \omega = 2\pi \mathrm{i} \mathsf{C} + \mathsf{D} B.$$

The canonical basis of $H^1(\mathcal{R}, \mathbb{C})$ dual to the basis $(\tilde{a}, \tilde{b})$ is given by the right multiplication

$$\tilde{\omega} = 2\pi \mathrm{i} \omega (2\pi \mathrm{i} \mathsf{A} + \mathsf{B} B)^{-1}.$$

This implies the following transformation formula for the period matrix.

Lemma 4. *The period matrices B and $\tilde{B}$ of the Riemann surface $\mathcal{R}$ corresponding to the homology basis (a, b) and $(\tilde{a}, \tilde{b})$ respectively are related by*

$$\tilde{B} = 2\pi \mathrm{i} (\mathsf{D} B + 2\pi \mathrm{i} \mathsf{C})(\mathsf{B} B + 2\pi \mathrm{i} \mathsf{A})^{-1}, \tag{1.64}$$

where $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ are the coefficients of the symplectic matrix (1.63).

Using the Riemann bilinear identity one can express the periods of the normalized Abelian differentials of the second and third kind in terms of the normalized holomorphic differentials. Choosing $\omega = \omega_j$ and $\omega' = \Omega_R^{(N)}$ or $\omega' = \Omega_{RQ}$ in the Riemann bilinear identity we obtain the following representations.

Lemma 5. *Let* $\omega_j, \Omega_R^{(N)}, \Omega_{RQ}$ *be the normalized Abelian differentials from Definition 21. Let* z *be a local parameter at* R *with* $z(R) = 0$ *and*

$$\omega_j = \sum_{k=0}^{\infty} \alpha_{k,j} z^k \mathrm{d}z \tag{1.65}$$

the representation of the normalized holomorphic differentials at R*. The periods of* $\Omega_R^{(N)}$*,* Ω_{RQ} *are equal to:*

$$\int_{b_j} \Omega_R^{(N)} = \frac{1}{N} \alpha_{N-1,j} \tag{1.66}$$

$$\int_{b_j} \Omega_{RQ} = \int_Q^R \omega_j \ , \tag{1.67}$$

where the integration path $[R, Q]$ *in (1.67) does not cross the cycles* a, b*.*

Let Λ be the lattice

$$\Lambda = \{2\pi \mathrm{i} N + BM, \ N, M \in \mathbb{Z}^g\}$$

generated by the periods of $\mathcal{R}$. It defines an equivalence relation in $\mathbb{C}^g$: two points of $\mathbb{C}^g$ are equivalent if they differ by an element of Λ.

Definition 23. *The complex torus*

$$\mathrm{Jac}(\mathcal{R}) = \mathbb{C}^g / \Lambda$$

is called the Jacobi variety *of* $\mathcal{R}$*. The map*

$$\mathcal{A} : \mathcal{R} \to \mathrm{Jac}(\mathcal{R}) \ , \qquad \mathcal{A}(P) = \int_{P_0}^P \omega \ , \tag{1.68}$$

where $\omega = (\omega_1, \ldots, \omega_g)$ *is the canonical basis of holomorphic differentials and* $P_0 \in \mathcal{R}$*, is called the* Abel map.

1.5 Meromorphic functions on compact Riemann surfaces

1.5.1 Divisors and the Abel theorem

In order to analyze functions and differentials on Riemann surfaces, one characterizes them in terms of their zeros and poles. It is convenient to consider formal sums of points on $\mathcal{R}$. (Later these points will become zeros and poles of functions and differentials).

Definition 24. *A formal linear combination*

$$D = \sum_{j=1}^{N} n_j P_j \ , \qquad n_j \in \mathbb{Z} \ , \ P_j \in \mathcal{R} \tag{1.69}$$

is called a divisor *on the Riemann surface* $\mathcal{R}$*. The sum*

$$\deg \, D = \sum_{j=1}^{N} n_j$$

is called the degree *of* D*.*

The set of all divisors with the obviously defined group operations

$$n_1 P + n_2 P = (n_1 + n_2) P \ , \qquad -D = \sum_{j=1}^{N} (-n_j) P_j$$

forms an Abelian group $\mathrm{Div}(\mathcal{R})$. A divisor (1.69) with all $n_j \geq 0$ is called *positive* (or integral, or effective). This notion allows us to define a partial ordering in $\mathrm{Div}(\mathcal{R})$

$$D \leq D' \iff D' - D \geq 0 \ .$$

Definition 25. *Let* f *be a meromorphic function on* $\mathcal{R}$ *and let* $P_1, \ldots, P_M$ *be its zeros with multiplicities* $p_1, \ldots, p_M > 0$ *and* $Q_1, \ldots, Q_N$ *its poles with multiplicities* $q_1, \ldots, q_N > 0$*. The divisor*

$$D = p_1 P_1 + \ldots + p_M P_M - q_1 Q_1 - \ldots - q_N Q_N = (f)$$

is called the divisor of f *and is denoted by* (f) *. A divisor* D *is called* principal *if there exists a function* f *with* $(f) = D$*.*

Obviously we have

$$(fg) = (f) + (g) \ ,$$

where f and g are two meromorphic functions on $\mathcal{R}$.

Definition 26. *Two divisors* D *and* D' *are called* linearly equivalent *if the divisor* $D - D'$ *is principal. The corresponding equivalence class is called the* divisor class*.*

We denote linearly equivalent divisors by $D \equiv D'$. Divisors of Abelian differentials are also well-defined. We have already seen that the order of the point $N(P)$ defined by (1.51) is independent of the choice of a local parameter and is a characteristic of the Abelian differential. The set of points $P \in \mathcal{R}$ with $N(P) \neq 0$ is finite.

Definition 27. *The divisor of an Abelian differential Ω is*

$$(\Omega) = \sum_{P \in \mathcal{R}} N(P)P \ ,$$

where $N(P)$ is the order of the point P of Ω.

Since the quotient of two Abelian differentials

$$\Omega_1/\Omega_2$$

is a meromorphic function any two divisors of Abelian differentials are linearly equivalent. The corresponding class is called *canonical*. We will denote it by $\mathcal{C}$.

Any principal divisor can be represented as the difference of two positive linearly equivalent divisors

$$(f) = D_0 - D_\infty \ , \qquad D_0 \equiv D_\infty \ ,$$

where D_0 is the zero divisor and D_∞ is the pole divisor of f. Corollary 2 implies that

$$\deg(f) = 0 \ ,$$

i.e., all principal divisors have zero degree. All canonical divisors have the same degree.

The Abel map is defined for divisors in a natural way

$$\mathcal{A}(D) = \sum_{j=1}^{N} n_j \int_{P_0}^{P_j} \omega \ . \tag{1.70}$$

If the divisor D is of degree zero, then $\mathcal{A}(D)$ is independent of P_0

$$D = P_1 + \ldots + P_N - Q_1 - \ldots - Q_N \ ,$$

$$\mathcal{A}(D) = \sum_{i=1}^{N} \int_{Q_i}^{P_i} \omega \ . \tag{1.71}$$

Theorem 17 (Abel's theorem). *The divisor $D \in$ Div $(\mathcal{R})$ is principal if and only if:*

1) $\deg\ D = 0$,
2) $\mathcal{A}(D) \equiv 0$.

The necessity of the first condition is already shown in Corollary 2. Let f be a meromorphic function with the divisor

$$(f) = P_1 + \ldots + P_N - Q_1 - \ldots - Q_N$$

(these points need not be different). Then

$$\Omega = \frac{\mathrm{d}f}{f} = \mathrm{d}(\log f)$$

is an Abelian differential of the third kind. Since periods of Ω are integer multiples of $2\pi \mathrm{i}$, it can be represented as

$$\Omega = \sum_{i=1}^{N} \Omega_{P_i Q_i} + \sum_{k=1}^{g} n_k \omega_k$$

with $n_k \in \mathbb{Z}$. The representation (1.67) for the b-periods of the normalized differentials of the third kind implies $\mathcal{A}(D) \equiv 0$.

Corollary 7. *All linearly equivalent divisors are mapped by the Abel map to the same point of the Jacobian*

$$\mathcal{A}((f) + D) = \mathcal{A}(D)\ .$$

The Abel theorem can be formulated in terms of any basis $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_g)$ of holomorphic differentials. In this case the second condition of the theorem reads

$$\sum_{i=1}^{N} \int_{Q_i}^{P_i} \tilde{\omega} \equiv 0 \quad (\text{mod periods of } \tilde{\omega}).$$

1.5.2 The Riemann-Roch theorem

Let D_∞ be a positive divisor on $\mathcal{R}$. A natural problem is to describe the vector space of meromorphic functions with poles at D_∞ only. More generally, let D be a divisor on $\mathcal{R}$. Let us consider the vector space

$$L(D) = \{f \text{ meromorphic functions on } \mathcal{R} \mid (f) \geq D \text{ or } f \equiv 0\}.$$

Let us split

$$D = D_0 - D_\infty$$

into negative and positive parts

$$D_0 = \sum n_i P_i\ , \qquad D_\infty = \sum m_k Q_k\ ,$$

where both D_0 and D_∞ are positive. The space $L(D)$ of dimension

$$l(D) = \dim L(D)$$

consists of the meromorphic functions with zeros of order at least n_i at P_i and with poles of order at most m_k at Q_k.

