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Lectures on String Theory

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ABSTRACT: The course covers the basic concepts of modern string theory. This includes covariant and light-cone quantisation of bosonic and fermionic strings, geometry and topology of string world-sheets, vertex operators and string scattering amplitudes, world-sheet and space-time supersymmetries, elements of conformal field theory, Green-Schwarz superstrings, strings in curved backgrounds, low-energy effective actions, D-brane physics.

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1. Introduction to the course

1.1 Historical remarks

String theory arose at the end of sixties in an attempt to describe the theory of strong interactions. In 1969 Veneziano found a beautiful formula for the scattering amplitude of four particles. This amplitude comprised many features that physicists expected to be found in the theory of strong interactions. It was realized very soon by Nambu and Susskind that the underlying dynamical object from which the Veneziano formula can be derived is a *relativistic string*. The fundamental difference of strings from the theory of point particles is that strings are extended, one-dimensional, objects; when string moves through space and time it sweeps two-dimensional surface called the string "world-sheet". The strings can be of two types: open – with topology of an interval and closed – with topology of a circle.

Subsequent investigations revealed however severe difficulties to treat the string as the theory of strong interactions. These difficulties are

1. Existence of a "critical dimension".

2. Existence of a massless spin two particle which is absent in the hadronic world.

If one tries to construct *the quantum mechanics* of relativistic strings one finds that mathematically consistent theory exists if and only if the dimension of space-time where string propagates is 26. The number 26 was named the *"critical dimension"*. On the one hand, it was pretty remarkable and unexpected to find an example of physical theory which puts restrictions on space-time where it is defined. On the other hand, it was certainly not clear why a theory that shared at least some qualitative features with hadronic physics should exist in 26 dimensions only. A subsequent discovery of QCD (Quantum Chromo Dynamics) as the most appropriate candidate to describe the theory of strong interactions led to a considerable loss of interest to string theory.

In 1974, Scherk and Schwarz came up with a proposal to completely alter the view on string theory. They suggested to consider the massless spin two particle absent in the hadronic world as the *graviton* – the quantum of the gravitational interaction. Indeed, this particle neatly fits the properties of the graviton – string theory predicts that this particle interacts according to the standard laws of General Relativity. Gravitational interactions have a natural scale, called the Planck mass, which is around 10^{19} GeV. This is a huge number in comparison with characteristic energies of hadronic physics, $100 - 200$ MeV. Thus, according to their view, string theory could provide the unifying description of all the particles and matter forces, including gravity and it operates on a new fundamental scale.

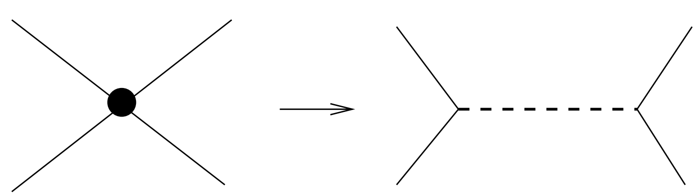
Even if one accepts that quantum mechanics of relativistic strings can be defined in the unusual number 26 of the space-time dimensions, another problem arises. Such string does not contain fermionic degrees of freedom and it predicts the existence of a particle with the negative mass squared: $m^2 < 0$. Such a particle, *tachyon*, is a source of instability and its existence indicates that either the theory is ill-defined or it is formulated around a "wrong" ground state, or as physicists say, around a "wrong" vacuum. Critical dimension, tachyon and absence of fermions were the puzzling features the string theory had to face.

The status of string theory changed again with the discovery of supersymmetry. All universe is made of two fundamental types of particles: bosons and fermions. Fermions constitute all the matter and bosons mediate interactions of the matter particles. Supersymmetry is a new type of symmetry between bosons and fermions (Wess and Zumino 1974). Many physicists hope today that supersymmetry could provide an underlying principle for unification of all interactions.

The first success in incorporating supersymmetry in string theory was achieved in 1971 by Ramond, who constructed a string analogue of the Dirac equation (the spinning string). Shortly afterwards, Neveu and Schwarz constructed a new bosonic string theory. They realized that the two constructions were different facets of a single theory - an interacting superstring theory containing Neveu and Schwarz's bosons and Ramond's fermions. The supersymmetry of the two-dimensional string world sheet was recognized by Gervais and Sakita in 1971. This was advent of the NSR (Neveu-Ramond-Schwarz) *superstring.*

In 1972 Schwarz demonstrated the consistency of the superstring theory in 10 dimensions. Instead of 26 found for purely bosonic string, the critical dimension for the NSR string appears to be 10. In 1977 Gliozzi, Scherk and Olive realized that further conditions should be imposed on the spectrum (the GSO projection mechanism) of the NSR string which lead to both the so-called *space-time supersymmetry* (to compare with the world-sheet supersymmetry mentioned above) and to the removal of tachyon. Thus, superstring theory has at least two advantages in comparison with bosonic strings: critical dimension $10 < 26$ and the absence of tachyon. It also turned out that the GSO projection can be imposed in two different ways which lead to two different types of superstrings, called the Type IIA and Type IIB.

String theories have a natural particle limit, when the length of string vanishes. In this limit superstrings give rise to the low-energy effective theories, known as *supergravities.* These theories can be defined in a way completely independent of string theory: they can be thought of as supersymmetric generalizations of the pure Einstein gravity. As is known, attempts to quantize gravity in the standard framework of quantum mechanics fail because gravity is a non-renormalizable theory (there are infinitely many divergent graphs with any number of external legs and with an arbitrarily high index of divergence, cf. the course on Quantum Field Theory). Supersymmetric theories tend to be less divergent than non-supersymmetric ones which gave initially a hope that supersymmetry could cure the nonrenormalizable infinities of the quantum gravity. It seems that supergravities themselves are still not capable to solve the divergency problem[1]. Quite remarkably, there is a strong evidence that the divergency problem of quantum (super)gravities is resolved by string theory.



Resolution of the four-fermi interaction. At high energies the weak force is mediated by a heavy boson.

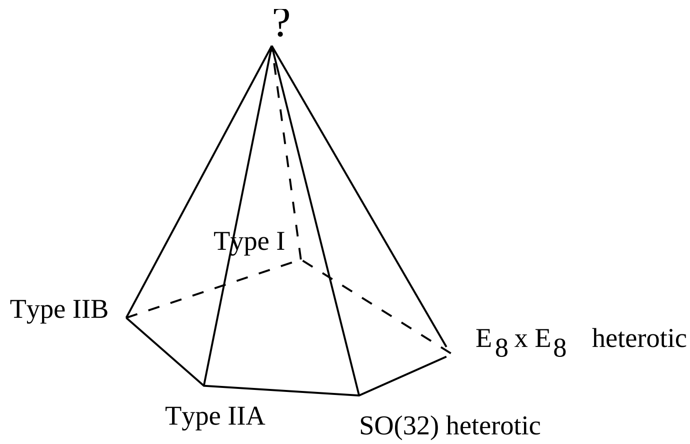
[1]For instance, it is unknown if the so-called $\mathcal{N} = 8$ supergravity is finite or not.

To get a better feeling why it happens it is convenient to envoke an analogy with the theory of weak interactions. Trying to describe the decay of the neutron by the Fermi-type Lagrangian containing the quartic, point-like, interaction vertex one finds irremovable ultraviolet divergencies at higher loop orders. Again, the reason is that the corresponding theory of Fermi-interactions in non-renormalizable. The solution of this problem lies in a fact that at higher energies (more than 100 GeV), the pointlike vertex gets resolved and the interaction is mediated by a heavy W-boson. In the new theory one has qubic vertices and this ultimately makes the theory renormalizable.

Very similar phenomenon occurs in string theory. Expanding the Einstein-Hilbert action $\sqrt{-g}R$ one gets more and more complicated point-like vertices which render the theory non-renormalizable, analogously to the old Fermi theory. In opposite, in string theory all these vertices get dissolved by the exchange of the massive string states. String states form an infinite tower of particles of arbitrarily high mass and spin and all of them participate in the interaction process. Resolving the ultraviolet divergency problem string theory appears to be a natural candidate for the theory of quantum gravity.

There is a still an important question how to relate the superstring theory defined in a ten-dimensional space-time with a our conventional four-dimensional physics. The basic approach to obtain four-dimensional theories is along the old ideas of Kaluza and Klein and it consists in *compactifying* of the ten-dimensional theory down to four dimensions. In this case, the four string coordinates remain uncompactified, while the other six are curled up and parametrize a compact space of a very small size (of the order of the Plank length). It appears that the internal space cannot be completely arbitrary – it must have vanishing Ricchi-curvature.

One of the major obstacles to build a unified theory is the left-right asymmetry recorded in the present days experiments. A theory in which there is an asymmetry between the left and the right must contain *chiral fermions*. Chiral fermions are usually a source of anomalies, i.e., of breakdown of classical symmetries by quantum effects. Anomalies render a theory eventually inconsistent. Only in some special, "rear" cases anomalies cancel (as it happens for instance for a standard generation of quarks and leptons, cf. the course on the Standard Model). In higher space-time dimensions it becomes even more non-trivial to achieve cancellation of anomalies.



String theories

Remarkably, in 1983, Alvarez-Gaumé and Witten demonstrated that anomaly cancelled out in the chiral Type IIB supergravity theory. This was nice, but the corresponding theory was still far from phenomenology. Shortly afterwards Green and Schwarz (1984) showed that cancellation of anomalies in the open superstring theories singles out two gauge groups, namely SO(32) and $E_8 \times E_8$. The gauge groups SO(32) and $E_8 \times E_8$ are realized in the so-called "heterotic string" which is a hybrid of the old $d = 26$ dimensional bosonic string and the $d = 10$ superstring. It was shown that compactifying the theory down to four dimensions one can obtain several generations of the chiral fermions. It was a first example of a string theory whose low-energy limit was not in an immediate conflict with all known physics.

One of the important questions is why there exists non a unique but several consistent string theories. With a recent advent of the string dualities and D-branes an interesting new picture start to emerge (but still very far from being complete), according to which different superstring theories provide different descriptions of the one and the same theory valid in different regimes of the coupling constant parameters.

Recommended literature:

1. M. Green, J. Schwarz and E. Witten, "Superstring Theory", volume I and II, *Cambridge University Press*, 1987.

2. D. Lüst and S. Theisen, "Lectures on string theory", *Lecture Notes in Physics, Springer Verlag*, 1989.

3. 't Hooft, "Introduction to String Theory", *Lecture Notes*, 2005.

4. B. Zwiebach, "A first course in string theory", *Cambridge university press*, 2004.

5. J. Polchinski, "String theory", volume I and II, *Cambridge university press*, 1998.

2. Relativistic particle

Consider relativistic particle of mass m moving in d-dimensional Minkowski space: $\eta_{\mu\nu} = (-1, +1, +1, \ldots, +1)$.

Action

$$S = -\alpha \int_{s_0}^{s} ds$$

Note that $\int_{s_0}^{s} ds$ has maximum along straight lines, this explains the sign "-" in front of the action.

Embedding $x^\mu \equiv x^\mu(\tau)$:

$$ds = \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}}d\tau \equiv \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

If $x^\mu = (c\tau, \vec{x})$ then

$$ds = \sqrt{c^2 - \vec{v}^2}\,, \qquad\qquad \vec{v} = \frac{d\vec{r}}{d\tau}$$

Thus, the action is

$$S = -\alpha c \int_{\tau_0}^{\tau_1} \sqrt{1 - \frac{\vec{v}^2}{c^2}}d\tau$$

The Lagrangian in the non-relativistic limit

$$\mathscr{L} = -\alpha c\sqrt{1 - \frac{\vec{v}^2}{c^2}}d\tau = -c\alpha\left(1 - \frac{\vec{v}^2}{2c^2}\right) + \ldots = -\alpha c + \frac{\alpha\vec{v}^2}{2c} + \ldots$$

To get the standard kinetic energy one has to identify

$$\alpha = mc$$

In what follows we will work in units in which $c = 1$.

The action is invariant under reparametrizations of τ:

$$\delta x^\mu = \xi(\tau)\partial_\tau x^\mu \quad \text{as long as} \quad \xi(\tau_0) = \xi(\tau_1) = 0$$

Let us show this

$$\begin{aligned}
\delta(\sqrt{-\dot{x}_\mu\dot{x}^\mu}) &= \frac{1}{2\sqrt{-\dot{x}_\mu\dot{x}^\mu}}(-2\dot{x}^\nu\delta\dot{x}_\nu) = -\frac{1}{\sqrt{-\dot{x}_\mu\dot{x}^\mu}}\dot{x}^\nu\partial_\tau(\xi\dot{x}_\nu) = \\
&= -\frac{1}{\sqrt{-\dot{x}_\mu\dot{x}^\mu}}\left[\dot{x}^\nu\dot{x}_\nu\dot{\xi} + \xi\dot{x}^\nu\ddot{x}_\nu\right] = -\frac{1}{\sqrt{-\dot{x}_\mu\dot{x}^\mu}}\dot{x}^\nu\dot{x}_\nu\dot{\xi} + \xi\partial_\tau(\sqrt{-\dot{x}_\mu\dot{x}^\mu}) \\
&= \sqrt{-\dot{x}_\mu\dot{x}^\mu}\dot{\xi} + \xi\partial_\tau(\sqrt{-\dot{x}_\mu\dot{x}^\mu}) = \partial_\tau(\xi\sqrt{-\dot{x}_\mu\dot{x}^\mu})
\end{aligned}$$

Therefore, we arrive at

$$\delta S = -m \int_{\tau_0}^{\tau_1} \partial_\tau(\xi\sqrt{-\dot{x}_\mu \dot{x}^\mu}) = -m\Big[\xi\sqrt{-\dot{x}_\mu \dot{x}^\mu}\Big]|_{\tau=\tau_0}^{\tau=\tau_1} = 0\,,$$

i.e., the action is indeed invariant w.r.t. the *local* reparametrization transformations.

The most elegant way to quantize a system is to use the Hamiltonian formalism[2]. Let us therefore try to develop the Hamiltonian (canonical) formalism for the relativistic particle. The canonical momenta

$$p_\mu = \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} = -m\frac{\partial}{\partial \dot{x}^\mu}(\sqrt{-\dot{x}_\mu \dot{x}^\mu}) = m\frac{\dot{x}_\mu}{\sqrt{-\dot{x}_\mu \dot{x}^\mu}}$$

We notice that

$$p^2 \equiv p_\mu p^\mu = -m^2$$

Thus, we see that the canonical momenta are *not independent*, rather they obey the constraint

$$\phi \equiv p^2 + m^2 = 0 \tag{2.1}$$

Constraints which follow just from the definition of the canonical momenta without using equations of motion are called primary constraints. Mass-shell condition for the point particle is a primary constraint.

In general the number of primary constraints is equal to the number of zero eigenvalues of the Hessian matrix:

$$H_{\mu\nu} = \frac{\partial p_\mu}{\partial \dot{x}^\nu} = \frac{\partial^2 \mathscr{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$$

For the relativistic particle we have only one eigenvector $\dot{x}^\mu$ with zero eigenvalue

$$\frac{\partial p_\mu}{\partial \dot{x}^\nu}\dot{x}^\nu = \left(\frac{m}{\sqrt{-\dot{x}_\mu \dot{x}^\mu}}\delta_{\mu\nu} + m\frac{\dot{x}^\mu \dot{x}^\nu}{(-\dot{x}_\mu \dot{x}^\mu)^{\frac{3}{2}}}\right)\dot{x}^\nu = 0$$

The inverse function theorem stats that absence of zero eigenvalues of the Hessian is a necessary condition to be able to express the velocities $\dot{x}^\mu$ via the canonical momenta p^μ. *Dynamical systems with the Hessian of non-maximal rank are called singular.*

Constraints which have vanishing Poisson bracket:

$$\{\phi_i, \phi_j\} = 0$$

[2] About the Hamiltonian approach to dynamical systems of classical mechanics, the reader may consult appendix A.

are called the first class constraints. The mass-shell constraint for the relativistic particle is of the first class.

There is another action for the relativistic particle. It has the following two characteristic features

- it does not contain square root
- it admits generalization to the massless case

Introduce $e(\tau)$ the auxiliary field on the world-line.

$$S=\frac{1}{2}\int_{\tau_0}^{\tau_1}d\tau\Big[\frac{1}{e}\Big(\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}\Big)-em^2\Big]$$

Equations of motion:

$$\begin{aligned}&\text{for } x^\mu && \frac{d}{d\tau}\Big(\frac{1}{e}\dot{x}^\mu\Big)=0\\ &\text{for } e(\tau) && -\frac{1}{2e^2}\dot{x}^\mu\dot{x}_\mu-\frac{1}{2}m^2=0 \quad\Longrightarrow \dot{x}^2+m^2e^2=0\end{aligned}$$

The last equation can be solved for e:

$$e^2=-\frac{1}{m^2}\dot{x}^2$$

which leads to

$$\frac{d}{d\tau}\Big[\frac{m}{\sqrt{-\dot{x}_\nu\dot{x}^\nu}}\dot{x}^\mu\Big]=0$$

which is nothing else as the old eom for x^μ. Also if we substitute solution for e into the new action then this action reduces to the old one:

$$S=\frac{1}{2}\int_{\tau_0}^{\tau_1}d\tau\Big[\frac{m}{\sqrt{-\dot{x}^2}}\dot{x}^2-\frac{\sqrt{-\dot{x}^2}}{m}m^2\Big]=-m\int_{\tau_0}^{\tau_1}d\tau\sqrt{-\dot{x}^2} \tag{2.2}$$

Also

$$p^\mu=\frac{1}{e}\dot{x}^\mu \quad\Longrightarrow\quad p^2=\frac{1}{e^2}\dot{x}^2=-m^2$$

but this time due to equations of motion for e. Equation of motion for e is purely algebraic. The Hessian

$$\frac{\partial^2\mathscr{L}}{\partial\dot{x}^\mu\partial\dot{x}^\nu}=\frac{1}{e}\frac{\partial}{\partial\dot{x}^\mu}\dot{x}_\nu=\frac{1}{e}\eta_{\mu\nu}$$

is of maximal rank.