Similarly, let us denote by

$$H(D) = \{\Omega \text{ Abelian differential on } \mathcal{R} \mid (\Omega) \geq D \text{ or } \Omega \equiv 0\}$$

the corresponding vector space of differentials, and by

$$i(D) = \dim H(D)$$

its dimension, which is called the *index of speciality* of D.

It is easy to see that $l(D)$ and $i(D)$ depend only on the divisor class of D, and

$$i(D) = l(D - C) , \tag{1.72}$$

where C is the canonical divisor class. Indeed, let Ω_0 be a non-zero Abelian differential and $C = (\Omega_0)$ be its divisor. The map $H(D) \to L(D - C)$ defined by

$$H(D) \ni \Omega \longrightarrow \frac{\Omega}{\Omega_0} \in L(D - C)$$

is an isomorphism of linear spaces, which implies $i(D) = l(D - C)$.

Theorem 18 (Riemann-Roch theorem). *Let $\mathcal{R}$ be a compact Riemann surface of genus g and D a divisor on $\mathcal{R}$. Then*

$$l(-D) = \deg D - g + 1 + i(D) . \tag{1.73}$$

This identity can be proved by an analysis of the singularities and the periods of the differential *df* for a function $f \in L(-D)$. However the proof is rather involved [FK92]. Many important results can be easily obtained from this fundamental theorem.

Since the index $i(D)$ is non-negative one has

Theorem 19 (Riemann's inequality). *For any divisor D*

$$l(-D) \geq \deg D + 1 - g .$$

This has the following immediate consequence.

Corollary 8. *For any positive divisor D with* $\deg D = g + 1$ *there exists a non-constant meromorphic function in $L(-D)$.*

Let us consider a divisor on a Riemann surface of genus zero which consists of one point $D = P$. Riemann's inequality implies $l(-P) \geq 2$. There exists a non-trivial function f with 1 pole on $\mathcal{R}$. It is a holomorphic covering $f : \mathcal{R} \to \hat{\mathbb{C}}$. Since f has only one pole, every value is assumed once (Corollary 2), therefore $\mathcal{R}$ and $\hat{\mathbb{C}}$ are conformally equivalent.

Corollary 9. *Any Riemann surface of genus* 0 *is conformally equivalent to the complex sphere $\hat{\mathbb{C}}$.*

Corollary 10. *The degree of the canonical class is*

$$\deg C = 2g - 2 . \tag{1.74}$$

The Riemann-Roch theorem implies for the canonical divisor

$$\deg C = l(-C) + g - 1 - i(C) \ .$$

Since the spaces of holomorphic differentials and functions are g- and 1-dimensional respectively, using (1.72), $l(-C) = i(0) = g, i(C) = l(0) = 1$, one arrives at (1.74).

Corollary 11. *On a compact Riemann surface there is no point where all holomorphic differentials vanish simultaneously.*

Indeed, suppose there exists such a point $P \in \mathcal{R}$, i.e., $i(P) = g$. The Riemann-Roch theorem for the divisor $D = P$ implies $l(-P) = 2$, i.e., there exists a non-constant meromorphic function f with only one simple pole. This implies that $f : \mathcal{R} \to \hat{\mathbb{C}}$ is bi-holomorphic, in particular $g = 0$.

1.5.3 Jacobi inversion problem

Now we come to more complicated properties of the Abel map. Let us fix a point $P_0 \in \mathcal{R}$. From corollary 11 of the Riemann-Roch theorem we know that all holomorphic differentials do not vanish simultaneously. Therefore $\mathrm{d}\mathcal{A}(P) = \omega(P) \neq 0$, which shows that the Abel map is an immersion (the differential of the map vanishes nowhere on $\mathcal{R}$).

Proposition 5. *The Abel map*

$$\begin{aligned} \mathcal{A} : \mathcal{R} &\to Jac(\mathcal{R}) \\ P &\mapsto \int_{P_0}^{P} \omega \end{aligned} \tag{1.75}$$

is an embedding, i.e., the map (1.75) is an injective immersion.

The injectivity follows from Abel's theorem. Suppose there exist $P_1, P_2 \in \mathcal{R}$ with $\mathcal{A}(P_1) = \mathcal{A}(P_2)$, i.e., the divisor $P_1 - P_2$ is principal. Functions with one pole do not exist for Riemann surfaces of genus $g > 0$, thus the points must coincide $P_1 = P_2$.

The Jacobi variety of a Riemann surface of genus one is a one-dimensional complex torus, which is itself a Riemann surface of genus one (see Sect. 1.1.2).

Corollary 12. *A Riemann surface of genus one is conformally equivalent to its Jacobi variety.*

Although the next theorem looks technical (see for example [FK92] for the proof), it is an important result often used.

Theorem 20 (Jacobi inversion problem). *Let $\mathcal{D}_g$ be the set of positive divisors of degree g. The Abel map on this set*

$$\mathcal{A} : \mathcal{D}_g \to Jac(\mathcal{R})$$

is surjective, i.e., for any $\xi \in Jac(\mathcal{R})$ *there exists a degree* g *positive divisor* $P_1 + \ldots + P_g \in \mathcal{D}_g$ *(the* P_i *need not be different) satisfying*

$$\sum_{i=1}^{g} \int_{P_0}^{P_i} \omega = \xi. \tag{1.76}$$

1.5.4 Special divisors and Weierstrass points

Definition 28. *A positive divisor* D *of degree* deg $D = g$ *is called* special *if* $i(D) > 0$*, i.e., there exists a holomorphic differential* ω *with*

$$(\omega) \geq D\ . \tag{1.77}$$

The Riemann-Roch theorem implies that (1.77) is equivalent to the existence of a non-constant function f with $(f) \geq -D$. Since the space of holomorphic differentials is g-dimensional, (1.77) is a homogeneous linear system of g equations in g variables. This shows that most of the positive divisors of degree g are non-special.

Definition 29. *A point* $P \in \mathcal{R}$ *is called the* Weierstrass point *if the divisor* $D = gP$ *is special.*

The Weierstrass points are special points of $\mathcal{R}$. Weierstrass points exists on Riemann surfaces of genus $g > 1$. They coincide with the zeros of the holomorphic q-differential Hdz^q with $q = g(g+1)/2$ and

$$H := \det \begin{pmatrix} h_1 & \ldots & h_g \\ h_1' & \ldots & h_g' \\ \vdots & & \vdots \\ h_1^{(g-1)} & \ldots & h_g^{(g-1)} \end{pmatrix}\ , \tag{1.78}$$

where $\omega_k = h_k(z)dz$ are the local representations of a basis of holomorphic differentials. Indeed, H vanishes at P_0 if and only if the matrix in (1.78) has a non-zero vector $(\alpha_1, \ldots, \alpha_g)^T$ in the kernel. In this case the differential $\sum_{k=1}^{g} \alpha_k h_k$ has a zero of order g at P_0, which implies $i(gP_0) > 0$.

The number of the Weierstrass points is bounded by the number of zeros of H, which is $g^3 - g$.

1.5.5 Hyperelliptic Riemann surfaces

Let $\mathcal{R}$ be a compact Riemann surface of a hyperelliptic curve as in Theorem 1. On this Riemann surface there exist meromorphic functions with precisely two poles counting multiplicities. Examples of such functions are λ and $\frac{1}{\lambda - \lambda_0}$ with arbitrary λ_0. This observation leads to an equivalent definition of hyperelliptic Riemann surfaces.

Definition 30. *A compact Riemann surface $\mathcal{R}$ of genus $g \geq 2$ is called* hyperelliptic *if there exists a positive divisor D on $\mathcal{R}$ with*

$$\deg\, D = 2\;, \qquad l(-D) \geq 2\;.$$

A non-constant meromorphic function Λ in $L(-D)$ defines a two-sheeted covering of the complex sphere

$$\Lambda : \mathcal{R} \to \hat{\mathbb{C}}\;. \tag{1.79}$$

All the ramification points of this covering have branch numbers 1.

It is not difficult to show that two hyperelliptic Riemann surfaces are conformally equivalent if and only if their branch points differ by a fractional linear transformation. The branch points can be used as parameters in the moduli space of hyperelliptic curves. The complex dimension of this space is $2g-1$. Indeed, there are $2g+2$ branch points and three of them can be normalized to $0, 1, \infty$ by a fractional linear transformation. We see that for $g = 2$ this dimension coincides with the complex dimension $3g-3$ of the space of Riemann surfaces of genus g (see Sect. 1.8.1). This simple observation shows that there exist non-hyperelliptic Riemann surfaces of genus $g \geq 3$.

Theorem 21. *Any Riemann surface of genus $g = 2$ is hyperelliptic.*

This is not difficult to prove. The zero divisor (ω) of a holomorphic differential on a Riemann surface $\mathcal{R}$ of genus 2 is of degree $2 = 2g-2$. Since $i((\omega)) > 0$, the Riemann-Roch theorem implies $l((\omega)) \geq 2$.

Special divisors on hyperelliptic Riemann surfaces are characterized by the following simple property.

Proposition 6. *Let $\mathcal{R}$ be a hyperelliptic Riemann surface and let $\lambda : \mathcal{R} \to \hat{\mathbb{C}}$ be the corresponding two-sheeted covering (1.5) with branch points $\lambda_k, k = 1\ldots, N$. A positive divisor D of degree g is singular if and only if it contains a pair of points $(\mu_0, \lambda_0),\ (-\mu_0, \lambda_0)$ with the same λ-coordinate $\lambda_0 \neq \lambda_k$ or a double branch point $2(0, \lambda_k)$.*

D being a special divisor implies that there exists a differential ω with $(\omega) \geq D$. The differential ω is holomorphic, and due to Theorem 14 it can be represented as

$$\omega = \frac{P_{g-1}(\lambda)}{\mu} \mathrm{d}\lambda\;,$$

where $P_{g-1}(\lambda)$ is a polynomial of degree $g-1$. The differential ω has $g-1$ pairs of zeros

$$(\mu_n, \lambda_n),\ (-\mu_n, \lambda_n)\;,\ n = 1, \ldots, g-1,\ P_{g-1}(\lambda_n) = 0\;.$$

Since D is of degree g it must contain at least one of these pairs.

1.6 Theta functions

1.6.1 Definition and simplest properties

Consider a g-dimensional complex torus $\mathbb{C}^g/\Lambda$ where Λ is a lattice of full rank:

$$\Lambda = AN + BM\ , \qquad A, B \in gl(g, \mathbb{C})\ ,\ N, M \in \mathbb{Z}^g\ , \tag{1.80}$$

and the $2g$ columns of A, B are $\mathbb{R}$-linearly independent. Non-constant meromorphic functions on $\mathbb{C}^g/\Lambda$ exist only (see, for example, [Sie71]) if the complex torus is an *Abelian torus*, i.e., if by an appropriate linear choice of coordinates on $\mathbb{C}^g$ the lattice (1.80) can be reduced to a special form: A is a diagonal matrix of the form

$$A = 2\pi\mathrm{i}\ \mathrm{diag}(a_1 = 1, \dots, a_g)\ , \quad a_k \in \mathbb{N}\ ,\ a_k \text{ divides } a_{k+1}\ ,$$

and B is a symmetric matrix with negative real part. An Abelian torus with $a_1 = \ldots = a_g = 1$ is called *principally polarized*. Jacobi varieties of Riemann surfaces are principally polarized Abelian tori. Meromorphic functions on Abelian tori are constructed in terms of theta functions, which are defined by their Fourier series.

Definition 31. *Let B be a symmetric $g \times g$ matrix with negative real part. The* theta function *is defined by the following series*

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp\{\frac{1}{2}(Bm, m) + (z, m)\}\ , \qquad z \in \mathbb{C}\ .$$

Here

$$(Bm, m) = \sum_{ij} B_{ij} m_i m_j\ , \quad (z, m) = \sum_j z_j m_j\ .$$

Since B has negative real part, the series converge absolutely and defines an entire function on $\mathbb{C}^g$.

Proposition 7. *The theta function is even,*

$$\theta(-z) = \theta(z)\ ,$$

and possesses the following periodicity property:

$$\theta(z+2\pi\mathrm{i}N+BM) = \exp\{-\frac{1}{2}(BM, M)-(z, M)\}\theta(z)\ , \quad N, M \in \mathbb{Z}^g\ . \tag{1.81}$$

More generally one introduces *theta functions with characteristics* $[\alpha, \beta]$

$$\begin{aligned}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) &= \sum_{m \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}(B(m+\alpha), m+\alpha) + (z + 2\pi\mathrm{i}\beta, m+\alpha)\right\}\\ &= \theta(z + 2\pi\mathrm{i}\beta + B\alpha)\exp\left\{\frac{1}{2}(B\alpha, \alpha) + (z + 2\pi\mathrm{i}\beta, \alpha)\right\}\ , \\ &\quad z \in \mathbb{C}^g,\ \alpha, \beta \in \mathbb{R}^g\ .\end{aligned} \tag{1.82}$$

with the transformation law

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z + 2\pi \mathrm{i} N + BM) = \tag{1.83}$$
$$\exp\left\{-\tfrac{1}{2}(BM, M) - (z, M) + 2\pi\mathrm{i}((\alpha, N) - (\beta, M))\right\} \ \theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z) \ .$$

1.6.2 Theta functions of Riemann surfaces

From now on we consider an Abelian torus which is a Jacobi variety, $\mathbb{C}/\Lambda = Jac(\mathcal{R})$. By combining the theta function with the Abel map, one obtains the following useful map on a Riemann surface:

$$\Theta(P) := \theta(\mathcal{A}_{P_0}(P) - d) \ , \qquad \mathcal{A}_{P_0}(P) = \int_{P_0}^{P} \omega \ . \tag{1.84}$$

Here we incorporated the base point $P_0 \in \mathcal{R}$ in the notation of the Abel map, and the parameter $d \in \mathbb{C}^g$ is arbitrary. The periodicity properties of the theta function (1.81) imply the following

Proposition 8. *$\Theta(P)$ is an entire function on the universal covering $\tilde{\mathcal{R}}$ of $\mathcal{R}$. Under analytical continuation $\mathcal{M}_{a_k}, \mathcal{M}_{b_k}$ along a- and b-cycles on the Riemann surface, it is transformed as follows:*

$$\mathcal{M}_{a_k} \ \Theta(P) = \Theta(P) \ ,$$
$$\mathcal{M}_{b_k} \ \Theta(P) = \exp\{-\tfrac{1}{2}B_{kk} - \textstyle\int_{P_0}^{P} \omega_k + d_k\} \ \Theta(P) \ . \tag{1.85}$$

The zero divisor (Θ) of $\Theta(P)$ on $\mathcal{R}$ is well defined.

Theorem 22. *The theta function $\Theta(P)$ either vanishes identically on $\mathcal{R}$ or has exactly g zeros (counting multiplicities):*

$$\deg(\Theta) = g \ .$$

Suppose $\Theta \not\equiv 0$. As in Sect. 1.4 consider the simply connected model F_g of the Riemann surface. The differential $\mathrm{d}\log\Theta$ is well defined on F_g and the number of zeros of Θ is equal to

$$\deg(\Theta) = \frac{1}{2\pi\mathrm{i}} \int_{\partial F_g} \mathrm{d}\log\Theta(P) \ .$$

using the periodicity properties of Θ we get (see Theorem 12 for notation) for the values of $d\log\Theta$ at the corresponding points

$$\mathrm{d}\log\Theta(Q_j') = \mathrm{d}\log\Theta(Q_j), \ \mathrm{d}\log\Theta(P_j') = \mathrm{d}\log\Theta(P_j) - \omega_j(P_j) \ . \tag{1.86}$$

For the number of zeros of the theta function this implies

$$\deg(\Theta) = \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{g} \int_{a_j} \omega_j = \ .$$

A similar but more involved computation [FK92] of the integral

$$I_k = \frac{1}{2\pi\mathrm{i}} \int_{\partial F_g} \mathrm{d}\log\Theta(P) \int_{P_0}^{P} \omega_k \ .$$

implies the following Jacobi inversion problem for the zeros of (Θ).

Proposition 9. *Suppose $\Theta \not\equiv 0$. Then its g zeros $P_1, \dots, P_g$ satisfy*

$$\sum_{i=1}^{g} \int_{P_0}^{P_i} \omega = d - K \ , \tag{1.87}$$

where K is the vector of Riemann constants

$$K_k = \pi\mathrm{i} + \frac{B_{kk}}{2} - \frac{1}{2\pi\mathrm{i}} \sum_{j \neq k} \int_{a_j} \omega_j \int_{P_0}^{P} \omega_k \ . \tag{1.88}$$

One can easily check that $K \in Jac(\mathcal{R})$ is well defined by (1.88), i.e., it is independent of the integration path. On the other hand, K depends on the choice of the base point P_0.