The constraint $p^2+m^2=0$ does not follow from the definition of the canonical momentum along, but one has to use equations of motion. Constraints which are satisfied as consequences of equations of motion are called secondary.

The action has local gauge symmetry which is now

$$\delta x^\mu = \xi \dot{x}^\mu$$
$$\delta e = \partial_\tau(\xi e)$$

The first symmetry transformation is generated by the constraint p^2+m^2:

$$\delta_\eta x^\mu = \eta\{p^2+m^2, x^\mu\} = 2\eta p^\mu = \frac{\eta}{e}\dot{x}^\mu = \xi\dot{x}^\mu\,, \qquad \eta \equiv e\xi\,.$$

The second transformation for e is easily derived from $e^2 = -\frac{1}{m^2}\dot{x}^2$. Thus, in our new formulation we have the set of fields (x_μ, e) and reparametrization symmetry which acts on them and leaves the action invariant. This reparametrization freedom can be used to put $e=\frac{1}{m}$ which results into the following equation

$$\ddot{x}_\mu = 0$$

This is not the end however, because there is eom for e which now reads as $\dot{x}^2=-1$. Complete eoms are

$$\ddot{x}_\mu = 0\,, \qquad \dot{x}^2 = -1$$

Thus, the relativistic particle moves freely in Minkowski space over time-like geodesics. Space-like and light-like straight lines are excluded by the constraint $\dot{x}^2=-1$.

In the case of the massless particle we can set $e=1$ and get eoms

$$\ddot{x}_\mu = 0\,, \qquad \dot{x}^2 = 0$$

In both, the massive and massless cases, the constraints are integral of motions: they are preserved in time due to the dynamical equation $\ddot{x}_\mu = 0$.

Finally, we treat the relativistic particle in the so-called first order (the Hamiltonian) formalism. To this end we have to represent the initial Lagrangian in the form

$$\mathscr{L} = p_\mu \dot{x}^\mu + \mathscr{L}_{\rm rest}$$

where $p_\mu = \frac{1}{e}\dot{x}^\mu$ and express in $\mathscr{L}_{rest}$ the derivatives $\dot{x}^\mu$ via p^μ. In doing so we obtain the phase-space Lagrangian

$$\mathscr{L} = p_\mu \dot{x}^\mu - \frac{e}{2}(p^2+m^2)$$

We clearly see that the auxiliary field e we introduced in our second formulation plays here the role of the lagrangian multiplier to the constraint $p^2+m^2=0$. By using the gauge freedom we can fix the gauge $e=\frac{1}{m}$ and the physical Hamiltonian becomes in this case

$$H = \frac{1}{2m}(p^2+m^2)$$

which is in complete agreement with our previous discussion (We actually have to identify $\theta = e$). This Hamiltonian[3] should be provided with the constraint $p^2+m^2 = 0$ which is the eom for e.

More generally, evolution of a singular system is governed by the Hamiltonian

$$H = H_{\text{can}} + \sum_n \chi_n \phi_n$$

Here $\{\phi_n\}$ is an irreducible set of primary constraints and H_{can} is the canonical Hamiltonian:

$$H_{\text{can}} = p^\mu \dot{x}_\mu - \mathscr{L}$$

Only on the constraint surface $\phi_n = 0$ the Hamiltonian H coincides with H_{can}. In our present case

$$H_{\text{can}} = m\frac{\dot{x}_\nu \dot{x}^\nu}{\sqrt{-\dot{x}_\mu \dot{x}^\mu}} - (-m\sqrt{-\dot{x}_\mu \dot{x}^\mu}) = 0$$

and Hamiltonian dynamics of the system is due to the mass-shell condition only. *The choice of the coefficients $\chi_n(\tau)$ in H is equivalent to the choice of the gauge.*

$$H = H_{\text{can}} + \chi\phi = \frac{\theta}{2m}(p^2+m^2)\,, \qquad \chi = \frac{\theta}{2m}$$

We get the time-evolution

$$\frac{dx^\mu}{d\tau} = \{\frac{\theta}{2m}(p^2+m^2), x^\mu\} = \frac{\theta}{m}p^\mu = \frac{\theta\dot{x}^\mu}{\sqrt{-\dot{x}_\mu \dot{x}^\mu}}$$

Therefore $\dot{x}^2 = -\theta^2$. Choosing $\theta = 1$ means that we identify the time variable with the proper time. This nicely illustrates the general point: in order to write down the evolution equations in a system with local gauge invariance one has first to identify the "time" variable.

Other gauge choices are possible. For instance, the static gauge consists in imposing the condition $t \equiv x^1 = \tau$. Equation for $p_t \equiv p_1$ allows us to determine e:

$$\frac{\delta \mathcal{L}}{\delta p_t} = \frac{dt}{d\tau} - ep_t = 0 \quad \Longrightarrow \quad e = \frac{1}{p_t}$$

The physical Hamiltonian dual to the world-line time τ coincides in this case with the momentum p_t conjugate to t: $H = p_t$. It can be found from the eom for e:

$$p^2 = m^2 = 0 \quad \Longrightarrow \quad -p_t^2 + \vec{p}^2 + m^2 = 0 \quad \Longrightarrow \quad p_t = \sqrt{\vec{p}^2+m^2}\,.$$

Note that here $\vec{p} = \{p_i\}$ with $i = 2, \ldots, d$.

[3]The gauge $e = \frac{1}{m}$ is a close analogue of the conformal gauge to be considered for the string case.

This gauge choice leads to the common Hamiltonian of the relativistic particle

$$H = \sqrt{\vec{p}^2 + m^2}\,.$$

This exercise with the relativistic particle also shows how sensitive is the Hamiltonian to the choice of the gauge. Fixing the gauge $e = 1$ we get the polynomial Hamiltonian, while fixing the static gauge the Hamiltonian appears to be a non-linear square root.

Finally, there is another type of gauge known as the light-cone gauge. We introduce the light-cone coordinates

$$\begin{aligned} t &= x^+ - x^- \qquad & x_d &= x^+ + x^- \\ p_t &= \frac{1}{2}(p^+ + p^-), \qquad & p_d &= \frac{1}{2}(p^+ - p^-) \end{aligned}$$

and denote the other "transverse coordinates" x_i and p_i with $i = 2, \ldots, d-1$ as $\vec{x}$ and $\vec{p}$. Then the phase space Lagrangian becomes

$$\mathscr{L} = \dot{x}^- p^+ - \dot{x}^+ p^- + \dot{\vec{x}}\vec{p} - \frac{e}{2}(-p^- p^+ + \vec{p}^2 + m^2)$$

The light-cone gauge consists in choosing $x^+ = \tau$. From the kinetic term of the Hamiltonian it is clear that the variable x^+ is conjugate to p^- and therefore p^- is the physical Hamiltonian. It can be easily found from the equation for e:

$$H = \frac{1}{p^+}(\vec{p}^2 + m^2)$$

The gauge-fixed Lagrangian becomes

$$\mathscr{L} = \dot{x}^- p^+ + \dot{\vec{x}}\vec{p} - p^- = \dot{\vec{x}}\vec{p} - H = \dot{\vec{x}}\vec{p} - \frac{1}{p^+}(\vec{p}^2 + m^2)$$

The variable p^+ is canonically conjugate to x^-.

Notice that both in the static and in the light-cone gauge the number of physical degrees of freedom is $2(d-1)$. The auxiliary field e was solved in terms of physical fields. The physical phase space inherits the canonical Poisson bracket.

In the next lecture we extend these different approaches to dynamics of relativistic particles to relativistic strings.

3. Classical relativistic bosonic string

3.1 Nambu-Goto string

Two-dimensional surface traced by string during its time evolution is called *world-sheet*. The action for a relativistic string should be a functional of a string trajectory, i.e. of the world-sheet.

The Nambu-Goto action

$$S_{NG} = -T\int dA = -T\int \mathrm{d}^2\sigma\sqrt{-\det\left(\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\right)} \tag{3.1}$$

This we can write as

$$-\det\left[\begin{pmatrix} \dot{X}^\mu\dot{X}_\mu & \dot{X}^\mu X'_\mu \\ \dot{X}^\mu\dot{X}_\mu & X'^\mu X'_\mu \end{pmatrix}\right] = (\dot{X}X')^2 - \dot{X}^2X'^2\,. \tag{3.2}$$

Thus,

$$S_{NG} = -T\int \mathrm{d}^2\sigma\sqrt{(\dot{X}X')^2 - \dot{X}^2X'^2} = -T\int \mathrm{d}^2\sigma\sqrt{-\Gamma}$$

Here $\Gamma = \det\Gamma_{\alpha\beta}$, where

$$\Gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \tag{3.3}$$

is the metric induced on the string world-sheet.

What is a local characteristic of the string world-sheet? Consider a point on the world-sheet and the space of all vectors tangent to the surface is at this point. These vectors sweep two-dim vector space. The physical propagation of the string requires that in these two-dim vector space there is a basis built over two vectors one of them is time-like and another is space-like.

Recall the standard definitions from the theory of special relativity. We have the invariant interval between two infinitezimal events:

$$-ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dx^0dx^0 + \sum_{i=1}^{3}dx_idx_i\,. \tag{3.4}$$

- If $ds^2 > 0$ the interval is called time-like. In this case the different events which happen in the same space-point are always time-separated.

- If $ds^2 < 0$ the interval is called space-like. In this case events which happen at the same time are space-separated.

- If $ds^2 = 0$ the interval is light-like. Vectors v^μ obeying the condition $v^2 = \eta_{\mu\nu}v^\mu v^\nu = 0$ are called *light-like or null.*

Identify $x^0 = t = \tau$. Then

$$\dot{X}^\mu\dot{X}_\mu = -1 + \vec{v}^2 \le 0 \Leftarrow \mathrm{time-like}$$

$$X'^\mu X'_\mu = (X'^i)^2 \ge 0 \Leftarrow \mathrm{space-like}$$

This situation should persist in any Lorentz frame. At any point on a string world-sheet one should always be able to find two vectors: one is time-like and another is space-like.

Consider $S^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$ must be space- or time-like as λ varies.

$$S^\mu S_\mu = \dot{X}^2 + 2\lambda \dot{X} X' + \lambda^2 X'^2 \equiv y(\lambda)\,.$$

Discriminant must be positive then there are two roots and therefore the regions of λ with time- and space-like vectors. Discriminant is

$$(\dot{X} X')^2 - \dot{X}^2 X'^2 > 0\,.$$

This condition guarantees the causal propagation of string.

Action is invariant under reparametrizations

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu\,, \qquad \xi^\alpha = 0 \;\; \text{on the boundary}$$

Two possibilities:

- Open strings: $0 \le \sigma \le \pi$
- Closed strings: $0 \le \sigma \le 2\pi$

Equations of motion:

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \mathscr{L}$$

Variation

$$\frac{\delta S}{\delta X^\mu} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left(\frac{\delta \mathscr{L}}{\delta \dot{X}^\mu} \partial_\tau \delta X^\mu + \frac{\delta \mathscr{L}}{\delta X'^\mu} \partial_\sigma \delta X^\mu \right) \tag{3.5}$$

$$= - \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left(\partial_\tau \frac{\delta \mathscr{L}}{\delta \dot{X}^\mu} + \partial_\sigma \frac{\delta \mathscr{L}}{\delta X'^\mu} \right) \tag{3.6}$$

$$+ \int_0^\pi d\sigma \frac{\delta \mathscr{L}}{\delta \dot{X}^\mu} \delta X^\mu |_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \frac{\delta \mathscr{L}}{\delta X'^\mu} \delta X^\mu |_{\sigma=0}^{\sigma=\pi} \tag{3.7}$$

- Open string boundary conditions: $\frac{\delta \mathscr{L}}{\delta X'^\mu}(\tau, \sigma = \pi) = \frac{\delta \mathscr{L}}{\delta X'^\mu}(\tau, \sigma = 0) = 0$
- $X^\mu(\sigma + 2\pi) = X^\mu(\sigma)$.

Canonical formalism. Momentum

$$P^\mu = \frac{\delta \mathscr{L}}{\delta \dot{X}^\mu} = -T \frac{(\dot{X} X') X'^\mu - (X')^2 \dot{X}^\mu}{\sqrt{(\dot{X} X')^2 - \dot{X}^2 X'^2}}$$

We see that

$$P^\mu X'_\mu = 0 \tag{3.8}$$

$$P^\mu P_\mu + T^2 X'^2 = 0 \tag{3.9}$$

Exercise Show that the Hessian matrix has for each σ two zero eigenvalues corresponding to $\dot{X}^\mu$ and X'^μ.

Equations of motion are very complicated:

$$\frac{\partial}{\partial\tau}\left(\frac{(\dot{X}X')X'^\mu-(X')^2\dot{X}^\mu}{\sqrt{(\dot{X}X')^2-\dot{X}^2X'^2}}\right)+\frac{\partial}{\partial\sigma}\left(\frac{(\dot{X}X')\dot{X}^\mu-(\dot{X})^2X'^\mu}{\sqrt{(\dot{X}X')^2-\dot{X}^2X'^2}}\right)=0\,.$$

Exercise Think how reparametrization invariance can be used to bring this equation to the simplest form.

3.2 The Polyakov action

Introduce a word-sheet metric $h_{\alpha\beta}(\sigma,\tau)$. Consider the action

$$S_p=-\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \tag{3.10}$$

Here $h=\mathrm{det}h_{\alpha\beta}$.

3.2.1 Symmetries

The reparametrization invariance

$$\begin{aligned}\delta X^\mu&=\xi^\alpha\partial_\alpha X^\mu\\ \delta h_{\alpha\beta}&=\xi^\gamma\partial_\gamma h_{\alpha\beta}+h_{\alpha\gamma}\partial_\beta\xi^\gamma+h_{\beta\gamma}\partial_\alpha\xi^\gamma\\ \delta h^{\alpha\beta}&=\xi^\gamma\partial_\gamma h^{\alpha\beta}-h^{\alpha\gamma}\partial_\gamma\xi^\beta-h^{\beta\gamma}\partial_\gamma\xi^\alpha\\ \delta(\sqrt{-h})&=\partial_\alpha(\xi^\alpha\sqrt{-h})\end{aligned}$$

The Weyl invariance

The Weyl invariance consists in rescaling the metric

$$h_{\alpha\beta}\to e^{-2\Lambda(\sigma,\tau)}h_{\alpha\beta}\,. \tag{3.11}$$

3.2.2 Equations of motion

First we discuss the equation of motion for the intrinsic metric $h_{\alpha\beta}$. This discussion amounts to the introduction of the two-dimensional stress-energy tensor which is a response of the action to the change of the metric

$$\delta S_p=-T\int d^2\sigma\sqrt{-h}\;T_{\alpha\beta}\delta h^{\alpha\beta}$$

that is

$$T_{\alpha\beta}=-\frac{1}{T\sqrt{-h}}\frac{\delta S_p}{\delta h^{\alpha\beta}}$$

Thus, eom for $h_{\alpha\beta}$ is

$$T_{\alpha\beta} = 0\,.$$

Performing a variation we obtain

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu\,. \tag{3.12}$$

From here we immediately notice that the stress-energy tensor is traceless

$$T^\alpha_\alpha = T_{\alpha\beta} h^{\alpha\beta} = 0$$

This is a direct consequence of the Weyl symmetry.

Due to the equations of motion for the scalar fields X^μ this tensor is covariantly conserved:

$$\nabla^\alpha T_{\alpha\beta} = 0\,.$$

This can be easily derived as follows. Consider a variation of the action:

$$\delta S_p = \int \mathrm{d}^2\sigma \Big(\frac{\delta \mathscr{L}}{\delta h^{\alpha\beta}} \delta h^{\alpha\beta} + \frac{\delta \mathscr{L}}{\delta X^\mu} \delta X^\mu \Big)$$

We see that on the equations of motion for X^μ, i.e. on $\frac{\delta \mathscr{L}}{\delta X^\mu} = 0$, we have

$$\delta S_p = -T \int \mathrm{d}^2\sigma \sqrt{-h}\, T_{\alpha\beta} \delta h^{\alpha\beta} = -2T \int \mathrm{d}^2\sigma \sqrt{-h}\, T_{\alpha\beta} \nabla^\alpha \xi^\beta\,,$$

where on the r.h.s. we specified the variation to be a diffeomorphism transformation. The last expression can be integrated by parts and, die to $\delta S_p = 0$, we conclude that $\nabla^\alpha T_{\alpha\beta} = 0$. This derivation is similar to the derivation of the charge conservation law in electrodynamics, the latter being a consequence of the gradient invariance.

Let us show directly that $\nabla^\alpha T_{\alpha\beta} = 0$. We have

$$2\nabla^\alpha T_{\alpha\beta} = \nabla^\alpha \partial_\alpha X^\mu \partial_\beta X_\mu + \partial_\alpha X^\mu \nabla^\alpha \partial_\beta X_\mu - \frac{1}{2} \nabla_\beta \Big(h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \Big) \tag{3.13}$$

Since $\nabla^\alpha \partial_\alpha X^\mu = 0$ is eom for X^μ we find

$$\begin{aligned} 2\nabla^\alpha T_{\alpha\beta} &= \partial_\alpha X^\mu \nabla^\alpha \partial_\beta X_\mu - \frac{1}{2} \partial_\beta \Big(h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \Big) = \\ &= \partial_\alpha X^\mu h^{\alpha\delta} \nabla_\delta \partial_\beta X_\mu - \frac{1}{2} \partial_\beta h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu - h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta \partial_\beta X_\mu = \\ &= \partial_\alpha X^\mu h^{\alpha\delta} \partial_\delta \partial_\beta X_\mu - \partial_\alpha X^\mu \partial_s X_\mu h^{\alpha\delta} \Gamma^s_{\delta\beta} - \frac{1}{2} \partial_\beta h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu - h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta \partial_\beta X_\mu\,. \end{aligned}$$

Here the first term cancels with the last one and we find

$$2\nabla^\alpha T_{\alpha\beta} = -\partial_\alpha X^\mu \partial_s X_\mu h^{\alpha\delta} \Gamma^s_{\delta\beta} - \frac{1}{2} \partial_\beta h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu$$

Only symmetric part of $h^{\alpha\delta}\Gamma^s_{\delta\beta} = \frac{1}{2} h^{\alpha\delta} h^{sp} \Big(\partial_\delta h_{p\beta} + \partial_\beta h_{p\delta} - \partial_p h_{\beta\delta} \Big)$ in the indices α, s matters! This translates into symmetry of β, δ. Finally,

$$2\nabla^\alpha T_{\alpha\beta} = -\frac{1}{2} h^{\alpha\delta} \partial_\beta h_{p\delta} h^{sp} \partial_\alpha X^\mu \partial_s X_\mu - \frac{1}{2} \partial_\beta h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu = 0\,. \tag{3.14}$$

Equations

$$T^\alpha_\alpha = 0\,, \qquad\qquad \nabla^\beta T_{\alpha\beta} = 0$$

are consequences of the Weyl and diffeomorphism symmetry, respectively. These two are gauge symmetries, and the relations above can be understood as the second Noether theorem.

3.2.3 Conformal gauge

Consider the conformal Killing equation

$$\xi^\gamma \partial_\gamma h_{\alpha\beta} + h_{\alpha\gamma}\partial_\beta \xi^\gamma + h_{\beta\gamma}\partial_\alpha \xi^\gamma = \Lambda h_{\alpha\beta} \tag{3.15}$$

and solve it assuming the conformal gauge

$$h_{\alpha\beta} = e^{2\phi}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.16}$$

This equation can be split as follows. First we take $\alpha = \tau$ and $\beta = \sigma$ and using $h_{\tau\sigma} = 0$ we find

$$h_{\tau\tau}\partial_\sigma \xi^\tau + h_{\sigma\sigma}\partial_\tau \xi^\sigma = 0 \quad \rightarrow \quad \partial_\sigma \xi^\tau - \partial_\tau \xi^\sigma = 0$$

Second, we take $\alpha, \beta = \tau$ and then $\alpha, \beta = \sigma$. We get

$$\begin{aligned} \xi^\gamma \partial_\gamma h_{\tau\tau} + 2h_{\tau\tau}\partial_\tau \xi^\tau &= \Lambda h_{\tau\tau} \\ \xi^\gamma \partial_\gamma h_{\sigma\sigma} + 2h_{\sigma\sigma}\partial_\sigma \xi^\sigma &= \Lambda h_{\sigma\sigma} \end{aligned}$$

i.e.

$$\begin{aligned} \xi^\gamma h^{\tau\tau}\partial_\gamma h_{\tau\tau} + 2\partial_\tau \xi^\tau &= \Lambda \\ \xi^\gamma h^{\sigma\sigma}\partial_\gamma h_{\sigma\sigma} + 2\partial_\sigma \xi^\sigma &= \Lambda \, . \end{aligned}$$

Subtracting one from the other we get

$$\partial_\tau \xi^\tau - \partial_\sigma \xi^\sigma = 0 \, .$$

Thus, the conformal Killing equations reduce in the conformal gauge to

$$\begin{aligned} \partial_\sigma \xi^\tau - \partial_\tau \xi^\sigma &= 0 \\ \partial_\tau \xi^\tau - \partial_\sigma \xi^\sigma &= 0 \end{aligned} \tag{3.17}$$

or equivalently

$$\begin{aligned} (\partial_\tau + \partial_\sigma)(\xi^\tau - \xi^\sigma) &= 0 \\ (\partial_\tau - \partial_\sigma)(\xi^\tau + \xi^\sigma) &= 0 \end{aligned} \tag{3.18}$$

By using the world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ this can be reformulated as[4]

$$\partial_+ \xi^- = 0 = \partial_- \xi^+ \, .$$

[4] The basic relations are $\partial_\tau = \partial_+ + \partial_-$, $\partial_\sigma = \partial_+ - \partial_-$, $\partial_\pm = \frac{1}{2}\Big(\partial_\tau \pm \partial_\sigma\Big)$.

Thus,

$$\xi^+ = \xi^+(\sigma^+)\,, \qquad \xi^- = \xi^-(\sigma^-)$$

solve the conformal Killing equations. This is a freedom (reparametrization + Weyl rescaling) which does not destroy the conformal gauge condition.

The final remark concerns restrictions on $\xi^\pm$ following from the periodicity conditions. Indeed, for the Weyl factor Λ we have

$$\xi^\tau \partial_\tau \phi + \xi^\sigma \partial_\sigma \phi + 2\partial_\tau \xi^\tau = \Lambda(\sigma, \tau)\,.$$

The factor Λ must be periodic in σ because the metric $h_{\alpha\beta}$ is. Since ϕ is periodic the allowed solutions $\xi^\pm$ of the conformal Killing equation are those which lead to periodic Λ. One can easily see that this implies that $\xi^\pm$ must be periodic in σ.

3.2.4 Polyakov string in the first order formalism.

In the first order formalism the density $\mathcal{L} \equiv \mathcal{L}(\sigma, \tau)$ of the Polyakov Lagrangian takes the form

$$\mathscr{L} = P_\mu \partial_\tau X^\mu + \frac{1}{2T\gamma^{\tau\tau}}\Big(P_\mu P^\mu + T^2 X'_\mu X'^\mu\Big) + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big(P_\mu X'^\mu\Big) \tag{3.19}$$

The conformal gauge consists in imposing the following two conditions

$$\gamma^{\tau\tau} = -1\,, \qquad \gamma^{\tau\sigma} = 0.$$

The gauged-fixed Lagrangian density is

$$\mathscr{L} = P_\mu \partial_\tau X^\mu - \frac{1}{2T}\Big(P_\mu P^\mu + T^2 X'_\mu X'^\mu\Big) \tag{3.20}$$

and the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2T}\Big(P_\mu P^\mu + T^2 X'_\mu X'^\mu\Big)$$

The phase-space Lagrangian shows that the variables (P_μ, X_μ) are canonical, i.e. the corresponding Poisson bracket is

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \{P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = 0\,, \tag{3.21}$$

$$\{X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = \eta^{\mu\nu}\delta(\sigma - \sigma')\,. \tag{3.22}$$

The dynamics of the system in the conformal gauge is governed by the Hamiltonian

$$\mathrm{H} = \int \mathrm{d}\sigma\; \mathcal{H} = \frac{1}{2T}\int \mathrm{d}\sigma\Big(P_\mu P^\mu + T^2 X'_\mu X'^\mu\Big)$$

Equations of motion

$$\dot{X}^\mu = \{X^\mu, \mathrm{H}\} = \frac{1}{T}P^\mu\,, \tag{3.23}$$

$$\dot{P}^\mu = \{P^\mu, \mathrm{H}\} = T X''^\mu\,, \tag{3.24}$$

which result into

$$\ddot{X}^\mu - X''^\mu = \Box X^\mu = 0$$

We see that we have two constraints

$$C_1 = P_\mu P^\mu + T^2 X'_\mu X'^\mu \,, \qquad C_2 = P_\mu X'^\mu \tag{3.25}$$

We find the following Poisson brackets

$$\begin{aligned}
\{C_1(\sigma), C_1(\sigma')\} &= 4T^2 \partial_\sigma C_2(\sigma)\delta(\sigma-\sigma') + 8T^2 C_2(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, && (3.26)\\
\{C_1(\sigma), C_2(\sigma')\} &= \partial_\sigma C_1(\sigma)\delta(\sigma-\sigma') + 2C_1(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, && (3.27)\\
\{C_2(\sigma), C_1(\sigma')\} &= \partial_\sigma C_1(\sigma)\delta(\sigma-\sigma') + 2C_1(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, && (3.28)\\
\{C_2(\sigma), C_2(\sigma')\} &= \partial_\sigma C_2(\sigma)\delta(\sigma-\sigma') + 2C_2(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,. && (3.29)
\end{aligned}$$

Instead of the constraints C_1 and C_2 we can equally consider their linear combinations

$$\begin{aligned}
T_{++} &= \frac{1}{8T^2}(C_1 + 2TC_2) = \frac{1}{8T^2}(P_\mu + TX'_\mu)^2\,, && (3.30)\\
T_{--} &= \frac{1}{8T^2}(C_1 - 2TC_2) = \frac{1}{8T^2}(P_\mu - TX'_\mu)^2\,. && (3.31)
\end{aligned}$$

Their Poisson algebra becomes

$$\begin{aligned}
\{T_{++}(\sigma), T_{++}(\sigma')\} &= \frac{1}{2T}\Big(\partial_\sigma T_{++}(\sigma)\delta(\sigma-\sigma') + 2T_{++}(\sigma)\partial_\sigma\delta(\sigma-\sigma')\Big)\,,\\
\{T_{--}(\sigma), T_{--}(\sigma')\} &= -\frac{1}{2T}\Big(\partial_\sigma T_{--}(\sigma)\delta(\sigma-\sigma') + 2T_{--}(\sigma)\partial_\sigma\delta(\sigma-\sigma')\Big)\,,\\
\{T_{++}(\sigma), T_{--}(\sigma')\} &= 0\,. && (3.32)
\end{aligned}$$

Constraints T_{++} and T_{--} Poisson commute and form two independent Poisson algebras!

Now one can easily find the evolution equations for T_{++} and T_{--}. We have

$$\begin{aligned}
\partial_\tau T_{++} &= \{T_{++}(\sigma), \mathrm{H}\} = \partial_\sigma T_{++} \qquad \Longrightarrow \qquad (\partial_\tau - \partial_\sigma)T_{++} = \partial_- T_{++} = 0\,,\\
\partial_\tau T_{--} &= \{T_{--}(\sigma), \mathrm{H}\} = -\partial_\sigma T_{--} \qquad \Longrightarrow \qquad (\partial_\tau + \partial_\sigma)T_{--} = \partial_+ T_{--} = 0\,,
\end{aligned}$$

Thus, we see that evolution equations imply that

$$T_{++} = T_{++}(\sigma^+)\,, \qquad\qquad T_{--} = T_{--}(\sigma^-)\,. \tag{3.33}$$

The Hamiltonian itself is

$$\mathrm{H} = 2T\int_0^{2\pi} \mathrm{d}\sigma (T_{++} + T_{--})\,,$$

i.e. it is a sum of zero modes of the left- and right-moving constraints.

3.3 Integrals of motion. Classical Virasoro algebra.

String has an infinite set of integrals of motion (quantities which are conserved in time due to equations of motion) which are constructed with the help of $T_{\pm\pm}$.

Note that $T_{\pm\pm}$ themselves are not the good conserved quantities as they depend on time! However, if we define the Fourier components

$$L_m = 2T \int_0^{2\pi} \mathrm{d}\sigma \; e^{im\sigma^-} T_{--}(\sigma,\tau)$$

then[5]

$$\begin{aligned}
\frac{dL_m}{d\tau} &= 2T \int_0^{2\pi} \mathrm{d}\sigma \; \Big(\partial_\tau e^{im(\tau-\sigma)} T_{--}(\sigma,\tau) + e^{im(\tau-\sigma)} \partial_\tau T_{--}(\sigma,\tau)\Big) \\
&= 2T \int_0^{2\pi} \mathrm{d}\sigma \; \Big(im e^{im(\tau-\sigma)} T_{--}(\sigma,\tau) - e^{im(\tau-\sigma)} \partial_\sigma T_{--}(\sigma,\tau)\Big) \\
&= 2T \int_0^{2\pi} \mathrm{d}\sigma \; \Big(im e^{im(\tau-\sigma)} T_{--}(\sigma,\tau) + \partial_\sigma e^{im(\tau-\sigma)} T_{--}(\sigma,\tau)\Big) = 0
\end{aligned} \tag{3.34}$$

Thus, the Fourier components of the stress-energy tensor provide an infinite set of the conserved constraints. Analogously, we define

$$\bar{L}_m = 2T \int_0^{2\pi} \mathrm{d}\sigma \; e^{im\sigma^+} T_{++}(\sigma,\tau) \,,$$

which is also an integral of motion. Note that in this derivation we never used the constraints $T_{++} = 0 = T_{--}$.

The Poisson brackets of the L_m and $\bar{L}_m$ generators are

$$\begin{aligned}
\{L_m, L_n\} &= -i(m-n)L_{m+n} \,, \\
\{\bar{L}_m, \bar{L}_n\} &= -i(m-n)\bar{L}_{m+n} \,, \\
\{L_m, \bar{L}_n\} &= 0 \,.
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\{L_m, L_n\} &= 4T^2 \int \mathrm{d}\sigma \mathrm{d}\sigma' e^{-im\sigma - in\sigma'} \{T_{--}(\sigma), T_{--}(\sigma')\} \\
&= -2T \int \mathrm{d}\sigma \mathrm{d}\sigma' e^{-im\sigma - in\sigma'} \Big(\partial_\sigma T_{--}(\sigma)\delta(\sigma-\sigma') + 2T_{--}(\sigma)\partial_\sigma \delta(\sigma-\sigma') \Big) = \\
&= -2T \int \mathrm{d}\sigma \Big(- T_{--}(\sigma)\partial_\sigma e^{-i(m+n)\sigma'} + 2e^{-im\sigma} T_{--}(\sigma)\partial_\sigma e^{-in\sigma} \Big) \\
&= -i(m-n)\; 2T \int \mathrm{d}\sigma e^{-i(m+n)\sigma} T_{--}(\sigma) = -i(m-n)L_{m+n}
\end{aligned}$$

This is the so-called Wit algebra. This algebra acts on $X^\mu(\sigma,\tau)$:

$$\begin{aligned}
\{L_m, X^\mu\} &= 2T \int_0^{2\pi} \mathrm{d}\sigma' e^{im\sigma'^-} \{T_{--}(\sigma'), X^\mu(\sigma)\} = \\
&= -\frac{1}{2} e^{im\sigma^-} (\dot{X}^\mu - X'^\mu) = -e^{im\sigma^-} \partial_- X^\mu \,.
\end{aligned} \tag{3.36}$$

[5] When finding the time dynamics of L_m one has to remember that the function L_m has an explicit time-dependence and, therefore $\frac{dL_m}{d\tau} = \partial_\tau L_m + \{L_m, \mathrm{H}\}$.

Analogously,

$$\{\bar{L}_m, X^\mu\} = 2T\int_0^{2\pi} \mathrm{d}\sigma' e^{im\sigma'^+}\{T_{++}(\sigma'), X^\mu(\sigma)\} =$$
$$= -\frac{1}{2}e^{im\sigma^+}(\dot{X}^\mu + X'^\mu) = -e^{im\sigma^+}\partial_+ X^\mu\,. \tag{3.37}$$

To summarize

$$\{L_m, X^\mu\} = -e^{im\sigma^-}\partial_- X^\mu\,, \qquad \{\bar{L}_m, X^\mu\} = -e^{im\sigma^+}\partial_+ X^\mu\,. \tag{3.38}$$

In particular,

$$\{\bar{L}_0 - L_0, X^\mu\} = \partial_\sigma X^\mu \tag{3.39}$$

i.e. $\bar{L}_0 - L_0$ generates rigid σ-translations. The transformations we consider transform a solution

$$\Box X^\mu = 0$$

into another solution of this equation. Indeed, we have for instance

$$\partial_+\partial_-\left(e^{im\sigma^-}\partial_- X^\mu\right) = im\sigma^- e^{im\sigma^-}\partial_+\partial_- X^\mu + e^{im\sigma^-}\partial_-\partial_+\partial_- X^\mu = 0$$

as the consequence of $\partial_+\partial_- X^\mu = 0$.