1.6.3 Theta divisor

Let us denote by J_k the set of equivalence classes (of linearly equivalent divisors, see Sect. 1.5.1) of divisors of degree k. The Abel theorem and the Jacobi inversion imply a canonical identification of J_g and the Jacobi variety

$$J_g \ni D \longleftrightarrow \mathcal{A}(D) \in Jac(\mathcal{R}) \ .$$

The zero set of the theta function of a Riemann surface, which is called the *theta divisor* can be characterized in terms of divisors on $\mathcal{R}$ as follows. Take a non-special divisor $\tilde{D} = P_1 + \ldots + P_g$ and distinguish one of its points $\tilde{D} = P_1 + D,\ D \in J_{g-1}$. Proposition 9 implies that $\theta(\int^P \omega - \mathcal{A}(\tilde{(D)}) - K)$ vanishes in particular at P_1, and one has $\theta(\mathcal{A}(D) - K) = 0$. Since non-special divisors form a dense set, one has this identify for any $D \in J_{g-1}$. Moreover, this identity gives a characterization of the theta divisor [FK92].

Theorem 23. *The theta divisor is isomorphic to the set J_{g-1} of equivalence classes of positive divisors of degree $g-1$:*

$$\theta(e) = 0 \Leftrightarrow \exists D \in J_{g-1}, D \geq 0 : \ e = \mathcal{A}(D) + K \ .$$

For any $D \in J_{g-1}$ the expression $\mathcal{A}(D) + K \in Jac(\mathcal{R})$ is independent of the choice of the initial integration point P_0.

Using this characterization of the theta divisor one can complete the description of Proposition 9 of the divisor of the function Θ

Theorem 24. *Let $\Theta(P) = \theta(\mathcal{A}_{P_0}(P) - d)$ be the theta function (1.84) on a Riemann surface and let the divisor $D \in J_g,\ D \geq 0$ be a Jacobi inversion (1.76) of $d - K$,*

$$d = \mathcal{A}(D) + K \ .$$

Then the following alternative holds:

(i) $\Theta \equiv 0$ iff $i(D) > 0$, i.e., if the divisor D is special,

(ii) $\Theta \not\equiv 0$ iff $i(D) = 0$, i.e., if the divisor D is non-special. In the last case, D is precisely the zero divisor of Θ.

The evenness of the theta function and Theorem 23 imply that $\theta(d - \mathcal{A}(P)) \equiv 0$ is equivalent to the existence (for any P) of a positive divisor D_P of degree $g-1$ satisfying $\mathcal{A}(D) + K - \mathcal{A}(P) = \mathcal{A}(D_P) + K$. Due to the Abel theorem the last identity holds if and only if the divisors D and $D_P + P$ are linearly equivalent, i.e., if there exists a function in $L(-D)$ vanishing at the (arbitrary) point P. In terms of the dimension of $L(-D)$ the last property can be formulated as $l(-D) > 1$, which is equivalent to $i(D) > 0$.

Suppose now that D is non-special. Then, as we have shown above, $\Theta \not\equiv 0$, and Proposition 9 implies for the zero divisor of Θ

$$\mathcal{A}((\Theta)) = \mathcal{A}(D) \ .$$

The non-speciality of D implies $D = (\Theta)$.

Although the vector of Riemann constants K appeared in Proposition 9 just as a result of computation, K plays an important role in the theory of theta functions. The geometrical nature of K is partially clarified by the following

Proposition 10.

$$2K = -\mathcal{A}(C) \ ,$$

where C is a canonical divisor.

Indeed, take an arbitrary positive $D_1 \in J_{g-1}$. Due to Theorem 23 the theta function vanishes at $e = \mathcal{A}(D_1) + K$. Theorem 23 applied to $\theta(-e) = 0$ implies the existence of a positive divisor $D_2 \in J_{g-1}$ with $-e = \mathcal{A}(D_2) + K$. For $2K$ this gives

$$2K = \mathcal{A}(D_1 + D_2) \ .$$

It is not difficult to show that this representation (where $D_1 \in J_{g-1}$ is arbitrary) implies $l(-D_1 - D_2) \geq g$, or equivalently $i(D_1 + D_2) > 0$, i.e., the divisor $D_1 + D_2$ is canonical.

The vanishing of theta functions at some points follows from their algebraic properties.

Definition 32. *Half-periods of the period lattice*

$$\Delta = 2\pi i\alpha + B\beta\ , \qquad \alpha = (\alpha_1, \dots, \alpha_g)\ , \beta = (\beta_1, \dots, \beta_g), \quad \alpha_k, \beta_k \in \left\{0, \frac{1}{2}\right\} .$$

are called half periods or theta characteristics. *A half period is called* even (odd) *if* $4(\alpha, \beta) = 4\sum \alpha_k\beta_k$ *is even (odd).*

We denote the theta characteristic by $\Delta = [\alpha, \beta]$. The theta function $\theta(z)$ vanishes in all odd theta characteristics

$$\theta(\Delta) = \theta(-\Delta + 4\pi i\alpha + 2B\beta) = \theta(-\Delta)\exp(-4\pi i(\alpha, \beta))\ .$$

To any odd theta characteristic Δ there corresponds a positive divisor D_Δ of degree $g - 1$,

$$\Delta = \mathcal{A}(D_\Delta) + K\ . \tag{1.89}$$

Since 2Δ belongs to the lattice of $Jac(\mathcal{R})$, doubling of (1.89) yields

$$\mathcal{A}(2D_\Delta) = -2K = \mathcal{A}(C)\ .$$

The next corollary follows from the Abel theorem.

Corollary 13. *For any odd theta characteristic Δ there exists a holomorphic differential ω_Δ with*

$$(\omega_\Delta) = 2D_\Delta\ . \tag{1.90}$$

In particular all zeros of ω_Δ are of even multiplicity.

Note, that identity (1.90) is an identity on divisors and not only on equivalence classes of divisors.

The differential ω_Δ can be described explicitly in theta functions.

To any point z of the Abelian torus one can associate a number $s(z)$ determined by the condition that all partial derivatives of θ up to order $s(z)-1$ vanish at z and at least one partial derivative of order $s(z)$ does not vanish at z. For most of the points $s = 0$. The points of the theta divisor are precisely those with $s > 0$. In particular, $s(\Delta) > 0$ for any odd theta characteristic Δ. An odd theta characteristic Δ is called *non-singular* iff $s(\Delta) = 1$.

Let $D = P_1 + \ldots + P_{g-1}$ be a positive divisor of degree $g-1$. Consider the function $f(P_1, \dots, P_{g-1}) = \theta(\mathcal{A}(D) + K)$ of $g - 1$ variables. Since f vanishes identically, differentiating it with respect to P_k one sees that the holomorphic differential

$$h = \sum_i \frac{\partial\theta}{\partial z_i}(e)\omega_i$$

with $e = \mathcal{A}(D) + K$ vanishes at all points P_k. Let Δ be an odd non-singular theta characteristic. Then $D_\Delta \in J_{g-1}$ is uniquely determined by the identity (1.89), i.e., $i(D_\Delta) = 1$. Indeed, if D_Δ is not determined by its Abel image then it is linearly equivalent to a divisor $P + D_{g-2},\ D_{g-2} \in J_{g-2}$ with an arbitrary point P. Repeating the arguments with the differential h above we see that it vanishes identically on $\mathcal{R}$, i.e., all the derivatives of the theta function $\theta(\Delta)$ vanish. This contradicts the non-singularity of Δ. Finally, we arrive at the following explicit description the one-dimensional ($i(D_\Delta) = 1$) space of holomorphic differentials vanishing at D_Δ.

Proposition 11. *Let Δ be a non-singular odd theta characteristic and D_Δ the corresponding (1.89) positive divisor of degree $g-1$. Then the holomorphic differential ω_Δ of Corollary 13 is given by the expression*

$$\omega_\Delta = \sum_{i=1}^{g} \frac{\partial \theta}{\partial z_i}(\Delta)\omega_i \ ,$$

where ω_i are normalized holomorphic differentials.

We finish this section with Riemann's complete description of the theta divisor. The proof of this classical theorem can be found for example in [FK92, Lew64]. It is based on considerations similar to the ones in this section, but technically more involved.

Theorem 25. *The following two characterizations of a point $e \in Jac(\mathcal{R})$ are equivalent:*

- *The theta function and all its partial derivatives up to order $s-1$ vanish in e and at least one partial derivative of order s does not vanish at e.*
- *$e = \mathcal{A}(D) + K$ where D is a positive divisor of degree g and $i(D) = s$.*

1.7 Holomorphic line bundles

In this section some results of the previous sections are formulated in the language of holomorphic line bundles. This language is useful for generalizations to manifolds of higher dimension, where one does not have concrete tools as in the case of Riemann surfaces, and where one has to rely on more abstract geometric constructions.

1.7.1 Holomorphic line bundles and divisors

Let (U_α, z_α) be coordinate charts of an open cover $\cup_{\alpha\in A} U_\alpha = \mathcal{R}$ of a Riemann surface. The geometric idea behind the concept of a holomorphic line bundle is the following. One takes the union $U_\alpha \times \mathbb{C}$ over all $\alpha \in A$ and "glues" them together by identifying $(P, \xi_\alpha) \in U_\alpha \times \mathbb{C}$ with $(P, \xi_\beta) \in U_\beta \times \mathbb{C}$ for $P \in U_\alpha \cap U_\beta$ linearly holomorphically, i.e., $\xi_\beta = g(P)\xi_\alpha$ where $g(P) : U_\alpha \cap U_\beta :\to \mathbb{C}$ is holomorphic.