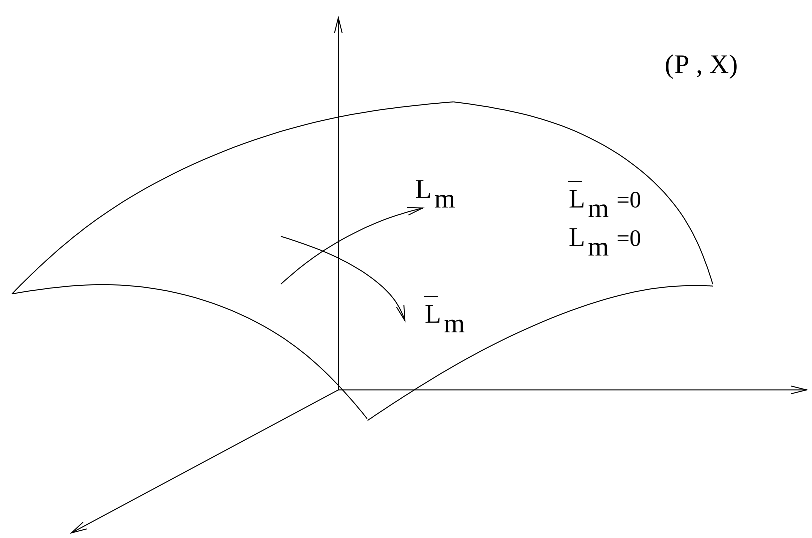


Fig. 1. The phase space of string. The Virasoro constraints $L_m = 0 = \bar{L}_m$ define a time-independent hypersurface on which the dynamics of string takes place. This hypersurface remains invariant under the action of L_m's and $\bar{L}_m$'s.

Even more generally, for any periodic function f we define

$$L_f = \int_0^{2\pi} \mathrm{d}\sigma f(\sigma^+) T_{++}\,.$$

Then we see that

$$0 = \int_0^{2\pi} \mathrm{d}\sigma \partial_- \Big(f(\sigma^+) T_{++} \Big) = \partial_\tau L_f - \int_0^{2\pi} \mathrm{d}\sigma \partial_\sigma \Big(f(\sigma^+) T_{++} \Big) \,.$$

Thus,

$$\partial_\tau L_f = \int_0^{2\pi} \mathrm{d}\sigma \partial_\sigma \Big(f(\sigma^+) T_{++} \Big) = 0 \,.$$

To summarize: we see that upon fixing conformal gauge we are still left with gauge freedom. It corresponds to reparametrizations of the special type (solutions to the conformal Killing equation):

$$\sigma^+ \to \xi^+(\sigma^+) \,, \qquad \sigma^- \to \xi^-(\sigma^-) \,,$$

where $\xi^\pm$ are two arbitrary functions periodic in σ.

Finally, for the case of open string we define

$$L_m = 2T \int_0^{\pi} \mathrm{d}\sigma \Big(e^{im\sigma^+} T_{++} + e^{im\sigma^-} T_{--} \Big) \,. \tag{3.40}$$

The point is that there appears additional boundary terms which show that the old L_m and $\bar{L}_m$ are not separately conserved. With our new definition of L_m we obtain

$$\begin{aligned}
\frac{dL_m}{d\tau} &= 2T \int_0^{\pi} \mathrm{d}\sigma \Big(im e^{im\sigma^+} T_{++} + e^{im\sigma^+} \partial_\tau T_{++} + im e^{im\sigma^-} T_{--} + e^{im\sigma^-} \partial_\tau T_{--} \Big) \\
&= 2T \int_0^{\pi} \mathrm{d}\sigma \Big(im e^{im\sigma^+} T_{++} + e^{im\sigma^+} \partial_\sigma T_{++} + im e^{im\sigma^-} T_{--} - e^{im\sigma^-} \partial_\sigma T_{--} \Big) \\
&= 2T \int_0^{\pi} \mathrm{d}\sigma \Big(im e^{im\sigma^+} T_{++} - \partial_\sigma e^{im\sigma^+} T_{++} + im e^{im\sigma^-} T_{--} + \partial_\sigma e^{im\sigma^-} T_{--} \Big) \\
&+ 2T e^{im(\tau+\pi)} \Big(T_{++}(\pi,\tau) - e^{-2\pi i m} T_{--}(\pi,\tau) \Big) - 2T e^{im\tau} \Big(T_{++}(0,\tau) - T_{--}(0,\tau) \Big)
\end{aligned}$$

The bulk term vanishes as before and we left with the boundary term

$$\frac{dL_m}{d\tau} = 2T e^{im(\tau+\pi)} \Big(T_{++}(\pi,\tau) - T_{--}(\pi,\tau) \Big) - 2T e^{im\tau} \Big(T_{++}(0,\tau) - T_{--}(0,\tau) \Big)$$

Due to the open string boundary conditions $X'^\mu(\pi,\tau) = 0 = X'^\mu(0,\tau)$ we obviously have

$$T_{++}(\pi,\tau) = T_{--}(\pi,\tau) \,, \qquad T_{++}(0,\tau) = T_{--}(0,\tau)$$

Therefore, the boundary term vanishes and, therefore, L_m are conserved quantities in the open string case.

3.3.1 Solutions of the equations of motion

Here we are going to discuss the solutions of the string equations off motion

$$\Box X^\mu = 0$$

which arises upon fixing the conformal gauge.

- *Closed strings.* We have

$$X^\mu(\sigma,\tau) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)$$

where

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{p^\mu}{4\pi T}(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)} \tag{3.41}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{p^\mu}{4\pi T}(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)} \tag{3.42}$$

Since $X^\mu(\sigma,\tau)$ are real then (x^μ, p^μ) are real as well and

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger\,, \qquad \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^\dagger$$

Let us define the zero modes as

$$\alpha_0^\mu = \bar{\alpha}_0^\mu = \frac{1}{\sqrt{4\pi T}}p^\mu\,.$$

Oscillators obey the Poisson relations

$$\begin{aligned}\{\alpha_m^\mu, \alpha_n^\nu\} &= \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\} = -im\delta_{m+n}\eta^{\mu\nu}\,,\\ \{\alpha_m^\mu, \bar{\alpha}_n^\nu\} &= 0\\ \{x^\mu, p^\nu\} &= \eta^{\mu\nu}\,.\end{aligned} \tag{3.43}$$

The Virasoro constraints become

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}^\mu\alpha_{n\mu}\,, \qquad \bar{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\bar{\alpha}_{m-n}^\mu\bar{\alpha}_{n\mu}\,. \tag{3.44}$$

- *Open strings* Solution of the wave equation with the open string boundary conditions is

$$X^\mu(\sigma,\tau) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma\,. \tag{3.45}$$

Oscillators obey the Poisson algebra

$$\begin{aligned}\{\alpha_m^\mu,\alpha_n^\nu\} &= -im\delta_{m+n}\eta^{\mu\nu}\,,\\ \{x^\mu,p^\nu\} &= \eta^{\mu\nu}\,.\end{aligned} \tag{3.46}$$

If we define the zero mode as $\alpha_0^\mu = \frac{1}{\sqrt{\pi T}}p^\mu$ then the generators of the open string classical Virasoro algebra are realized as

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha^\mu_{m-n}\alpha_{n\mu}\,. \tag{3.47}$$

The hermiticity property of the non-zero modes are $\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger$.

3.3.2 Poincaré symmetry. String tension.

The Noether currents of the Poincaré symmetry

$$P_\mu^\alpha = -T\sqrt{-h}h^{\alpha\beta}\partial_\beta X_\mu \tag{3.48}$$

$$J_\alpha^{\mu\nu} = -T\sqrt{-h}h^{\alpha\beta}(X_\mu\partial_\beta X_\nu - X_\nu\partial_\beta X_\mu) \tag{3.49}$$

Here P_α is a current corresponding to translational invariance $X^\mu \to X^\mu + a$, where a is an arbitrary constant, and $J^{\mu\nu}$ is a current corresponding to Lorentz rotations $X^\mu \to \Lambda^\mu_\nu X^\nu$, where Λ^μ_ν is a (constant) matrix comprising the parameters of the Lorentz transformation. We see that

$$J_\alpha^{\mu\nu} = X_\mu P_\nu^\alpha - X_\nu P_\mu^\alpha\,.$$

Both currents are conserved due to equations of motion of X^μ and their τ-components integrated over σ define the conserved charges (for the open string case integration rans from 0 to π)

$$P^\mu = \int_0^{2\pi}\mathrm{d}\sigma P_\tau^\mu\,, \qquad J^{\mu\nu} = \int_0^{2\pi}\mathrm{d}\sigma J_\tau^{\mu\nu}\,.$$

Imposing the conformal gauge and using the fundamental brackets for (X, P) one finds the following Poisson algebra

$$\begin{aligned}\{P^\mu,P^\nu\} &= 0\\ \{P^\mu,J^{\rho\sigma}\} &= \eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}P^\sigma\\ \{J^{\mu\nu},J^{\rho\sigma}\} &= \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho}\,.\end{aligned} \tag{3.50}$$

Substituting solution for X^μ one finds

$$P^\mu = T\int_0^{2\pi}\mathrm{d}\sigma\dot{X}^\mu = p^\mu\,,$$

i.e. the total mass momentum of string coincides with p^μ.

The total angular momentum of closed string in the conformal gauge is is defined as

$$J^{\mu\nu} = T\int_0^{2\pi} d\sigma (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)\,. \tag{3.51}$$

Substituting the oscillator expansion we get

$$J^{\mu\nu} = \underbrace{x^\mu p^\nu - x^\nu p^\mu}_{\ell^{\mu\nu}} + S^{\mu\nu} + \bar{S}^{\mu\nu}\,,$$

where

$$S^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\big(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\big)\,, \tag{3.52}$$

$$\bar{S}^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\big(\bar{\alpha}^\mu_{-n}\bar{\alpha}^\nu_n - \bar{\alpha}^\nu_{-n}\bar{\alpha}^\mu_n\big)\,. \tag{3.53}$$

For the case of open string expressions are the same (again integration runs from 0 to π) except $\bar{S}^{\mu\nu}$ is absent. Here $\ell^{\mu\nu}$ is the *angular momentum* of string and $S^{\mu\nu} + \bar{S}^{\mu\nu}$ is its *internal spin*.

Let us show that both P^μ and $J^{\mu\nu}$ are invariant under the action of the Virasoro algebra. We have

$$\{L_m, P^\mu\} = \{L_m, p^\mu\} = 0 \tag{3.54}$$

as L_m does not contain x^μ. Let us separate the zero mode part[6] of L_m

$$L_m = \alpha^\rho_0\alpha_{m\rho} + \frac{1}{2}\sum_{n\neq 0,m}\alpha^\rho_{m-n}\alpha_{n\rho}\,.$$

We first compute

$$\begin{aligned}\{L_m, \ell^\mu\} &= \{\alpha^\rho_0\alpha_{m\rho}, x^\mu p^\nu - x^\nu p^\mu\} = \frac{1}{\sqrt{4\pi T}}\alpha_{m\rho}\{p^\rho, x^\mu p^\nu - x^\nu p^\mu\} = \\ &= \alpha^\nu_m\alpha^\mu_0 - \alpha^\mu_m\alpha^\nu_0\,. \end{aligned} \tag{3.55}$$

Second, since $S^{\mu\nu} = -\sum_{k\neq 0}\frac{i}{k}\alpha^\mu_{-k}\alpha^\nu_k$ we have

$$\sum_{n,k\neq 0; n\neq m}\{\frac{1}{2}\alpha^\rho_{m-n}\alpha_{n\rho}, -\frac{i}{k}\alpha^\mu_{-k}\alpha^\nu_k\} = 0\,,$$

$$\sum_{k\neq 0}\{\alpha^\rho_0\alpha_{m\rho}, -\frac{i}{k}\alpha^\mu_{-k}\alpha^\nu_k\} = \alpha^\mu_m\alpha^\nu_0 - \alpha^\nu_m\alpha^\mu_0\,,$$

[6] We assume here that $m \neq 0$.

therefore, $\{L_m, \ell^{\mu\nu} + S^{\mu\nu}\} = 0$. Thus, Poincaré symmetry descends on the space of physical states. Physical states will be classified in terms of representations of the Poincaré group of d-dimensional Minkowski space.

An important invariant of the Poincaré group is the square of the momentum $P_\mu P^\mu$. Indeed,

$$\{P_\mu P^\mu, P^\nu\} = \{P_\mu P^\mu, J^{\rho\lambda}\} = 0\,.$$

It coincides with the mass: $M^2 = -P_\mu P^\mu$. Of course, P^2 is also invariant w.r.t. to the action of L_m and $\bar{L}_m$. Consider now one of the constraints, $L_0 = 0$. Separating the zero mode

$$L_0 = \frac{1}{2}\sum_{n=-\infty}^{n=\infty} \alpha_n \alpha_{-n} = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_n \alpha_{-n}\,, \qquad \alpha_0^\mu = \frac{1}{\sqrt{4\pi T}} p^\mu$$

we have

$$p_\mu p^\mu + 8\pi T \sum_{n=1}^{\infty} \alpha_n \alpha_{-n} = 0\,.$$

From here we deduce the mass

$$M^2 = -p^2 = 8\pi T \sum_{n=1}^{\infty} \alpha_n \alpha_{-n}\,.$$

Mass is created due to internal excitations of string! From this expression it is not obvious that M^2 is non-negative.

Another invariant of the Poincaré group is

$$J^2 = \frac{1}{2}\left(J_{\alpha\beta}J^{\alpha\beta} + \frac{2}{M^2} P_\alpha J^{\alpha\lambda} P^\beta J_{\beta\lambda}\right)\,.$$

Checking Poincaré invariance of this expression is straightforward, as an intermediate step we note the relations

$$\begin{aligned}
\{J^{\mu\nu}, P_\rho J^{\rho\sigma}\} &= \eta^{\nu\sigma} J^{\mu\rho} P_\rho - \eta^{\mu\sigma} J^{\nu\rho} P_\rho \\
\{P^\mu, J_{\alpha\beta} J^{\alpha\beta}\} &= -4 J^{\mu\rho} P_\rho\,.
\end{aligned}$$

The terms $J_{\alpha\beta}J^{\alpha\beta}$ and $P_\alpha J^{\alpha\lambda} P^\beta J_{\beta\lambda}$ separately Poisson commute with $J^{\mu\nu}$.

Consider an open string which has $P^i = p^i = 0$ for all $i = 1, \ldots, d-1$. By using reparametrization invariance fix the static gauge $X^0 = \tau$. The angular moment $\ell^{\mu\nu}$ of this string is zero. Also $P_\alpha J^{\alpha\lambda} P^\beta J_{\beta\lambda} = 0$ and, therefore,

$$J^2 = \frac{1}{2} J_{ij} J^{ij}\,,$$

where $i, j = 1, \ldots d-1$. We have

$$\begin{aligned}
J^2 &= -\frac{1}{2}\sum_{n,m=1}^{\infty} \frac{1}{nm} \Big(\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i\Big)\Big(\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i\Big) \\
&= \sum_{n,m=1}^{\infty} \frac{1}{nm}\Big((\alpha_n^* \alpha_m)(\alpha_m^* \alpha_n) - (\alpha_n^* \alpha_m^*)(\alpha_n \alpha_m)\Big) \le \sum_{n,m=1}^{\infty} \frac{1}{nm}(\alpha_n^* \alpha_m)(\alpha_m^* \alpha_n)\,.
\end{aligned}$$

It follows from the Schwarz inequality that

$$(\alpha_n^*\alpha_m)(\alpha_m^*\alpha_n) = |(\alpha_n^*\alpha_m)|^2 \leq (\alpha_n^*\alpha_n)(\alpha_m^*\alpha_m) \leq nm(\alpha_n^*\alpha_n)(\alpha_m^*\alpha_m)\,.$$

Therefore,

$$J^2 \leq \sum_{n,m=1}^{\infty}(\alpha_n^*\alpha_n)(\alpha_m^*\alpha_m) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\alpha_n\alpha_{-n}\sum_{m=-\infty}^{\infty}\alpha_m\alpha_{-m} = \frac{1}{(2\pi T)^2}M^4$$

because in the open string case

$$M^2 = \pi T\sum_{n=-\infty}^{\infty}\alpha_n\alpha_{-n} = 2\pi T\sum_{n>0}\alpha_n\alpha_{-n}\,.$$

Thus, for an open string motion we found inequality

$$J \equiv \sqrt{J^2} \leq \frac{1}{2\pi T}M^2\,.$$

Here the parameter

$$\alpha' = \frac{1}{2\pi T}$$

is called a slope of the Regge trajectory. The function $J = \alpha' M^2$ is a straight line in the (M^2, J) plane whose slope is α'.

Consider a closed (pulsating) string solution

$$x = R\cos\sigma\cos\tau\,, \quad y = R\sin\sigma\cos\tau\,, \quad t = R\tau\,.$$

We see that

$$P^0 = 2\pi RT \quad\Longrightarrow\quad T = \frac{P^0}{2\pi R} \equiv \frac{E}{2\pi R}\,,$$

i.e. *tension is energy per unit length.*

3.4 Strings in physical gauge

As we have seen upon fixing conformal gauge we are still left with the gauge freedom. It corresponds to reparametrizations of the special type (solutions to the conformal Killing equation):

$$\sigma^+ \to \xi^+(\sigma^+)\,, \qquad \sigma^- \to \xi^-(\sigma^-)\,,$$

where $\xi^\pm$ are two arbitrary functions (periodic in σ). This freedom can be further fixed leaving only physical excitations. This is achieved by imposing the so-called *light-cone gauge.*

3.4.1 First order formalism

Introduce the light-cone coordinates in the d-dimensional Minkowski space

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}), \qquad X^i, \qquad i = 1, \ldots, d-2\,.$$

Consider the Polyakov action and introduce the light-cone momenta conjugate to the light-cone coordinates

$$P_\pm = \frac{\partial\mathcal{L}}{\partial\dot{X}^\pm}\,, \qquad P_i = \frac{\partial\mathcal{L}}{\partial\dot{X}^i} \tag{3.56}$$

Express now the velocities via the corresponding momenta

$$\dot{X}^{\pm} = \frac{1}{T\gamma^{\tau\tau}}(P_{\mp} - T\gamma^{\tau\sigma}X'^{\pm})\,, \qquad \dot{X}^i = \frac{1}{T\gamma^{\tau\tau}}(-P^i - T\gamma^{\tau\sigma}X'^i)\,.$$

Note that the light-cone indices are raised and lowered according to the rule

$$P_- = -P^+\,, \qquad P_+ = -P^-\,.$$

Now we can construct the phase-space Lagrangian in the light-cone coordinates:

$$\mathcal{L} = P_i\dot{X}^i + P_+\dot{X}^+ + P_-\dot{X}^- + \text{rest}\,.$$

After explicit computation we find

$$\begin{aligned}\mathcal{L} &= P_i\dot{X}^i + P_+\dot{X}^+ + P_-\dot{X}^- + \frac{1}{2T\gamma^{\tau\tau}}\Big(-2P_-P_+ + P_iP^i + T^2X'_iX'^i - 2T^2X'^-X'^+\Big)\\ &\quad + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big(P_iX'^i + P_-X'^- + P_+X'^+\Big)\,. \end{aligned} \tag{3.57}$$

The phase-space light-cone gauge consists in imposing the following two conditions

1. *Closed string*

$$X^+ = \frac{p^+}{2\pi T}\tau\,, \qquad P^+ = \text{const} \equiv \frac{1}{2\pi}p^+\,. \tag{3.58}$$

2. *Open string*

$$X^+ = \frac{p^+}{\pi T}\tau\,, \qquad P^+ = \text{const} \equiv \frac{1}{\pi}p^+\,. \tag{3.59}$$

This gauge choice is done to remove completely the gauge degrees of freedom (recall that in the conformal gauge we still had a gauge freedom left which was generated by solutions of the conformal Killing equation).

We further consider the closed string case in detail. We will derive and solve all the constraints followed from the Lagrangian (3.57) in several steps.

1. Varying the Lagrangian w.r.t. $\gamma^{\tau\tau}$ and imposing the light-cone gauge allows one to solve for P^-:

$$P^- = \frac{\pi}{p^+}\Big(P_iP^i + T^2X'_iX'^i\Big)\,. \tag{3.60}$$

Thus, equation of motion for $\gamma^{\tau\tau}$ allows one to determine P^-.

2. Varying w.r.t. $\gamma^{\tau\sigma}$ leads to determination of X'^-:

$$X'^- = -\frac{1}{P_-} P_i X'^i = \frac{2\pi}{p^+} P_i X'^i \,. \tag{3.61}$$

If we integrate the last equation over σ we obtain

$$X^-(2\pi) - X^-(0) = \frac{2\pi}{p^+} \int_0^{2\pi} \mathrm{d}\sigma P_i X'^i \,. \tag{3.62}$$

The closed string periodicity condition requires the fulfillment of the following constraint

$$\mathcal{V} = \int_0^{2\pi} \mathrm{d}\sigma P_i X'^i = 0 \,. \tag{3.63}$$

This is the only constraint which remains unsolved and it is known as the *level matching condition.* We will impose it on physical states of the theory.

3. Now we can determine the world-sheet metric $\gamma^{\alpha\beta}$. Equation of motion for P_+ is

$$0 = \frac{\delta \mathcal{L}}{\delta P_+} = \dot{X}^+ - \frac{P_-}{T\gamma^{\tau\tau}} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} X'^+ \,,$$

which with our gauge choice gives

$$\gamma^{\tau\tau} = -1 \,.$$

4. Equation of motion for $\dot{X}^-$ gives

$$0 = \frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{X}^-} - \frac{\delta \mathcal{L}}{\delta X^-} = -\frac{d}{dt}\frac{p^+}{2\pi} - \frac{\delta \mathcal{L}}{\delta X^-} \quad \Longrightarrow \quad \frac{\delta \mathcal{L}}{\delta X^-} = 0 \,,$$

which gives

$$\partial_\sigma \left(\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} P_- \right) = 0 \quad \Longrightarrow \quad \partial_\sigma \gamma^{\tau\sigma} = 0 \,.$$

For the closed string case this implies that $\gamma^{\tau\sigma} = \gamma^{\tau\sigma}(\tau)$ is an arbitrary function of τ. The presence of this function signals a residual symmetry. Indeed, on the solutions of the level-matching constraint $\mathcal{V} = 0$ the ratio $\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}$ can be shifted by an arbitrary function $f(\tau)$ of τ without affecting the Lagrangian.

5. Varying w.r.t P_- we find an evolution equation for X^-:

$$0 = \frac{\delta \mathcal{L}}{\delta P_-} = \dot{X}^- - \frac{P_+}{T\gamma^{\tau\tau}} = 0 \quad \Longrightarrow \quad \dot{X}^- = \frac{\pi}{Tp^+}(P_i P^i + T^2 X'_i X'^i) \,.$$

Thus, the variable X^- is not physical and can be solved from the following two equations we found above

$$\dot{X}^- = \frac{\pi}{Tp^+}(P_i P^i + T^2 X_i' X'^i) \tag{3.64}$$

$$X'^- = \frac{2\pi}{p^+} P_i X'^i \tag{3.65}$$

These two equations can be rewritten as

$$\partial_\pm X^- = \frac{2\pi T}{p^+}(\partial_\pm X^i)^2 \,. \tag{3.66}$$

It is worth noting that two equations (3.64), (3.65) are compatible. Indeed, the σ-derivative of the first equation must be equal the τ-derivative of the second one. One can see that this is indeed so due to equations of motion for physical fields.

Now we are ready to construct the gauge-fixed Lagrangian. Substituting solutions of all the constraints and the gauge conditions into eq.(3.57) we obtain the density

$$\mathcal{L} = P_i \dot{X}^i - \frac{1}{2\pi} p^+ \dot{X}^- - \frac{p^+}{2\pi T} P^- - \gamma^{\tau\sigma}(\tau)\Big(P_i X'^i - \frac{p^+}{2\pi} X'^-\Big) \,. \tag{3.67}$$

Thus, the Lagrangian itself is

$$L = \int_0^{2\pi} \mathrm{d}\sigma \mathcal{L} = \underbrace{-p^+ \dot{x}^- + \int_0^{2\pi} \mathrm{d}\sigma \; P_i \dot{X}^i}_{\text{defines Poisson structure}} - \mathrm{H} - \gamma^{\tau\sigma}(\tau)\mathcal{V} \,. \tag{3.68}$$

Here x^- denotes the *zero (constant) mode* of the variable X^- and the Hamiltonian is

$$\mathrm{H} = \frac{1}{2T} \int_0^{2\pi} \mathrm{d}\sigma \Big(P_i P^i + T^2 X_i' X'^i\Big) \,.$$

We also see that $\gamma^{\tau\sigma}(\tau)$ plays the role of the Lagrangian multiplier to the level-matching constraint $\mathcal{V}$. Without loss of generality we will choose $\gamma^{\tau\sigma} = 0$ which corresponds the conformal gauge condition discussed above.

From the gauge-fixed Lagrangian we conclude that our physical variables are (P_i, X_i), where $i = 1, \ldots, d-2$, and also (x^-, p^+) and they have the following Poisson brackets

$$\begin{aligned} &\{X^i(\sigma,\tau), X^j(\sigma',\tau)\} = \{P^i(\sigma,\tau), P^j(\sigma',\tau)\} = 0 \,, \\ &\{X^i(\sigma,\tau), P^j(\sigma',\tau)\} = \delta^{ij}\delta(\sigma - \sigma') \,, \\ &\{p^+, x^-\} = 1 \,. \end{aligned} \tag{3.69}$$

The physical Hamiltonian and the Poisson brackets look the same as the ones in the conformal gauge, however, the important difference is that now they involve $2(d-2)$ physical fields only plus two additional degrees of freedom (x^-, p^+). First, from eq.(3.65) we find that the zero mode of X^- evolves as

$$\dot{x}^- = \frac{1}{p^+}\mathrm{H} \quad \Longrightarrow \quad x^-(\tau) = x^- + \frac{\mathrm{H}}{p^+}\tau\,.$$

It is this τ-independent mode x^- which is conjugate to p^+. Second, equations

$$\partial_\pm X^- = \frac{2\pi T}{p^+}(\partial_\pm X^i)^2 \tag{3.70}$$

can be solved for the longitudinal oscillators α_n^-, $\bar{\alpha}_n^-$ with $n \neq 0$ by substituting an expansion

$$X^-(\tau,\sigma) = x^- + \underbrace{\frac{p^-}{2\pi T}}_{\frac{\mathrm{H}}{p^+}}\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\Big(\alpha_n^- e^{-in\sigma^-} + \bar{\alpha}_n^- e^{-in\sigma^+}\Big)\,. \tag{3.71}$$

We find $p^- = \frac{2\pi T}{p^+}\mathrm{H}$ and

$$\alpha_n^- = \frac{\sqrt{\pi T}}{p^+}\sum_{m=-\infty}^{\infty}\alpha_{n-m}^i\alpha_m^i\,, \qquad n \neq 0, \tag{3.72}$$

$$\bar{\alpha}_n^- = \frac{\sqrt{\pi T}}{p^+}\sum_{m=-\infty}^{\infty}\bar{\alpha}_{n-m}^i\bar{\alpha}_m^i\,, \qquad n \neq 0\,. \tag{3.73}$$

These formulae give a complete solution for X^-.

Thus, the light-cone gauge allows for the explicit solution of the Virasoro constraints. The variables (P_i, X_i) are physical excitations while $X^\pm, P^+$ were removed by the light-cone gauge choice and by solving the constraints. The variable P^- plays the role of the Hamiltonian for physical excitations! Equations of motion for physical fields are the same as before

$$\ddot{X}^i - X''^i = 0\,, \qquad i = 1, \ldots, d-2.$$

The variables (P_i, X_i), where $i = 1, \ldots, d-2$, are called *transversal*, while $X^\pm$, $P^\pm$ are *longitudinal*. The only constraint which we were not able to solve explicitly is the level-matching constraint $\mathcal{V} = 0$. It is easy to check that

$$\{X^i, \mathcal{V}\} = \partial_\sigma X^i\,, \qquad \{P^i, \mathcal{V}\} = \partial_\sigma P^i\,, \tag{3.74}$$

i.e. $\mathcal{V}$ generates the rigid σ-rotations. We also have the evolution equations

$$\{X^i, \mathrm{H}\} = \frac{1}{T}P^i = \partial_\tau X^i\,, \qquad \{P^i, \mathrm{H}\} = T\partial_\sigma^2 X^i = \partial_\tau P^i\,. \tag{3.75}$$

One can also see that the τ- and σ-flows generated by H and $\mathcal{V}$ respectively commute with each other because

$$\{\mathrm{H}, \mathcal{V}\} = 0\,.$$

Imposition of the light-cone gauge is possible because in the conformal gauge the field X^+ satisfies the wave equation: $\Box X^+ = 0$. The corresponding solution for X^+ is

$$X^+(\tau,\sigma) = x^+ + \frac{p^+}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\Big(\alpha_n^+ e^{-in\sigma^-} + \bar{\alpha}_n^+ e^{-in\sigma^+}\Big)\,. \tag{3.76}$$

Our gauge choice (3.58) is taken to be compatible with the form (3.76). Effectively, it means taken equal to zero all oscillators

$$\alpha_n^+ = 0 = \bar{\alpha}_n^+\,, \qquad n \neq 0$$

and also $x^+ = 0$ or

$$\alpha_n^+ = \bar{\alpha}_n^+ = \frac{p^+}{\sqrt{4\pi T}}\delta_{n,0}\,.$$

One may wonder why one cannot completely remove X^+, i.e. to choose a gauge $X^+ = 0$. To understand this point one has to remember that the infinitesimal conformal transformations are of the form

$$X \to X + \sum_n a_n e^{in\sigma^-}\partial_- X\,, \qquad X \to X + \sum_n \bar{a}_n e^{in\sigma^+}\partial_+ X\,,$$

where $a_n, \bar{a}_n$ are arbitrary constants. In other words these transformations can be written as

$$X \to X + \xi^-(\sigma^-)\,, \qquad X \to X + \xi^+(\sigma^+)\,,$$

where $\xi^\pm$ are arbitrary functions obeying only one requirement: they must be periodic in σ. These functions can be used to remove all oscillator modes and the zero mode x^+ but they cannot remove

$$p^+\tau = \tfrac{p^+}{2}(\tau - \sigma) + \tfrac{p^+}{2}(\tau + \sigma)$$

because the functions $\tau - \sigma$ and $\tau + \sigma$ are not periodic in σ.

String in the light-cone gauge can be treated in the standard framework of the Hamiltonian reduction. The Hamiltonian $\mathrm{H} = L_0 + \bar{L}_0$ is invariant under the symmetry algebra on the surface $L_m = 0 = \bar{L}_m$:

$$\{\mathrm{H}, L_m\} = imL_m\,, \qquad \{\mathrm{H}, \bar{L}_m\} = im\bar{L}_m\,.$$

The symmetry algebra itself is

$$\{L_n, L_m\} = -i(m-n)L_{n+m}\,, \qquad \{\bar{L}_n, \bar{L}_m\} = -i(m-n)\bar{L}_{n+m}\,.$$

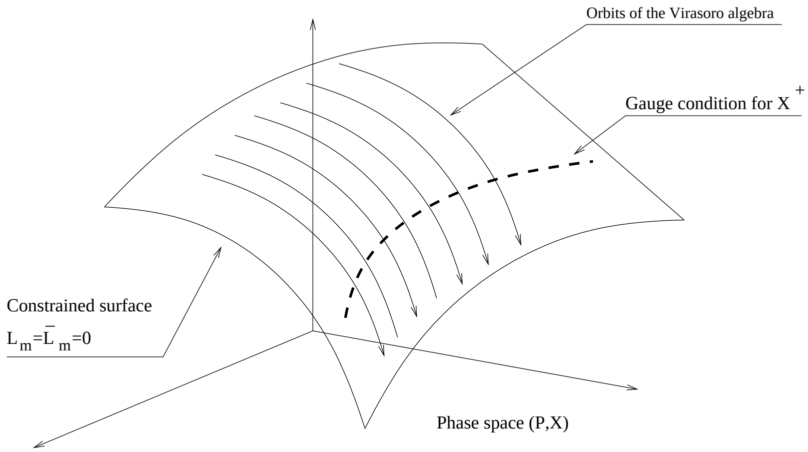


Fig. 2. The physical phase space is obtained by solving the Virasoro constraints $L_m = 0 = \bar{L}_m$ and reducing the action of the Virasoro algebra on the constrained surface by imposing the light-cone gauge.

One would like to reduce the dynamics of the system over the action of the symmetry algebra. In the framework of the Hamiltonian reduction conditions

$$L_m = 0 = \bar{L}_m$$

correspond to fixing the moment map. The reduced phase space $\mathcal{P}$ is defined as a quotient space

$$\mathcal{P} = \frac{\text{solutions of } L_m = 0 = \bar{L}_m}{\text{isotropy subalgebra}}\,,$$

where the isotropy subalgebra is a subalgebra of the Virasoro algebra which leaves the surface $L_m = 0 = \bar{L}_m$ invariant. In our present situation this subalgebra coincides with the algebra itself and, therefore,

$$\mathcal{P} = \frac{\text{solutions of } L_m = 0 = \bar{L}_m}{\text{action of Virasoro}}\,.$$

The action of the Virasoro algebra is factored out by imposing the light-cone gauge, which simultaneously leads to solving the Virasoro constraints. The transversal coordinates introduced above provide the description of the reduced phase space.