Let us make this "constructive" definition rigorous. Denote by

$$\mathcal{O}^*(U) \subset \mathcal{O}(U) \subset \mathcal{M}(U)$$

the sets of nowhere vanishing holomorphic functions, of holomorphic functions and of meromorphic functions on $U \subset \mathcal{R}$, respectively. A holomorphic line bundle is given by its *transition functions*, which are holomorphic non-vanishing functions $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying

$$g_{\alpha\beta}(P)g_{\beta\gamma}(P) = g_{\alpha\gamma}(P) \qquad \forall P \in U_\alpha \cap U_\beta \cap U_\gamma \; . \tag{1.91}$$

This identity implies in particular $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta}g_{\beta\alpha} = 1$.

Introduce on triples $[P, U_\alpha, \xi],\; P \in U_\alpha, \alpha \in A, \xi \in \mathbb{C}$ the following equivalence relation:

$$[P, U_\alpha, \xi] \sim [Q, U_\beta, \eta] \Leftrightarrow P = Q \in U_\alpha \cap U_\beta,\; \eta = g_{\beta\alpha}\xi \; . \tag{1.92}$$

Definition 33. *The union of* $U_\alpha \times \mathbb{C}$ *identified by the equivalence relation (1.92) is called a* holomorphic line bundle $L = L(\mathcal{R})$*. The mapping* $\pi : L \to \mathcal{R}$ *defined by* $[P, U_\alpha, \xi] \mapsto P$ *is called the canonical projection. The linear space* $L_P := \pi^{-1}(P) \cong \{P\} \times \mathbb{C}$ *is called the* fibre *of* L *over* P*.*

The line bundle with all $g_{\alpha\beta} = 1$ is called *trivial*.

A set of meromorphic functions $\phi_\alpha \in \mathcal{M}(U_\alpha)$ such that $\phi_\alpha/\phi_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ for all α, β is called a *meromorphic section* ϕ of a line bundle $L(\mathcal{R})$ defined by the transition functions

$$g_{\alpha\beta} = \phi_\alpha/\phi_\beta \; .$$

Note that the divisor (ϕ) of the meromorphic section ϕ is well defined by $(\phi)\Big|_{U_\alpha} = (\phi_\alpha)\Big|_{U_\alpha}$. In the same way one defines a line bundle $L(U)$ and its sections on an open subset $U \subset \mathcal{R}$. Bundles are locally trivializable, i.e., there always exist local sections: a local holomorphic section over U_α can be given simply by

$$U_\alpha \ni P \mapsto [P, U_\alpha, 1]. \tag{1.93}$$

One immediately recognizes that holomorphic (Abelian) differentials (see Definitions 17, 19) are holomorphic (meromorphic) sections of a holomorphic line bundle. This line bundle, given by the transition functions

$$g_{\alpha\beta}(P) = \frac{dz_\beta}{dz_\alpha}(P) \; ,$$

is called the *canonical bundle* and denoted by K.

Obviously a line bundle is completely determined by a meromorphic section. In Sect. 1.4 and 1.5.5 we dealt with meromorphic sections directly and formulated results in terms of sections without using the bundle language.

Let L be a holomorphic line bundle (1.92) with trivializations (1.93) on U_α. Local sections

$$U_\alpha \ni P \mapsto [P, U_\alpha, h_\alpha(P)] \; ,$$

where $h_\alpha \in \mathcal{O}^*(U_\alpha)$ define another holomorphic line bundle L' which is called (holomorphically) *isomorphic* to L. We see that fibres of isomorphic holomorphic line bundles can be holomorphically identified $h_\alpha : L(U_\alpha) \to L'(U_\alpha)$. This is equivalent to the following definition.

Definition 34. *Two holomorphic line bundles L and L' are isomorphic if their transition functions are related by*

$$g'_{\alpha\beta} = g_{\alpha\beta}\frac{h_\alpha}{h_\beta} \tag{1.94}$$

with some $h_\alpha \in \mathcal{O}^*(U_\alpha)$.

We have seen that holomorphic line bundles can be described by their meromorphic sections. Therefore it is not suprising that holomorphic line bundles and divisors are intimately related. To each divisor one can naturally associate a class of isomorphic holomorphic line bundles. Let D be a divisor on $\mathcal{R}$. Consider a covering $\{U_\alpha\}$ such that each point of the divisor belongs to only one U_α. Take $\phi_\alpha \in \mathcal{M}(U_\alpha)$ such that the divisor of ϕ_α is presicely the part of D lying in U_α,

$$(\phi_\alpha) = D_\alpha := D\,|_{U_\alpha} \quad .$$

For example take $\phi_\alpha = z_\alpha^{n_i}$, where z_α is a local parameter vanishing at the point $P_i \in U_\alpha$ of the divisor $D = \sum n_i P_i$. The so defined meromorphic section ϕ determines a line bundle L associated with D. If $\phi'_\alpha \in \mathcal{M}(U_\alpha)$ are different local sections with the same divisor $D = (\phi')$, then $h_\alpha = \phi'_\alpha/\phi_\alpha \in \mathcal{O}^*(U_\alpha)$ and ϕ' determines a line bundle L' isomorphic to L. We see that a divisor D determines not a particular line bundle but a class of isomorphic line bundles together with corresponding meromorphic sections ϕ such that $(\phi) = D$. This relation is clearly an isomorphism. Let us denote by $L[D]$ isomorphic line bundles determined by D.

It is natural to get rid of sections in this relation and to describe line bundles in terms of divisors.

Lemma 6. *The holomorphic line bundles $L[D]$ and $L[D']$ are isomorphic if and only the corresponding divisors D and D' are linearly equivalent.*

Indeed, choose a covering $\{U_\alpha\}$ such that each point of D and D' belongs to only one U_α. Take $h \in \mathcal{M}(\mathcal{R})$ with $(h) = D - D'$. This function is holomorphic on each $U_\alpha \cap U_\beta,\ \alpha \neq \beta$. If ϕ is a meromorphic section of $L[D]$ then $h\phi$ is a meromorphic section of $L[D']$, which implies (1.94) for the transition functions. Conversely, let ϕ and ϕ' be meromorphic sections of isomorphic line bundles $L[D]$ and $L[D']$ respectively, $(\phi) = D, (\phi') = D'$. Identity (1.94) implies that $\phi_\alpha h_\alpha/\phi'_\alpha$ is a meromorphic function on $\mathcal{R}$. The divisor of this function is $D - D'$, which yields $D \equiv D'$.

It turns out that Lemma 6 implies a classification of holomorphic line bundles. Namely, every holomorphic line bundle L comes as a bundle associated to the divisor $L = L[(\phi)]$ of a meromorphic section ϕ, and every holomorphic line bundle possesses a meromorphic section. The proofs of the last fact are based on homological methods and are rather involved [GH94, Gu66, Spr81]. We arrive at the following fundamental classification theorem.

Theorem 26. *There is a one to one correspondence between classes of isomorphic holomorphic line bundles and classes of linearly equivalent divisors.*

The degree $\deg D$ is called the *degree* of the line bundle $L[D]$.

Thus, holomorphic line bundles are classified by elements of J_n (see Sect. 1.6.3), where n is the degree of the bundle $n = \deg L$. Due to the Abel theorem and Jacobi inversion, elements of J_n can be identified with the points of the Jacobi variety. Namely, choose some $D_0 \in J_n$ as a reference point. Then due to the Abel theorem the class of divisor $D \in J_n$ is given by the point

$$\mathcal{A}(D - D_0) = \int_{D_0}^{D} \omega \in Jac(\mathcal{R}) \ .$$

Conversely, due to the Jacobi inversion, given some $D_0 \in J_n$, to any point $d \in Jac(\mathcal{R})$ there corresponds $D \in J_n$ satisfying $\mathcal{A}(D - D_0) = d$.

1.7.2 Picard group. Holomorphic spin bundle.

We will not distinguish isomorphic line bundles and denote by $L[D]$ the isomorphic line bundle associated with the divisor class D.

The set of line bundles can be equipped with an Abelian group structure. If L and L' are bundles with transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ respectively, then the line bundle. $L'L^{-1}$ is defined by the transition functions $g'_{\alpha\beta}g^{-1}_{\alpha\beta}$.

Definition 35. *The Abelian group of line bundles on $\mathcal{R}$ is called the* Picard group *of $\mathcal{R}$ and denoted by $Pic(\mathcal{R})$*

Let ϕ and ϕ' be meromorphic sections of L and L' respectively. Then ϕ'/ϕ is a meromorphic section of $L'L^{-1}$. For the divisors of the sections one has $(\phi'/\phi) = (\phi') - (\phi)$. The classification theorem 26 implies the following

Theorem 27. *The Picard group $Pic(\mathcal{R})$ is isomorphic to the group of divisors $Div(\mathcal{R})$ modulo linear equivalence.*

Holomorphic q-differentials of Sect. 1.5.4 are holomorphic sections of the bundle K^q.