Mass of the string in the light-cone gauge is computed as follows (recall that mass is a quadratic Casimir of the Poincaré group). Since we have found that $p^- = \frac{2\pi T}{p^+}\mathrm{H}$ we get for the mass

$$M^2 = -p_\mu p^\mu = -(p^i)^2 + 2p^+p^- = -(p^i)^2 + 4\pi T\mathrm{H}\,. \tag{3.77}$$

The physical Hamiltonian is

$$\mathrm{H} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\alpha^i_n\alpha^i_{-n} + \bar{\alpha}^i_n\bar{\alpha}^i_{-n}\right) = \frac{(p^i)^2}{4\pi T} + \sum_{n=1}^{\infty}\left(\alpha^i_n\alpha^i_{-n} + \bar{\alpha}^i_n\bar{\alpha}^i_{-n}\right). \tag{3.78}$$

Thus, we obtain

$$M^2 = 4\pi T\sum_{n=1}^{\infty}\left(\alpha^i_n\alpha^i_{-n} + \bar{\alpha}^i_n\bar{\alpha}^i_{-n}\right) = \frac{2}{\alpha'}\sum_{n=1}^{\infty}\left(\alpha^i_n\alpha^i_{-n} + \bar{\alpha}^i_n\bar{\alpha}^i_{-n}\right). \tag{3.79}$$

This clearly shows positivity of M^2, a property which was not obvious in the conformal gauge.

Finally, we can write the level-matching condition in terms of transversal oscillators. We find

$$\mathcal{V} = \frac{1}{2T}\sum_{n\neq 0}\left(\bar{\alpha}^i_n\bar{\alpha}^i_{-n} - \alpha^i_n\alpha^i_{-n}\right) = 0\,. \tag{3.80}$$

Thus, the level-matching condition tells that the left- and right-moving oscillators contribute the same amount of energy.

3.4.2 Poisson structure of the light-cone theory

Using the Poisson brackets of physical fields we can now establish the Poisson relations between all quantities of interest. We first summarize the basic Poisson relations for the closed string case

$\{,\}$	p^+	p^-	p^j	x^j	x^-	α^j_m	α^-_m
p^+	0	0	0	0	1	0	0
p^-	0	0	0	$-\frac{p^i}{p^+}$	$-\frac{p^-}{p^+}$	$\frac{2\pi iT}{p^+}m\alpha^j_m$	$\frac{2\pi iT}{p^+}m\alpha^-_m$
p^i	0	0	0	$-\delta^{ij}$	0	0	0
x^i	0	$\frac{p^i}{p^+}$	δ^{ij}	0	0	$\frac{\delta^{ij}\delta_m}{4\pi T}$	$\frac{\alpha^i_m}{p^+}$
x^-	-1	$\frac{p^-}{p^+}$	0	0	0	0	$\frac{\alpha^-_m}{p^+}$
α^i_n	0	$-\frac{2\pi iT}{p^+}n\alpha^i_n$	0	$-\frac{\delta^{ij}\delta_n}{4\pi T}$	0	$-in\delta^{ij}\delta_{n+m}$	$-i\frac{\sqrt{4\pi T}}{p^+}n\alpha^i_{n+m}$
α^-_n	0	$-\frac{2\pi iT}{p^+}n\alpha^-_n$	0	$-\frac{\alpha^i_n}{p^+}$	$-\frac{\alpha^-_n}{p^+}$	$i\frac{\sqrt{4\pi T}}{p^+}m\alpha^i_{n+m}$	$\{\alpha^-_n,\alpha^-_m\}$

Tab. 1. Poisson brackets of the light-cone modes. The variable p^- is essentially the Hamiltonian: $p^- = 2\pi T\frac{\mathrm{H}}{p^+}$. The brackets involving $\bar{\alpha}$ variables are the same.

These relations are easy to derive. For instance,

$$\{p^-,x^-\} = \{2\pi T\frac{\mathrm{H}}{p^+},x^-\} = -2\pi T\mathrm{H}\frac{1}{(p^+)^2}\{p^+,x^-\} = -2\pi T\frac{\mathrm{H}}{(p^+)^2} = -\frac{p^-}{p^+}\,.$$

Also, one has to remember that the variable α^-_n contains the zero mode α^i_0

$$\alpha^-_n = \frac{\sqrt{\pi T}}{p^+}2\alpha^i_0\alpha^i_n + \ldots = \frac{p^i\alpha^i_n}{p^+} + \ldots$$

and, therefore,

$$\{x^i,\alpha^-_n\} = \frac{1}{p^+}\{x^i,p^j\alpha^j_n\} = \frac{1}{p^+}\alpha^i_n\,.$$

The most complicated bracket is $\{\alpha^-_m,\alpha^-_n\}$.

Let us outline the computation of this bracket.

$$
\begin{aligned}
&\{\alpha_m^-,\alpha_n^-\}=\Big\{\frac{p^i\alpha_m^i}{p^+}+\frac{\sqrt{\pi T}}{p^+}\sum_{k\neq m,0}\alpha_{m-k}^i\alpha_k^i,\ \frac{p^j\alpha_n^j}{p^+}+\frac{\sqrt{\pi T}}{p^+}\sum_{l\neq n,0}\alpha_{n-l}^j\alpha_l^j\Big\}=\\
&\Big\{\frac{p^i\alpha_m^i}{p^+},\frac{p^j\alpha_n^j}{p^+}\Big\}+\frac{\sqrt{\pi T}}{p^+}\Big\{\frac{p^i\alpha_m^i}{p^+},\sum_{l\neq n,0}\alpha_{n-l}^j\alpha_l^j\Big\}-\frac{\sqrt{\pi T}}{p^+}\Big\{\frac{p^i\alpha_n^i}{p^+},\sum_{l\neq m,0}\alpha_{m-l}^j\alpha_l^j\Big\}+\frac{\pi T}{(p^+)^2}\Big\{\sum_{k\neq m,0}\alpha_{m-k}^i\alpha_k^i,\sum_{l\neq n,0}\alpha_{n-l}^j\alpha_l^j\Big\}\\
&\underbrace{-im\frac{p^ip^i}{(p^+)^2}\delta_{n+m}}_{\text{first bracket}}\underbrace{-2i(m-n)\frac{\sqrt{\pi T}}{(p^+)^2}p^i\alpha_{n+m}^i}_{\text{second and third brackets}}\underbrace{+\frac{2\pi T}{(p^+)^2}\left(-i(m-n)\sum_{k\neq m+n,0}\alpha_{n+m-k}^i\alpha_k^i\right)}_{\text{forth bracket}}\,.
\end{aligned}
\tag{3.81}
$$

Thus, we are getting

$$
\{\alpha_m^-,\alpha_n^-\}=-im\frac{p^ip^i}{(p^+)^2}\delta_{n+m}+\frac{2\pi T}{(p^+)^2}\left(-i(m-n)\sum_{k=-\infty}^{\infty}\alpha_{n+m-k}^i\alpha_k^i\right)\,,
$$

where due to $\alpha_0^i=\frac{p^i}{\sqrt{4\pi T}}$ we combined the second and a third terms into one sum. If $m+n\neq 0$ then the first term vanishes and we can rewrite the last formula as

$$
\{\alpha_m^-,\alpha_n^-\}=\frac{\sqrt{4\pi T}}{p^+}\,\square\,-i(m-n)\alpha_{m+n}^-\,\square\,.
$$

If $n=-m$ then in eq.(3.81) contribution from the second and the third term vanishes and we get

$$
\{\alpha_m^-,\alpha_{-m}^-\}=-im\frac{p^ip^i}{(p^+)^2}+\frac{2\pi T}{(p^+)^2}\left(-2im\sum_{k\neq 0}\alpha_{-k}^i\alpha_k^i\right)\equiv-2im\frac{\sqrt{4\pi T}}{p^+}\left(\frac{\sqrt{\pi T}}{p^+}\Big[\ \ \frac{p^i}{\sqrt{4\pi T}}\Big)^2+\sum_{k\neq 0}\alpha_k^i\alpha_{-k}^i\Big]\right)\,.
$$

Therefore,

$$
\{\alpha_m^-,\alpha_{-m}^-\}=-2im\frac{\sqrt{4\pi T}}{p^+}\left(\frac{\sqrt{\pi T}}{p^+}\sum_{k=-\infty}^{\infty}\alpha_k^i\alpha_{-k}^i\right)\,.
$$

It is therefore natural to define

$$
\alpha_0^-\equiv\frac{\sqrt{\pi T}}{p^+}\sum_{k=-\infty}^{\infty}\alpha_k^i\alpha_{-k}^i\,.
$$

With this definition we obtained a universal formula (valid for all indices m and n):

$$
\{\alpha_m^-,\alpha_n^-\}=\frac{\sqrt{4\pi T}}{p^+}\,\square\,-i(m-n)\alpha_{m+n}^-\,\square\,.
$$

Also we conclude that with this definition

$$
p^-=\sqrt{\pi T}(\alpha_0^-+\bar{\alpha}_0^-)\,.
$$

Thus, one finds the following result

$$
\{\alpha_m^-,\alpha_n^-\}=\frac{\sqrt{4\pi T}}{p^+}\left(-i(m-n)\alpha_{m+n}^-\right)\,.
$$

If we introduce

$$
L_n=\frac{p^+}{\sqrt{4\pi T}}\alpha_n^-\,.
$$

we therefore find

$$
\{L_n,L_m\}=-i(n-m)L_{n+m}
$$

which is the classical Virasoro algebra! Thus, in the light-cone gauge the Virasoro algebra is carried over by the longitudinal oscillators α_n^-.

3.4.3 Lorentz symmetry

The light-cone gauge manifestly breaks the d-dimensional Lorentz invariance of the theory. We have the generators of the Lorentz algebra which in terms of transversal oscillators become realized non-linearly. For instance, the generator

$$J^{i-} = \underbrace{x^i p^- - x^- p^i}_{\ell^{i-}} - i\sum_{n=1}^{\infty}\frac{1}{n}\big(\alpha^i_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^i_n\big) - i\sum_{n=1}^{\infty}\frac{1}{n}\big(\bar{\alpha}^i_{-n}\bar{\alpha}^-_n - \bar{\alpha}^-_{-n}\bar{\alpha}^i_n\big)$$

is not anymore quadratic in oscillators because p^- and α^- are non-trivial function of the transversal oscillators.

In spite of rather non-linear realization one can check that all the Poisson relations involving J^{i-} are still satisfied (this is also consequence of the fact that $\{L_m, J^{\mu\nu}\} = 0 = \{\bar{L}_m, J^{\mu\nu}\}$, which means that the Poisson bracket of $J^{\mu\nu}$ admits a reduction on the constraint surface $L_m = 0 = \bar{L}_m$). In particular, from the Poincaré algebra we must have

$$\{J^{i-}, J^{j-}\} = 0\,.$$

Let us check this explicitly. First we have

$$\begin{aligned}\{\ell^{i-}, \ell^{j-}\} &= \{x^i p^- - x^- p^i, x^j p^- - x^- p^j\} = \\ &= \{x^i p^-, x^j p^-\} - \{x^- p^i, x^j p^-\} - \{x^i p^-, x^- p^j\} + \{x^- p^i, x^- p^j\}\,.\end{aligned}$$

By using the Poisson brackets from Table 1 we obtain

$$\{\ell^{i-}, \ell^{j-}\} = \underbrace{\frac{p^i}{p^+}p^- x^j - \frac{p^j}{p^+}p^- x^i}_{\text{first bracket}} \underbrace{-p^i x^j \frac{p^-}{p^+} + x^- p^- \delta^{ij}}_{\text{second bracket}} \underbrace{+p^j x^i \frac{p^-}{p^+} - x^- p^- \delta^{ij}}_{\text{third bracket}}\,.$$

and the forth bracket vanishes. Thus,

$$\{\ell^{i-}, \ell^{j-}\} = 0\,.$$

Consider the Poisson bracket of the internal spin components

$$\begin{aligned}\{S^{i-}, S^{j-}\} &= -\sum_{n,m\neq 0}\frac{1}{nm}\{\alpha^i_{-n}\alpha^-_n, \alpha^j_{-m}\alpha^-_m\} = \\ &= -\sum_{n,m\neq 0}\frac{1}{nm}\Big(\{\alpha^i_{-n}, \alpha^j_{-m}\}\alpha^-_n\alpha^-_m + \{\alpha^i_{-n}, \alpha^-_m\}\alpha^-_n\alpha^j_{-m} + \{\alpha^-_n, \alpha^j_{-m}\}\alpha^i_{-n}\alpha^-_m + \{\alpha^-_n, \alpha^-_m\}\alpha^i_{-n}\alpha^j_{-m}\Big) = \\ &= i\delta^{ij}\sum_{n\neq 0}\frac{\alpha^-_n\alpha^-_{-n}}{n} + i\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n\neq 0}\Big(-\frac{1}{m}\alpha^i_{-n+m}\alpha^j_{-m}\alpha^-_n + \frac{1}{n}\alpha^i_{-n}\alpha^j_{n-m}\alpha^-_m + \frac{n-m}{nm}\alpha^i_{-n}\alpha^j_{-m}\alpha^-_{m+n}\Big)\,.\end{aligned}$$

Here the first first sum is zero (proved by changing n for $-n$). In the second and the third summonds one makes the change of summation indices, $n \to n+m$ and $m \to m+n$ respectively. After this change these terms cancel exactly against the last one. However, there are terms which are still left, these are the terms in the second and third summonds containing zero modes, i.e. terms for which $-n+m = 0$ and also the term of the last summond for which $m+n = 0$. Thus,

$$\{S^{i-}, S^{j-}\} = -\frac{i}{p^+}\sum_{n\neq 0}\frac{1}{n}\Big(p^i\alpha^j_{-n} - p^j\alpha^i_{-n}\Big)\alpha^-_n + 2i\frac{\sqrt{4\pi T}}{p^+}\alpha^-_0\sum_{n\neq 0}\frac{1}{n}\alpha^i_n\alpha^j_{-n}\,. \tag{3.82}$$

Analogously, we will get the following contribution from the left-moving modes Then we compute

$$\{\ell^{i-}, S^{j-}\} = -i\sum_{n\neq 0}\frac{1}{n}\{\underbrace{x^i p^-}_{\mathrm{A}} - \underbrace{x^- p^i}_{B}, \alpha^j_{-n}\alpha^-_n\} = -i\sum_{n\neq 0}\frac{1}{n}\left(\underbrace{\alpha^j_{-n}p^-\frac{\alpha^i_n}{p^+} + x^i\Big(-\frac{2\pi i T}{p^+}n\alpha^j_{-n}\alpha^-_n + \frac{2\pi i T}{p^+}n\alpha^j_{-n}\alpha^-_n\Big)}_{\text{from A}} \underbrace{-p^i\alpha^j_{-n}\frac{\alpha^-_n}{p^+}}_{\text{from B}}\right)\,.$$

Thus, we arrive at

$$\{\ell^{i-}, S^{j-}\} + \{S^{i-}, \ell^{j-}\} = -2i\frac{p^-}{p^+}\sum_{n\neq 0}\frac{1}{n}\alpha^i_n\alpha^j_{-n} + \frac{i}{p^+}\sum_{n\neq 0}\frac{1}{n}\left(p^i\alpha^j_{-n} - p^j\alpha^i_{-n}\right)\alpha^-_n\,. \tag{3.83}$$

Now summing up equations (3.82) and (3.83) we obtain

$$\{\ell^{i-}, S^{j-}\} + \{S^{i-}, \ell^{j-}\} + \{S^{i-}, S^{j-}\} = i\left(\frac{4\sqrt{\pi T}}{p^+}\alpha^-_0 - \frac{\alpha^-_0 + \bar{\alpha}^-_0}{2}\right)\sum_{n\neq 0}\frac{1}{n}\alpha^i_n\alpha^j_{-n}\,. \tag{3.84}$$

At first glance this expression is non-zero and it can not be compensated by the contribution of left-moving modes

$$\{\ell^{i-}, \bar{S}^{j-}\} + \{\bar{S}^{i-}, \ell^{j-}\} + \{\bar{S}^{i-}, \bar{S}^{j-}\} = i\left(\frac{4\sqrt{\pi T}}{p^+}\bar{\alpha}^-_0 - \frac{\alpha^-_0 + \bar{\alpha}^-_0}{2}\right)\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^i_n\bar{\alpha}^j_{-n}\,. \tag{3.85}$$

However, we have to invoke the level-matching constraint which simply tells that $\alpha^-_0 = \bar{\alpha}^-_0$ and makes both eqs.(3.84) and (3.85) separately vanish. Thus, we have indeed shown that the most non-trivial relation $\{J^{i-}, J^{j-}\} = 0$ is indeed satisfied.