Corollary 14. *Holomorphic line bundles L_1, L_2, L_3 satisfy*

$$L_3 = L_2 L_1^{-1}$$

if and only if

$$\deg L_3 = \deg L_2 - \deg L_1 \quad \textit{and} \quad \mathcal{A}(D_3 - D_2 + D_1) = 0 \ ,$$

where D_i are the divisors corresponding to $L_i = L[D_i]$.

For the proof one uses the characterization of line bundles via their meromorphic sections ϕ_1, ϕ_2, ϕ_3 and applies the Abel theorem to the meromorphic function $\phi_3\phi_1/\phi_2$.

Since the canonical bundle K is of even degree one can define a "square root" of it.

Definition 36. *A holomorphic line bundle S satisfying*

$$SS = K$$

is called a holomorphic spin bundle*. Holomorphic (meromorphic) sections of S are called holomorphic (meromorphic)* spinors*.*

Spinors are differentials of order 1/2. In local coordinates they are given by expressions $s(z)\sqrt{dz}$ where $s(z)$ is holomorphic (meromorphic) for holomorphic (meromorphic) spinors.

Proposition 12. *There exist exactly 4^g non-isomorphic spin bundles on a Riemann surface of genus g.*

This fact can be shown using the description of classes of isomorphic holomorphic line bundles by the elements of the Jacobi variety, see the end of Sect. 1.7.1. The classes of linearly equivalent divisors are isomorphic to points of the Jacobi variety

$$D \in J_n \leftrightarrow d = \mathcal{A}(D - nP_0) \in Jac(\mathcal{R}) \ ,$$

where P_0 is a reference point $P_0 \in \mathcal{R}$. For the divisor class D_S of a holomorphic spin bundle Corollary 14 implies

$$\deg D_S = g - 1 \quad \text{and} \quad 2\mathcal{A}(D_S) = \mathcal{A}(C) \ ,$$

where C is the canonical divisor. Proposition 10 provides us with the general solution to this problem,

$$\mathcal{A}(D_S) = -K + \Delta \ ,$$

where K is the vector of Riemann constants and Δ is one of the 4^g half-periods of Definition 32. Due to the Jacobi inversion the last equation is solvable (the divisor $D_S \in J_{g-1}$ is not necessarily positive) for any Δ. We denote by $D_\Delta \in J_{g-1}$ the divisor class corresponding to the half-period Δ and by S_Δ the corresponding holomorphic spin bundle $S_\Delta := L[D_\Delta]$. The line bundles with different half-periods can not be isomorphic since the images of their divisors in the Jacobi variety are different.

Note that we obtained a geometrical interpretation for the vector of Riemann constants.

Corollary 15. *Up to a sign the vector of Riemann constants is the image under the Abel map of the divisor of the holomorphic spin bundle with the zero theta characteristic*

$$K = -\mathcal{A}(D_{[0,0]} - (g-1)P_0) \,.$$

This corollary clarifies the dependence of K_{P_0} on the base point and on the choice of the canonical homology basis.

Finally, let us give a geometric interpretation of the Riemann-Roch theorem. Denote by $h^0(L)$ the dimension of the space of holomorphic sections of the line bundle L.

Theorem 28 (Riemann-Roch theorem). *For any holomorphic line bundle $\pi : L \to \mathcal{R}$ over a Riemann surface $\mathcal{R}$ of genus g*

$$h^0(L) = \deg L - g + 1 + h^0(KL^{-1}) \,. \tag{1.95}$$

This theorem is just a reformulation of Theorem 18. Indeed, let $D = (\phi)$ be the divisor of a meromorphic section of the line bundle $L = L[D]$ and let h be a holomorphic section of L. The quotient h/ϕ is a meromorphic function with the divisor $(h/\phi) \geq -D$. On the other hand, given $f \in \mathcal{M}(\mathcal{R})$ with $(f) \geq -D$ the product $f\phi$ is a holomorphic section of L. We see that the space of holomorphic sections of L can be identified with the space of meromorphic functions $L(-D)$ defined in Sect. 1.5.2. Similarly, holomorphic sections of KL^{-1} can be identified with Abelian differentials with divisors $(\Omega) \geq D$. This is the space $H(D)$ of Sect. 1.5.2 and its dimension is $i(D)$. Now the claim follows from (1.73).

The Riemann-Roch theorem does not allow us to compute the number of holomorphic sections of a spin bundle. The identity (1.95) implies only that $\deg S = g - 1$. A computation of $h^0(S)$ is a rather delicate problem. It turns out that the dimension of the space of holomorphic sections of S_Δ depends on the theta characteristic Δ and is even for even theta characteristics and odd for odd theta characteristics [Ati71]. Spin bundles with non-singular theta characteristics have no holomorphic sections if the characteristic is even and have a unique holomorphic section if the characteristic is odd.

Results of Sect. 1.6.3 allow us to show this easily for odd theta characteristics. Take the differential ω_Δ of Corollary 13. The square root of it $\sqrt{\omega_\Delta}$ is a holomorphic section of S_Δ.

Proposition 13. *Spin bundles S_Δ with odd theta characteristics Δ possess global holomorphic sections.*

If Δ is a non-singular theta characteristic then the corresponding positive divisor D_Δ of degree $g-1$ is unique (see the proof of Proposition 11). This implies the uniqueness of the differential with $(\omega) = D_\Delta$ and $h^0(S_\Delta) = 1$. This holomorphic section is given by

$$\sqrt{\sum_{i=1}^{g} \frac{\partial\theta}{\partial z_i}(\Delta)\omega_i} \; .$$

1.8 Schottky Uniformization

1.8.1 Schottky group

Let $C_1, C_1', \ldots, C_N, C_N'$ be a set of $2N$ mutually disjoint Jordan curves on $\hat{\mathbb{C}}$. They comprise the boundary of a domain $\Pi \subset \hat{\mathbb{C}}$ which is a topological sphere with $2N$ holes (see Fig. 1.22). Let us assume that the curves C_n and C_n' are identified by $\sigma C_n = C_n'$ where σ is a loxodromic transformation,

$$\frac{\sigma_n z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n} \; , \quad |\mu_n| < 1, \quad n = 1, \ldots, N \; , \tag{1.96}$$

which maps the exterior of C_n to the interior of C_n'. The points A_n, B_n are the fixed points of this transformation.

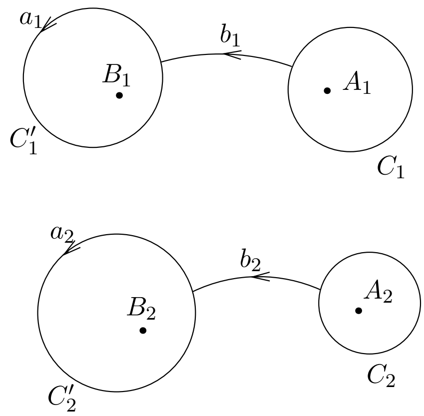

Fig. 1.22. The fundamental domain Π of a Schottky group with a canonical homology basis. The cycles a_n coincide with the positively oriented C_n' ; b_n run on Π between the points $z_n \in C_n$ and $\sigma_n z_n \in C_n'$.

Fractional-linear transformations can be canonically identified with the elements of the matrix group PSL(2,$\mathbb{C}$):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\leftrightarrow\; \sigma z = \frac{az - b}{cz - d} \; , \quad a, b, c, d \in \mathbb{C} \; , \quad ad - bc = 1 \; . \tag{1.97}$$

The canonical matrix representation of the transformation (1.96) is as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{A-B} \begin{pmatrix} A\sqrt{\mu} - \frac{B}{\sqrt{\mu}} & AB(\frac{1}{\sqrt{\mu}} - \sqrt{\mu}) \\ \sqrt{\mu} - \frac{1}{\sqrt{\mu}} & \frac{A}{\sqrt{\mu}} - B\sqrt{\mu} \end{pmatrix} .$$

The derivative of the transformation (1.97) is $\sigma' z = (cz+d)^{-2}$. It is an isometry for the points satisfying $|cz+d| = 1$. These points comprise the *isometric circle* C_σ of σ with the center at $-\frac{d}{c}$ and the radius $\frac{1}{|c|}$. The isometric circles C_σ and $C_{\sigma^{-1}}$ of the transformations (1.96) have equal radii and are disjoint.

Definition 37. *The group G generated by the transformations $\sigma_1, \ldots, \sigma_N$ is called a* Schottky group. *If all the boundary curves C_n, C'_n are circles the Schottky group is called classical.*

These groups were introduced in [Sch87] by Schottky who has established their fundamental properties and investigated their automorphic functions. The domain Π is the fundamental domain of the group G. The existence of nonclassical (for an arbitrary system of generators) Schottky groups is shown in [Mar74]. A special case of classical Schottky groups are the Schottky groups with fundamental domains bounded by isometric circles of their generators. General Schottky groups can be characterized as free, purely loxodromic finitely generated discontinuous groups [Mas67].

The limit set $\Lambda(G)$ of the Schottky group is the closure of the fix points of all its elements. The discontinuity set $\Omega(G) = \mathbb{C} \setminus \Lambda(G)$ factorized with respect to G is a compact Riemann surface $\Omega(G)/G$ of genus N. According to the classical uniformization theorem [Fo29] any compact Riemann surface of genus N can be represented in this form.

Theorem 29 (Schottky uniformization). *Let $\mathcal{R}$ be a Riemann surface of genus N with a set of homologically independent simple disjoint loops $v_1, \ldots, v_N$. Then there exists a Schottky group G such that*

$$\mathcal{R} = \Omega(G)/G ,$$

and the fundamental domain $\Pi(G)$ is conformally equivalent to $\mathcal{R}$ cut along the loops $v_1, \ldots, v_N$.

Under the Schottky uniformization the loops v_n are mapped to the boundary curves C_n, C'_n. The loop system $v_1, \ldots, v_N$ generates a subgroup of the homology group $H_1(\mathcal{R}, \mathbb{Z})$. Two loop systems generating the same subgroup determine the same Schottky group but with a different choice of generators. The Schottky groups G and G' corresponding to the loop systems generating different subgroups of $H_1(\mathcal{R}, \mathbb{Z})$ are different.

The Schottky group is parametrized by the fix points $A_1, B_1, \ldots, A_N, B_N$ of the generators and their trace parameters $\mu_1, \ldots, \mu_N$. The Schottky groups G and G' with the parameters $A_1, B_1, \ldots, A_N, B_N$ and $A'_1, B'_1, \ldots, A'_N, B'_N$ which differ by a common fractional-linear transformation uniformize the same

Riemann surface. This parameter counting gives the correct number $3N-3$ for the complex dimension of the moduli space of Riemann surfaces of genus N.

It is unknown whether every Riemann surface can be uniformized by a classical Schottky group.

The Schottky uniformization of $\mathcal{R}$ is determined by a half basis of $H_1(\Omega(G)/G,\mathbb{Z})$, and it is natural to choose a canonical basis of $H_1(\mathcal{R},\mathbb{Z})$ respecting this structure. Such a canonical basis of cycles is illustrated in Fig. 1.22: the cycle a_n coincides with the positively oriented curve C_n', and the cycle b_n connects the points $z_n \in C_n$ and $\sigma_n z_n \in C_n'$, and the b-cycles are mutually disjoint.

1.8.2 Holomorphic differentials as Poincaré series

Denote by G_n the subgroup of the Schottky group G generated by σ_n. The cosets G/G_n and $G_m\backslash G/G_n$ are the sets of all elements

$$\sigma = \sigma_{i_1}^{j_1}\dots\sigma_{i_k}^{j_k}\ , \quad i\in\{1,\dots,N\}\ ,\ j\in\mathbb{Z}\setminus\{0\}\ ,$$

such that $i_k \neq n$ and for $G_m\backslash G/G_n$ in addition $i_1\neq m$.

The following theorem is classical (see [Bur92, Bak97]).

Theorem 30. *If the Poincaré series*

$$\omega_n = \sum_{\sigma\in G/G_n}\left(\frac{1}{z-\sigma B_n}-\frac{1}{z-\sigma A_n}\right)\mathrm{d}z \tag{1.98}$$

are absolutely convergent on $\Pi(G)$, they are holomorphic differentials of the Riemann surface $\Omega(G)/G$ normalized in the canonical basis shown in Fig. 1.22. The period matrix is

$$\begin{aligned} B_{nm} &= \sum_{\sigma\in G_m\backslash G/G_n}\log\{B_m,\sigma B_n,A_m,\sigma A_n\}\ ,\quad m\neq n\ ,\\ B_{nn} &= \log\mu_n + \sum_{\sigma\in G_n\backslash G/G_n}\log\{B_n,\sigma B_n,A_n,\sigma A_n\}\ , \end{aligned} \tag{1.99}$$

where the curly brackets denote the cross-ratio

$$\{z_1,z_2,z_3,z_4\} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}\ .$$

The series (1.98) have no poles in $\Pi(G)$. The normalization $\int_{a_m}\omega_n = 2\pi\mathrm{i}\delta_{nm}$ follows from a computation of residues. Indeed, if $\sigma=\sigma_{i_1}^{j_1}\dots\sigma_{i_k}^{j_k}$, $\sigma\neq I$, then both the points σB_n and σA_n are inside of C_{i_1}' when $j_1>0$, and inside of C_{i_1} when $j_1<0$. Only for $\sigma=I$ the images σB_n and σA_n are separated: B_n is inside of C_n' and A_n is inside of C_n.

The series (1.98) are (-2)-dimensional Poincaré series and can be written in a slightly different form

$$\omega_n = \sum_{\sigma \in G_n \backslash G} \left(\frac{1}{\sigma z - B_n} - \frac{1}{\sigma z - A_n} \right) \sigma' z \mathrm{d}z \ , \quad \sigma' z = \frac{1}{(cz+d)^2} \ .$$

In this form it is easy to see that $\omega_n(\sigma z) = \omega_n(z)$, so ω_n is a holomorphic differential on $\mathcal{R} = \Omega(G)/G$. Using the invariance of the cross-ratio with respect to fractional-linear transformations

$$\{\sigma z_1, \sigma z_2, \sigma z_3, \sigma z_4\} = \{z_1, z_2, z_3, z_4\}$$

one can derive (1.99) from the definition of the period matrix.

The problem of convergence of Poincaré series describing holomorphic differentials is a non-trivial problem of crucial importance since one cannot rely on computations with divergent series. This problem is ignored in some applied papers. For general Schottky groups and even for classical ones the (-2)-dimensional Poincaré theta series can be divergent. However, the convergence of these series is guaranteed if the fundamental domain $\Pi(g)$ is "circle decomposable":

Assume that the Schottky group is classical and that $2N-3$ circles $L_1, \ldots, L_{2N-3}$ can be fixed on the fundamental domain $\Pi(G)$ so that the following conditions are satisfied:

- (i) The circles $L_1, \ldots, L_{2N-3}, C_1, C_1', \ldots, C_N, C_N'$ are mutually disjoint,
- (ii) The circles $L_1, \ldots, L_{2N-3}$ divide $\Pi(G)$ into $2N-2$ regions $T_1, \ldots, T_{2N-2}$,
- (iii) Each T_i is bounded by exactly three circles.

Such Schottky groups are called *circle decomposable* (see Fig. 1.23 for an example of a circle decomposable Schootky group). In particular, each Schottky group which has an invariant circle is circle decomposable by circles orthogonal to the invariant circle.

The following elegant geometric convergence result is due to Schottky [Sch87] (see also [FK65] for a proof).

Theorem 31 (Schottky condition). *(-2)-dimensional Poincaré theta series corresponding to a circle decomposable Schottky group is absolutely convergent on the whole fundamental domain of G.*

The convergence of (-2)-dimensional Poincaré theta series can be proved also in the case when the circles $C_n, C_n', \ n = 1, \ldots, N$ are small and far enough apart. The corresponding estimations can be found in [Bur92, Bak97] and in the contribution by Schmies in this volume.

The convergence of the Poincaré theta series can be characterized in terms of the metrical properties of the limiting set $\Lambda(G)$. If ν is the minimal dimension for which the $(-\nu)$-dimensional Poincaré theta series converge absolutely, then the Hausdorff measure of $\Lambda(G)$ is equal to $\frac{\nu}{2}$. In particular, the

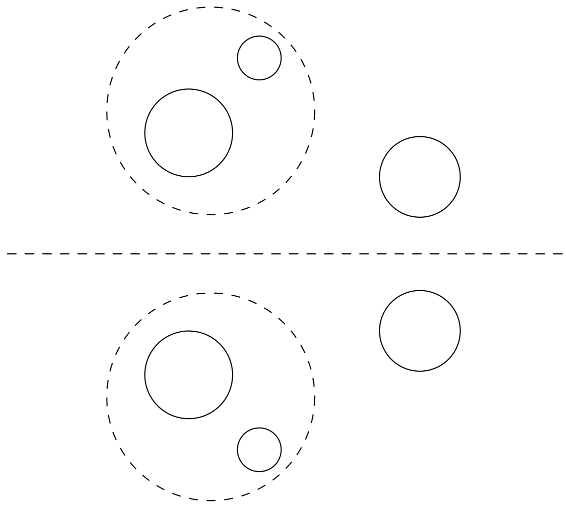

Fig. 1.23. The fundamental domain of a circle decomposable Schottky group.

1-dimensional Hausdorff measure of $\Lambda(G)$ of a Schottky group with divergent (-2)-dimensional Poincaré theta series is infinite. Examples of such classical Schottky groups with fundamental domains bounded by isometric circles can be found in [Myr16, Aka67]. The class of Schottky groups with convergent (-2)-dimensional Poincaré theta series is geometrically characterized in [Bow79].