4. Quantization of bosonic string

4.1 Remarks on canonical quantization

According to the standard principles of quantum mechanics canonical quantization consists in replacing the Poisson brackets of the fundamental phase space variables by commutators

$$\{\ ,\ \} \to \frac{1}{i\hbar}[\ ,\ ]\,,$$

where $\hbar$ is the Plank constant. Thus, we consider now $X(\sigma,\tau)$ and $P(\sigma,\tau)$ as the quantum mechanical operators which obey the following commutation relations[7]

$$\begin{aligned} &[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = 0\,,\\ &[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma')\,, \end{aligned} \tag{4.1}$$

These commutation relations induce the commutation relations on the Fourier coefficients

$$\begin{aligned} &[\alpha^\mu_m, \alpha^\nu_n] = [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = \hbar m\delta_{m+n}\eta^{\mu\nu}\,,\\ &[\alpha^\mu_m, \bar{\alpha}^\nu_n] = 0\,,\\ &[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}\,. \end{aligned} \tag{4.2}$$

For the case of open string the modes $\bar{\alpha}_n$ are absent. In what follows we will work in units in which $\hbar = 1$, so that the commutation relations read as

$$\begin{aligned} &[\alpha^\mu_m, \alpha^\nu_n] = [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,,\\ &[\alpha^\mu_m, \bar{\alpha}^\nu_n] = 0\,,\\ &[x^\mu, p^\nu] = i\eta^{\mu\nu}\,. \end{aligned} \tag{4.3}$$

[7]Here the indices μ, ν run from 0 to $d-1$.

To restore $\hbar$, one has to simply rescale the modes as $\alpha \to \frac{1}{\sqrt{\hbar}}\alpha$, $\bar{\alpha} \to \frac{1}{\sqrt{\hbar}}\bar{\alpha}$.

Let us take $m > 0$ and rescale $a_m = \frac{1}{\sqrt{m}}\alpha_m$. Then using the hermiticity property the commutation relations can be rewritten as

$$
\begin{aligned}
[a_n^\mu, a_m^{\dagger\nu}] &= \eta^{\mu\nu}\delta_{n,m} \\
[\bar{a}_n^\mu, \bar{a}_m^{\dagger\nu}] &= \eta^{\mu\nu}\delta_{n,m}
\end{aligned}
$$

which are the standard commutation relations of two infinite sets of independent quantum harmonic oscillators.

Introduce the number operator for m's mode

$$N_m =: \alpha_m^\mu \alpha_{-m,\mu} :$$

Then we see that for $m > 0$

$$
\begin{aligned}
[N_m, \alpha_m] &= -m\alpha_m \,, \\
[N_m, \alpha_{-m}] &= m\alpha_{-m} \,.
\end{aligned}
$$

From here we conclude that

- Modes with $m > 0$ should be identified with the lowering operators
- Modes with $m < 0$ should be identified with the raising operators

Construction of the representation of the canonical commutation relations is completed by introducing the *ground state* which satisfies the following properties

$$
\begin{aligned}
\alpha_m^\mu |p^\nu\rangle &= 0 \,, \\
\hat{p}^\mu |p^\nu\rangle &= p^\mu |p^\nu\rangle \,.
\end{aligned}
\tag{4.4}
$$

The whole infinite-dimensional (Hilbert) space of states is obtained by acting on the ground state with creation operators.

This construction brings us to the major problem of canonical quantization. Consider for m positive the following commutator

$$[\alpha_m^0, \alpha_{-m}^0] = [\alpha_m^0, (\alpha_m^0)^\dagger] = m\eta^{00} = -m \,.$$

Thus, we induce from here (let even a ground state carries zero momentum p^μ)

$$\langle 0|[\alpha_m^0, (\alpha_m^0)^\dagger]|0\rangle = \langle 0|\alpha_m^0 \; (\alpha_m^0)^\dagger|0\rangle = ||(\alpha_m^0)^\dagger|0\rangle||^2 = -m < 0 \,.$$

Thus, Minkowskian type of the target-space metric leads to the existence in the Hilbert space the states with *negative* norm. States with negative norm are sometimes called "ghosts" and they do not allow for probability interpretation of the corresponding quantum-mechanical system.

One can correctly anticipate that the problem with negative norm states arose because we did not take into account the Virasoro constraints. The covariant approach to quantization consists in defining the subspace of physical states in the original Hilbert space which obey the Virasoro constraints. One can further show that in a special dimension of space-time ($d = 26$) the negative norm states decouple from the physical Hilbert space.

In the classical theory we have the constraints $L_m = 0 = \bar{L}_m$. However, in quantum theory expressions for L_m and $\bar{L}_m$ are *quadratic* in oscillators and might involve operators (quantum oscillators) which do not commute with each other! From all L_m a constraint which suffers from ordering ambiguity is L_0 as

$$L_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \alpha_{-n,\mu} \tag{4.5}$$

and oscillators α_n^μ and α_{-n}^μ do not commute with each other. The standard way to deal with this ambiguity in quantum field theory is to use the normal ordering prescription

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{m-n}^\mu \alpha_{n,\mu} : \, . \tag{4.6}$$

The normal ordering prescription means that

$$: \alpha_{m_1} \dots \alpha_{m_k} := \underbrace{\alpha_{n_1} \dots \alpha_{n_p}}_{\text{all creation}} \underbrace{\alpha_{s_1} \dots \alpha_{s_r}}_{\text{all annihilation}}$$

in the operators are ordered in such a fashion that *all* annihilation operators are put on the right from *all* the creation operators. The order of the creation (or annihilation) operators between themselves does not matter because these operators commute between themselves and therefore their expression does not have ordering ambiguity. In particular, for L_0 we have

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n,\mu} - a \, , \tag{4.7}$$

where we include a so far *unknown normal ordering constant* a. As to the zero modes, the normal-ordering prescription here is

$$: p^\mu x^\nu := x^\nu p^\mu \, .$$

Since the ground state obeys $\hat{p}^\mu|p\rangle = p^\mu|p\rangle$ it can be regarded as the usual quantum-mechanical eigenstate of the momentum operator $\hat{p}^\mu = -i\frac{\partial}{\partial x_\mu}$ which is in the momentum representation has the form of the plane-wave

$$|p\rangle \equiv e^{ip^\mu x_\mu}|0\rangle \, ,$$

where on the r.h.s. p^μ is not an operator and $|0\rangle$ denotes the *zero-momentum* ground state. Plane-waves are not square-integrable functions but they form a basis of a generalized Hilbert space and they are assumed to be normalized as

$$\langle p|p'\rangle = \delta(p-p')\,.$$

4.1.1 Virasoro algebra

Let us now investigate the algebra of the operators L_m in the quantum case. We first assume that the normal ordering constant $a=0$. We start by computing

$$[\alpha^\mu_m, L_n] = \tfrac{1}{2}\sum_{p=-\infty}^{\infty}[\alpha^\mu_m, :\alpha^\nu_p\alpha_{n-p,\nu}:] = \tfrac{1}{2}\sum_{p=-\infty}^{\infty}\Big(m\delta_{m+p}\alpha^\mu_{n-p} + m\alpha^\mu_p\delta_{m+n-p}\Big) = m\alpha^\mu_{n+m}\,.$$

Then we have

$$[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{\infty}[:\alpha^\mu_p\alpha_{m-p,\mu}:, L_n]\,.$$

We write down the normal ordering explicitly (to simplify the notation we write the Lorentz summation index on the same level)

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{0}[\alpha^\mu_p\alpha^\mu_{m-p}, L_n] + \frac{1}{2}\sum_{p=1}^{\infty}[\alpha^\mu_{m-p}\alpha^\mu_p, L_n] = \\
&= \frac{1}{2}\sum_{p=-\infty}^{0}\underbrace{p\alpha^\mu_{p+n}\alpha^\mu_{m-p}}_{p=q-n} + (m-p)\alpha^\mu_p\alpha^\mu_{m-p+n} \\
&+ \frac{1}{2}\sum_{p=1}^{\infty}(m-p)\alpha^\mu_{m-p+n}\alpha^\mu_p + \underbrace{p\alpha^\mu_{m-p}\alpha^\mu_{n+p}}_{p=q-n}\,.
\end{aligned}$$

In the underbraced terms we make a change of summation index $p=q-n$ and get

$$\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Big(&\sum_{p=-\infty}^{0}(m-p)\alpha^\mu_p\alpha^\mu_{m+n-p} + \sum_{q=-\infty}^{n}(q-n)\alpha^\mu_q\alpha^\mu_{m+n-q} \\
&+ \sum_{p=1}^{\infty}(m-p)\alpha^\mu_{m+n-p}\alpha^\mu_p + \sum_{q=n+1}^{+\infty}(q-n)\alpha^\mu_{m+n-q}\alpha^\mu_q\Big)\,.
\end{aligned}$$

Without loss of generality we assume that $n>0$. then we have

$$\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Big(&\sum_{p=-\infty}^{0}(m-n)\alpha^\mu_p\alpha^\mu_{m+n-p} + \sum_{q=1}^{n}(q-n)\underbrace{\alpha^\mu_q\alpha^\mu_{m+n-q}}_{\text{not ordered!}} \\
&+ \sum_{p=n+1}^{\infty}(m-n)\alpha^\mu_{m+n-p}\alpha^\mu_p + \sum_{q=1}^{n}(m-q)\alpha^\mu_{m+n-q}\alpha^\mu_q\Big)\,.
\end{aligned}$$

Using $\alpha^\mu_q \alpha^\mu_{m+n-q} = \alpha^\mu_{m+n-q}\alpha^\mu_q + q\delta_{m+n}\delta^\mu_\mu = \alpha^\mu_{m+n-q}\alpha^\mu_q + qd\delta_{m+n}$, where d is the dimension of the Minkowskian space-time where string propagates. Thus, the algebra relation is

$$[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{\infty}(m-n) : \alpha^\mu_p \alpha^\mu_{m+n-p} : + \frac{d}{2}\delta_{n+m}\sum_{q=1}^{n}(q^2 - nq)\,.$$

Since

$$\sum_{q=1}^{n} q^2 = \frac{1}{6}n(n+1)(2n+1)\,, \qquad \sum_{q=1}^{n} q = \frac{1}{2}n(n+1)\,.$$

one finds the final result

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\delta_{m+n}$$

which is the famous Virasoro algebra. We see that it is different from the classical Virasoro (Wit) algebra by the presence of the *central* term.

In a more general setting the algebra is written as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,,$$

where the constant term c is known as the *central charge*.

If we introduce a normal ordering constant a by shifting the definition of L_m as $L_m \to L_m - \delta_{m,0}$ then the linear term in m in the central term changes

$$[L_m, L_n] = (m-n)L_{m+n} + \Big(\frac{c}{12}m^3 + \big(2a - \frac{c}{12}\big)m\Big)\delta_{m+n}\,.$$

We see that the central term has an invariant meaning and cannot be removed for all L_m by adjusting the normal ordering constant a.

Finally, we comment on the relation to semiclassics. If we restore the Plank the algebra relations take the form

$$[L_m, L_n] = \hbar(m-n)L_{m+n} + \hbar^2\frac{c}{12}m(m^2-1)\delta_{m+n}\,.$$

We see that one can define the Poisson bracket

$$\{L_m, L_n\} = \lim_{\hbar\to 0}\frac{1}{i\hbar}[L_m, L_n] = -i(m-n)L_{m+n}\,,$$

which coincides with the Wit algebra. The central term obviously vanishes in the semi-classical limit.

4.1.2 Virasoro constraints in quantum theory

At first sight it seems that the natural analog of the classical equations $L_n = 0 = \bar{L}_n$ in quantum theory is to require that a physical state should be annihilated by all Virasoro generators:

$$L_n|\Phi\rangle = 0 = \bar{L}_n|\Phi\rangle\,, \qquad n \in \mathbb{Z}\,.$$

Due to the normal-ordering ambiguity in the definition of L_0 in quantum theory the classical conditions $L_0 = 0 = \bar{L}_0$ are replaced now by

$$(L_0 - a)|\Phi\rangle = 0\,, \qquad (\bar{L}_0 - \bar{a})|\Phi\rangle = 0\,, \tag{4.8}$$

where in fact the normal-orderings constants a and $\bar{a}$ must be equal to each other[8] and L_0, $\bar{L}_0$ are understood as the normal-ordered generators. It is easy to see, however, that eqs.(4.8) cannot be consistently imposed for all m. Indeed, if eqs.(4.8) would be satisfied for all m we would have

$$[L_n, L_{-n}]|\Phi\rangle = 2nL_0|\Phi\rangle + \frac{d}{12}n(n^2-1)|\Phi\rangle\,,$$

i.e.

$$0 = \left(2na + \frac{d}{12}n(n^2-1)\right)|\Phi\rangle \qquad \text{for any } n\,.$$

This is obviously not possible to satisfy unless $|\Phi\rangle = 0$. The physical reason for impossibility to impose in quantum theory the same set of constraints as in the classical one is an *anomaly*. Because of the anomaly term the first-class Virasoro constrains of the classical theory turn upon quantization into the constraints of the second class!

From the experience with the quantum electrodynamics one can try to impose only "half" of the constraints, i.e.

$$\begin{aligned}(L_0 - a)|\Phi\rangle &= 0\,,\\ L_n|\Phi\rangle &= 0\,, \qquad n > 0\,.\end{aligned} \tag{4.9}$$

The conjugate state then obeys $\langle\Phi|L_{-n} = 0$ for $n > 0$ and we see that $\langle\Phi|L_n|\Phi\rangle$ for all $n \neq 0$, i.e. expectation values of L_n vanish for all nonnegative n.

Let us recall that the mass operator is obtained from the constraint $L_0 - a = 0$, We have

$$M^2 = -p^2 = 4\pi T(-a+N)\,, \qquad N = \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n,\mu}\,.$$

[8] This follows from the constraint $(L_0 - \bar{L}_0)|\Phi\rangle = 0$.

Here N is the number operator. It turns out that all eigenvalues of the number operator N are non-negative. Indeed,

$$N = \sum_{n=1}^{\infty} \Big(- \alpha^0_{-n} \alpha^0_n + \sum_{i=1}^{d-1} \alpha^i_{-n} \alpha^i_n \Big) .$$

We see the "-" sign coming from the time-like oscillators. However, the time-like oscillators themselves provide only non-negative contribution to N, because for any $m > 0$

$$[N, a^0_{-m}] = - \sum_{n=1}^{\infty} [\alpha^0_{-n} \alpha^0_n, a^0_{-m}] = - \sum_{n=1}^{\infty} \alpha^0_{-n} [\alpha^0_n, a^0_{-m}] = m \alpha^0_{-m}$$

since commutator of two time-like oscillators contributes with the negative sign. Thus, time-like creation operators contribute positively to N.

Virasoro primaries, descendents and physical states

Let us introduce the following useful definitions.

1. States which are annihilated by all positively moded Virasoro operators and are eigenstates of the operator L_0 with an eigenvalue a are called *Virasoro primaries*. Number a is called a *weight* of the Virasoro primary.

2. A *Virasoro descendent* of a given primary is a state that can be written as a finite linear combination of products of negatively moded Virasoro operators acting on the primary state.

3. A state which is both primary and descendent is called a *null* state.

If $|\Phi\rangle$ is a primary state then $L_{-1}|\Phi\rangle$ is its descendent. If $N|\Phi\rangle = N_\Phi|\Phi\rangle$ then

$$N L_{-1}|\Phi\rangle = (N_\Phi + 1) L_{-1}|\Phi\rangle.$$

There are two basis descendents with the number $N_\Phi + 2$, namely $L_{-2}|\Phi\rangle$ and $L_{-1}L_{-1}|\Phi\rangle$. The counting of descendents changes at $N_\Phi + 3$. Here the candidate descendents are

$$L_{-3}|\Phi\rangle \, , \qquad L_{-2}L_{-1}|\Phi\rangle \, , \qquad L_{-1}L_{-2}|\Phi\rangle \, , \qquad L^3_{-1}|\Phi\rangle \, .$$

The second and the third states are not identical because the Virasoro operators do not commute. However, due to the Virasoro algebra there is one relation between the above states

$$L_{-1}L_{-2} = [L_{-1}, L_{-2}] + L_{-2}L_{-1} = L_{-3} + L_{-2}L_{-1} \, .$$

Thus, there are only three descendents with number $N_\Phi + 3$.

In general, for any fixed level $N_\Phi + n$, one can choose an independent basis of descendents of the form

$$L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Phi\rangle\,, \qquad \text{where} \qquad n_1 \geq n_2 \geq \ldots \geq n_k \quad \text{and} \quad \sum_{i=1}^{k} n_i = n\,. \tag{4.10}$$

It is a conventional ordering of the Virasoro operators. Note that the number of descendents in this basis is equal to a *number of partitions* of integer n.

For any given primary, there can be a linear combination of descendents (4.10) which vanishes. The vanishing of such combinations happens *not* due to the Virasoro algebra relations but rather due to specific properties of the primary state $|\Phi\rangle$. For instance, one can realize that for the zero-momentum ground (primary) state $|0\rangle$ the descendent $L_{-1}|0\rangle$ vanishes identically.

An important property of the descendents is that they are all orthogonal to any primary. Indeed, a descendent $|\text{des}\rangle$ can be written as $L_{-n_i}|\chi\rangle$ for some $n_i > 0$ and some state $|\chi\rangle$. Then for any primary state

$$\langle\text{des}|\text{prime}\rangle = \langle\chi|L_{n_i}|\text{prime}\rangle = 0\,,$$

since a primary $|\text{prime}\rangle$ is annihilated by all positively moded oscillators.