1.8.3 Schottky uniformization of real Riemann surfaces

As it was mentioned in the preceding sections the problem of convergence of the Poincaré theta series for the holomorphic differentials is of crucial importance. Another important problem for the practical application of Schottky uniformization is to determine the *Schottky space* $S = (A_1, B_1, \mu_1, \dots, A_N, B_N, \mu_N) \subset \mathbb{C}^{3N}$ of the uniformizing Schottky groups. Both problems are so difficult that solving them for general Riemann surfaces seems hopeless.

The situation is more fortunate in the case of real Riemann surfaces, which is the most important for applications. In this case one can find a Schottky uniformization with convergent Poincaré series and describe the Schottky space S [Bob88, BBE+94]. Here we present the main ideas of this method.

Definition 38. *A Riemann surface* $\mathcal{R}$ *with an anti-holomorphic involution* $\tau : \mathcal{R} \to \mathcal{R}$ *is called a* real Riemann surface. *The connected components* $X_1, \dots, X_m$ *of the set of fix points of* τ *are called real ovals. If* $\mathcal{R} \backslash \{X_1, \dots, X_m\}$ *has two connected components the Riemann surface is called* of decomposing type.

There are real Riemann surfaces without fix point of τ. Let $\mathcal{R}$ be of decomposing type and $\mathcal{R}_+$ and $\mathcal{R}_-$ be two components of $\mathcal{R} \setminus \{X_1, \ldots, X_m\}$. Both $\mathcal{R}_\pm$ are Riemann surfaces of type (g, m), i.e., they are homeomorphic to a surface of genus $g = \frac{N+1-m}{2}$ with m boundary components.

Theorem 32. *Every real Riemann surface of decomposing type possesses a Schottky uniformization by a Fuchsian group G of the second kind. The Poincaré theta series of dimension -2 of G are absolutely convergent.*

The main idea behind this theorem is that in this case the Schottky group is of Fuchsian type. Indeed, consider the classical Fuchsian uniformization of the surface $\mathcal{R}_+ = H/G$. Here H is the upper half plane $H = \{z \in \mathbb{Z}, \Im z > 0\}$ and the group G is a purely hyperbolic Fuchsian group of the second kind [Fo29]. The matrix elements (1.97) of all the group elements of G are real and satisfy $|a+d| > 2$. The group G is generated by the hyperbolic transformations $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ and $\gamma_1, \ldots, \gamma_m,\ m > 0$, satisfying the constraint

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \gamma_1 \ldots \gamma_m = I \ .$$

Extending the action of G to the lower half-plane $\overline{H} = \{z \in \mathbb{Z}, \Im z < 0\}$ we obtain another component $\mathcal{R}_- = \overline{H}/G$. The elements

$$\sigma_i = \alpha_i, \sigma_{g+i} = \beta_i, \sigma_{2g+j} = \gamma_j, \quad i = 1, \ldots, g;\ j = 1, \ldots, m-1$$

acting on the whole Riemann sphere $\hat{\mathbb{C}}$ generate a free, purely hyperbolic group, which is a Schottky group uniformizing the Riemann surface $\mathcal{R}$. It possesses an invariant circle, which is the real line, and therefore is circle decomposable. The convergence of the Poincaré series follows from Theorem 31. Note that the Schottky group is classical since the fundamental domain of the Fuchsian group can be chosen to be bounded by geodesics in the hyperbolic geometry. These geodesics are arcs of circles orthogonal to the real line.

The Schottky uniformization of real Riemann surfaces of decomposing type described above looks as follows. The circles $C_n, C_n',\ n = 1, \ldots, N$ are orthogonal to the real axis and their discs are disjoint (see Fig. 1.24 for the special case of M-curves). The order of circles is arbitrary. Every pair C_n, C_n' determines a hyperbolic transformation σ_n. The transformations $\sigma_1, \ldots, N$ generate a Schottky group uniformizing a real Riemann surface of decomposing type. The number of real ovals is determined by the arrangement of the circles $C_n, C_n',\ n = 1, \ldots, N$. The Schottky parameters are real,

$$(A_1, B_1, \mu_1, \ldots, A_N, B_N, \mu_N) \in \mathbb{R}^{3N}, \quad 0 < \mu_n < 1, n = 1, \ldots, N \ .$$

The description of the Schottky space can be obtained from an analysis of the invariant lines of the group elements of G (see [Kee65, Nat04]). We present here the result for the case of M-curves.

A real Riemann surface $\mathcal{R}$ of genus N with maximal possible number $m = N + 1$ of real ovals is called an *M-curve*. The real ovals decompose it

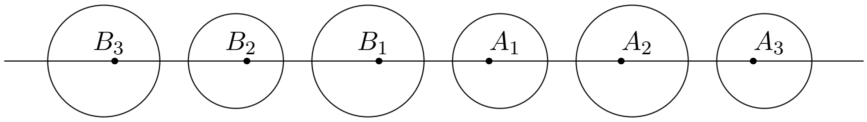

Fig. 1.24. Schottky uniformization of an M-curve.

into two components $\mathcal{R}_\pm$ which are topological spheres with $N+1$ holes. The Schottky space S is described as follows [Bob88, BBE+94]:

$$B_N < B_{N-1} < \ldots < B_1 < A_1 < \ldots < A_N \ , \quad 0 < \sqrt{\mu_n} < 1, n = 1, \ldots, N \ ,$$
$$\{B_n, B_{n+1}, A_n, A_{n+1}\} > \left(\frac{\sqrt{\mu_n} + \sqrt{\mu_{n+1}}}{1 + \sqrt{\mu_n \mu_{n+1}}} \right)^2 , \quad n = 1, \ldots, N-1 \ .$$

1.8.4 Schottky uniformization of hyperelliptic M-curves

By imposing an additional symmetry to the Schottky data one obtains hyperelliptic Riemann surfaces. The additional constraint

$$B_n = -A_n, \quad n = 1, \ldots, N \ , \tag{1.100}$$

in the previous description of M-curves gives all hyperelliptic M-curves. All the considerations simplify in this case (see [BBE+94] for details).

In particular the fundamental domain Π of the Schottky group can be chosen symmetric with respect to the involution $\pi z = -z$. The boundary circles C_n, C'_n can be chosen to be the isometric circles of the corresponding hyperbolic generators. The center c_n and the radius r_n of C_n are as follows:

$$c_n = A_n \frac{1 + \mu_n}{1 - \mu_n} \ , \quad r_n = 2A_n \frac{\sqrt{\mu_n}}{1 - \mu_n} \ .$$

The involutions $\tau z = \bar{z}$ and $\tilde{\tau} = \tau \pi z = -\bar{z}$ are anti-holomorphic, and τ is the one with $N+1$ real ovals. The Schottky group G is the subgroup of index 2 of the group generated by the inversions i_n in the circles C_n and $\tilde{\tau}$, in particular $\sigma_n = \tilde{\tau} i_n$, $\sigma_n^{-1} = i_n \tilde{\tau}$. The intersection points of the circles C_n with the real axis

$$z_n^\pm = A_n \frac{1 \pm \sqrt{\mu_n}}{1 \mp \sqrt{\mu_n}}$$

as well as $z = 0$ and $z = \infty$ are the fix points of the hyperelliptic involution π.

The reduction (1.100) simplifies the period matrix,

$$B_{nm} = \sum_{\sigma \in G_m \backslash G / G_n} \log \left(\frac{A_m - \sigma(A_n)}{A_m - \sigma(-A_n)} \right)^2 ,$$
$$B_{nn} = \log \mu_n + \sum_{\sigma \in G_n \backslash G / G_n} \log \left(\frac{A_n - \sigma(A_n)}{A_n - \sigma(-A_n)} \right)^2$$

and the description of the Schottky set,

$$0 < A_1 < \ldots < A_N, \quad 0 < \sqrt{\mu_n} < 1 \;, n = 1, \ldots, N \;,$$
$$\left(\frac{1-\sqrt{\mu_n}}{1-\sqrt{\mu_n}}\right)\left(\frac{1-\sqrt{\mu_{n+1}}}{1-\sqrt{\mu_{n+1}}}\right) > \frac{A_n}{A_{n+1}}, \quad n = 1, \ldots, N-1 \;.$$

A meromorphic function $\lambda : \mathcal{R} \to \hat{\mathbb{C}}$ with double pole (at $z = \infty$) defining a two-sheeted covering (see Sect. 1.5.5) is given by the Poincaré theta series

$$\lambda(z) = \sum_{\sigma \in G} ((\sigma z)^2 - (\sigma 0)^2) \;.$$

The corresponding hyperelliptic curve is

$$\mu^2 = \lambda \prod_{n=1}^{N} (\lambda - \lambda(z_n^-))(\lambda - \lambda(z_n^+)) \;.$$

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