Null states, which are both primary and descendents, correspond to pure gauge degrees of freedom. Any null state has a vanishing norm and it is orthogonal to any primary and to any descendent. If we alter a primary state by adding a null state, then the new primary state has the same inner products with all primary states as the original one. Adding null states to primaries cannot change any physical expectation value. This motivates the following definition of a physical state:

A physical state is an equivalence class of a primary state with the weight $a = 1$ modulo the null states.

The choice $a = 1$ will be motivated by studying the quantization of strings in the physical (light-cone) gauge. Note that here we talk about equivalent classes precisely because of the ambiguity created by the null states. Primary states which differ by a null state are physically indistinguishable. In the next section studying the spectrum of open string we will see that the null states are indeed responsible for the gauge degrees of freedom.

4.1.3 The spectrum

The classical strings cannot provide a reasonable particle physics because the masses of string states take continuous values. Only the ground state is massless in the classical open string theory but because it carries no spin we are not able to identify it with photon. Quantization procedure – this is what alters the nature of the classical string spectrum and makes it discrete. Simultaneously, states to be identified with photon emerge in the quantum spectrum because of the downward shift of the M^2 producing thereby the massless states with a proper spin labels.

Consider open strings. The Hamiltonian is

$$\mathrm{H} = L_0 - 1 = \alpha' p^2 + N - 1 \,.$$

The basis vectors of the Hilbert space are

$$|\psi\rangle = \prod_{n=1}^{\infty}\prod_{\mu=0}^{25}(\alpha_n^{\mu\dagger})^{\lambda_{n,\mu}}|p\rangle \,,$$

are non-negative integers. Generic state $|\psi\rangle$ is not physical. A physical state is a state which obeys the Virasoro constraints (4.9) and is not a descendent.

Let us look for some examples of physical states. The first one is the ground state $|p\rangle$. The only non-trivial constraint is

$$(L_0 - a)|p\rangle = (\alpha' p^2 - a)|p\rangle = 0 \,.$$

Since $p^2 = -M^2$ we get that the on-shell condition for this state is $M^2 = -\frac{a}{\alpha'}$. Later on studying the light-cone quantization we find that the normal-ordering constant a must be equal to one. Thus, the mass-squared of the ground state is negative: $M^2 = -\frac{1}{\alpha'}$. The corresponding hypothetic particle moving faster than light is called *tachyon*.

The next state to consider is $\zeta_\mu \alpha_{-1}^{\mu}|p\rangle$. We have

$$(L_0 - 1)\zeta_\mu \alpha_{-1}^{\mu}|p\rangle = (\alpha' p^2 + N - 1)\zeta_\mu \alpha_{-1}^{\mu}|p\rangle = \alpha' p^2 \zeta_\mu \alpha_{-1}^{\mu}|p\rangle = 0$$

from which we deduce the on-shell condition $p^2 = 0$, i.e. the corresponding particle is massless. Further condition gives

$$L_1 \zeta_\mu \alpha_{-1}^{\mu}|p\rangle = (\alpha_0\alpha_1 + \alpha_{-1}\alpha_2 + \cdots)\zeta_\mu \alpha_{-1}^{\mu}|p\rangle = \sqrt{2\alpha'}\zeta_\mu p^\mu|p\rangle = 0 \,.$$

Thus, for a physical state the momentum p^μ and the polarization vector ζ^μ must be related as $\zeta_\mu p^\mu = 0$ which is nothing else as the Lorentz gauge condition. All higher Virasoro modes L_n, $n \geq 2$ are automatically annihilate the state. We, however, have not described the physical state completely. The massless vector particle which is photon must have $d-2$ independent polarizations while the Lorentz gauge lives $d-1$ polarizations only. We should now recall that physical states are defined modulo the null states.

Consider a state (κ is any constant)

$$|d\rangle = \frac{\kappa}{\sqrt{2\alpha'}}L_{-1}|p\rangle = \kappa p_\mu \alpha_{-1}^{\mu}|p\rangle \,, \qquad p^2 = 0 \,.$$

This state has the same form as before with $\zeta_\mu = \kappa p_\mu$ (longitudinally polarized photons). It is physical, because $\zeta_\mu p^\mu = \kappa p^2 = 0$. On the other hand, it is null as it appears to be a descendent of L_{-1}. Thus, the states

$$\zeta_\mu \qquad \text{and} \qquad \zeta_\mu + \kappa p_\mu$$

should be identified. The reason for this identification can be understood as follows. If $\zeta_\mu p^\mu = 0$ then $(\zeta_\mu + \kappa p_\mu)p^\mu = 0$ as well, because $p^2 = 0$. As the result, this identification reduces the number of independent polarizations to $d-2$, as it should be for a massless photon. Indeed, for any vector ζ_μ obeying $\zeta_\mu p^\mu = 0$, one can use the shift freedom $\zeta_\mu \longrightarrow \zeta_\mu + \kappa p_\mu$ to put, e.g., one of the components of ζ_μ to zero.

4.1.4 Propagators

Here we introduce the concept of propagator or, equivalently, the two-point Green function.

Consider first the right-moving fields of the closed string. Their propagator is defined as

$$\langle X_R(\tau,\sigma)X_R(\tau',\sigma')\rangle = T\big(X_R(\tau,\sigma)X_R(\tau',\sigma')\big) - :X_R(\tau,\sigma)X_R(\tau',\sigma'):\,, \tag{4.11}$$

where T means time-ordering prescription. Thus,

$$T\big(X_R(\tau,\sigma)X_R(\tau',\sigma')\big) = \begin{cases} X_R(\tau,\sigma)X_R(\tau',\sigma')\,, & \text{for } \ \tau > \tau'\,, \\ X_R(\tau',\sigma')X_R(\tau,\sigma)\,, & \text{for } \ \tau < \tau'\,. \end{cases}$$

For $\tau > \tau'$ we have

$$\begin{aligned} &\langle X_R(\tau,\sigma)X_R(\tau',\sigma')\rangle = \\ &= \Big(\frac{1}{2}x^\mu + \frac{p^\mu}{4\pi T}(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}\Big)\Big(\frac{1}{2}x^\nu + \frac{p^\nu}{4\pi T}(\tau'-\sigma') + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_m e^{-im(\tau'-\sigma')}\Big) \\ &- :\Big(\frac{1}{2}x^\mu + \frac{p^\mu}{4\pi T}(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}\Big)\Big(\frac{1}{2}x^\nu + \frac{p^\nu}{4\pi T}(\tau'-\sigma') + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_m e^{-im(\tau'-\sigma')}\Big):\,. \end{aligned}$$

Most of the terms cancelled and we are left with

$$\begin{aligned} \langle X_R(\tau,\sigma)X_R(\tau',\sigma')\rangle &= \frac{1}{8\pi T}x^\mu p^\nu(\tau'-\sigma') + \frac{1}{8\pi T}p^\mu x^\nu(\tau-\sigma) \\ &- \frac{1}{8\pi T}:x^\mu p^\nu:(\tau'-\sigma') - \frac{1}{8\pi T}:p^\mu x^\nu:(\tau-\sigma) \\ &- \frac{1}{4\pi T}\sum_{n\neq 0}\sum_{m\neq 0}\frac{1}{nm}(\alpha^\mu_n\alpha^\nu_m - :\alpha^\mu_n\alpha^\nu_m:)e^{-in(\tau-\sigma)}e^{-im(\tau'-\sigma')}\,. \end{aligned}$$

Thus,

$$\langle X_R(\tau,\sigma)X_R(\tau',\sigma')\rangle = \frac{1}{8\pi T}[p^\mu, x^\nu](\tau-\sigma) - \frac{1}{4\pi T}\sum_{n>0}\sum_{m<0}\frac{1}{nm}[\alpha^\mu_n, \alpha^\nu_m]e^{-in(\tau-\sigma)}e^{-im(\tau'-\sigma')}\,.$$

Substituting the commutators we obtain

$$\begin{aligned} \langle X_R(\tau,\sigma)X_R(\tau',\sigma')\rangle &= -\frac{i}{8\pi T}\eta^{\mu\nu}(\tau-\sigma) - \frac{\eta^{\mu\nu}}{4\pi T}\sum_{m<0}\frac{1}{m}e^{im(\tau-\sigma)}e^{-im(\tau'-\sigma')} \\ &= -\frac{i}{8\pi T}\eta^{\mu\nu}(\tau-\sigma) + \frac{\eta^{\mu\nu}}{4\pi T}\sum_{m>0}\frac{1}{m}\left(\frac{e^{i(\tau'-\sigma')}}{e^{i(\tau-\sigma)}}\right)^m \\ &= -\frac{i}{8\pi T}\eta^{\mu\nu}(\tau-\sigma) - \frac{\eta^{\mu\nu}}{4\pi T}\ln\left(1 - \frac{e^{i(\tau'-\sigma')}}{e^{i(\tau-\sigma)}}\right)\,. \end{aligned}$$

Thus, for $\tau > \tau'$ one obtains the propagators[9]

$$\begin{aligned}
\langle X_R(\tau,\sigma)X_R(\tau',\sigma')\rangle &= \frac{\eta^{\mu\nu}}{8\pi T}\ln z - \frac{\eta^{\mu\nu}}{4\pi T}\ln(z-z')\,, \\
\langle X_L(\tau,\sigma)X_L(\tau',\sigma')\rangle &= \frac{\eta^{\mu\nu}}{8\pi T}\ln \bar{z} - \frac{\eta^{\mu\nu}}{4\pi T}\ln(\bar{z}-\bar{z}')\,, \\
\langle X_R(\tau,\sigma)X_L(\tau',\sigma')\rangle &= -\frac{\eta^{\mu\nu}}{8\pi T}\ln z\,,
\end{aligned}$$

where we made an identification

$$z = e^{i(\tau-\sigma)}\,, \qquad \bar{z} = e^{i(\tau+\sigma)}\,.$$

Computation for the case of open string is similar. For $\tau > \tau'$ we have

$$\langle X(\tau,\sigma)X(\tau',\sigma')\rangle = -i\frac{\eta^{\mu\nu}}{\pi T}\tau + \frac{1}{\pi T}\sum_{n=1}^{\infty}\frac{1}{n}e^{-in\tau+in\tau'}\cos n\sigma\cos n\sigma'\,.$$

Performing the sum one finds

$$\begin{aligned}
\langle X(\tau,\sigma)X(\tau',\sigma')\rangle = -\frac{\eta^{\mu\nu}}{4\pi T}\Big[&\log\left(e^{i\tau}-e^{-i(\sigma-\sigma')}e^{i\tau'}\right)+\log\left(e^{i\tau}-e^{i(\sigma-\sigma')}e^{i\tau'}\right) \\
&+\log\left(e^{i\tau}-e^{-i(\sigma+\sigma')}e^{i\tau'}\right)+\log\left(e^{i\tau}-e^{i(\sigma+\sigma')}e^{i\tau'}\right)\Big]\,.
\end{aligned}$$

4.1.5 Vertex operators. Tachyon scattering amplitude

Here we approach for the first time the question about string interactions. It is important to realize that the situation here is different to what one usually accounters in QFT. The interaction of strings cannot be introduced by adding non-linear terms to the string Lagrangian; in the latter case one would obtain non-linear interacting theory but still of a single string.

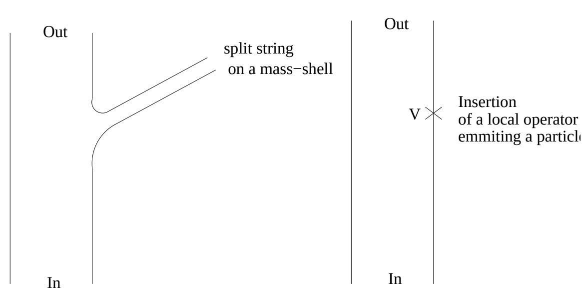


Fig. 3. Open (closed) strings interact by means of joining and splitting. Emission of a point particle on mass-shell is represented by insertion of a local vertex operator.

[9]Rigorous justification of these formulae requires an introduction of an IR regularization, because correlation functions of the massless field in two-dimensions suffer from IR divergencies.

String interactions are introduced by allowing a topology of a wold-sheet to change. If a split string is on-shell it can be thought as an infinite collection of particles which form its spectrum. In the limiting case of single emitted particle the process of scattering can be viewed as application to an initial state $|\mathrm{In}\rangle$ of a local *vertex* operator $V \equiv V(\tau, \sigma)$ which depends on the emitted state and transforms $|\mathrm{In}\rangle$ into the outgoing state $|\mathrm{Out}\rangle$:

$$|\mathrm{Out}\rangle = V|\mathrm{In}\rangle\,.$$

Thus, to any physical state $|\Phi\rangle$ from the spectrum one can put in correspondence a local vertex operator V_Φ.

Conformal operators

Operator $A(\tau)$ is called conformal with the conformal dimension Δ if[10]

$$A'(\tau') = \Big(\frac{d\tau}{d\tau'}\Big)^{\Delta} A(\tau)\,.$$

Under infinitezimal variation $\tau \to \tau' = \tau + \epsilon(\tau)$ one gets

$$\delta A(\tau) = -\epsilon\frac{dA}{d\tau} - \Delta A\frac{d\epsilon}{d\tau}$$

With $\epsilon = -ie^{im\tau}$

$$[L_m, A(\tau)] = e^{im\tau}(-i\partial_\tau + m\Delta)A(\tau)$$

If the operator A is expandable as $A(\tau) = \sum A_n e^{-in\tau}$, for its Fourier modes the last relation implies

$$[L_m, A_n] = (m(\Delta - 1) - n)A_{n+m}$$

Thus, if A has conformal weight $\Delta = 1$ then it's zero mode commutes with all L_m. Therefore, A_0 maps physical states into physical states:

$$|\Phi'\rangle = A_0|\Phi\rangle\,.$$

An alternative way to understand this is to notice that for $\Delta = 1$ we have

$$[L_m, A(\tau)] = e^{im\tau}(-i\partial_\tau + m\Delta)|_{\Delta=1}A(\tau) = -i\frac{\partial}{\partial\tau}\Big(e^{im\tau}A(\tau)\Big)\,,$$

i.e. the r.h.s. is the total derivative. Thus, $A_0 = \int d\tau A(\tau)$ will transform as

$$[L_m, A_0] = -i\int d\tau\frac{\partial}{\partial\tau}\Big(e^{im\tau}A(\tau)\Big)\,,$$

[10] This transformation law can be written as

$$A(\tau')(d\tau')^{\Delta} = A(\tau)(d\tau)^{\Delta}\,.$$

For $\Delta = 1$ this implies that $A(\tau)d\tau$ is a one-form.

where the expression on the r.h.s. is specified by the periodicity/boundary properties of $A(\tau)$.

Explicit example of a vertex operator

Consider the following operator acting in the Hilbert space of open string

$$V(k,\tau,\sigma) = e^{\frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\frac{k_\mu\alpha_{-n}^\mu}{n}e^{in\tau}\cos n\sigma}e^{ik_\mu(x^\mu+\frac{p^\mu}{\pi T}\tau)}e^{-\frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\frac{k_\mu\alpha_n^\mu}{n}e^{-in\tau}\cos n\sigma}\,.$$

At $\sigma=0$ this simplifies to

$$V(k,\tau) = \underbrace{e^{\frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\frac{k_\mu\alpha_{-n}^\mu}{n}e^{in\tau}}}_{V_-}\;\underbrace{e^{ik_\mu(x^\mu+\frac{p^\mu}{\pi T}\tau)}}_{V_0}\;\underbrace{e^{-\frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\frac{k_\mu\alpha_n^\mu}{n}e^{-in\tau}}}_{V_+}\,.$$

This operator is "almost" normal-ordered and can be concisely written as

$$V(k,\tau) = V_-V_0V_+$$

Here only zero modes entering V_0 are not normal-ordered. Indeed, according to our conventions $:p^\mu x^\nu: = x^\nu p^\mu$, the operator p^μ should be always on the right from x^μ which is not the case for V_0.

The normal-ordering of V_0 can be achieved with the help of the Baker-Campbell-Hausdorff formula[11]

$$e^Ae^B = e^{A+B+\frac{1}{2}[A,B]}$$

Thus, we have

$$:e^{ik_\mu x^\mu}e^{i\frac{k_\mu p^\mu}{\pi T}\tau}: = e^{ik_\mu x^\mu}e^{i\frac{k_\mu p^\mu}{\pi T}\tau} = e^{ik_\mu x^\mu + i\frac{k_\mu p^\mu}{\pi T}\tau - \frac{\tau}{2\pi T}k_\mu k_\nu[x^\mu,p^\nu]}$$

Recalling that $[x^\mu,p^\nu]=i\eta^{\mu\nu}$ we therefore find that

$$e^{ik_\mu x^\mu + i\frac{k_\mu p^\mu}{\pi T}\tau} = e^{i\alpha' k^2\tau}\;:e^{ik_\mu x^\mu}e^{i\frac{k_\mu p^\mu}{\pi T}\tau}: = e^{i\alpha' k^2\tau}\;:V_0:$$

Thus, we obtain completely normal-ordered vertex operator

$$V = e^{i\alpha' k^2\tau}V_-:V_0:V_+$$

We would like to investigate the transformation properties of this operator under conformal transformations. To this end we need to compute by using the Leibnitz rule

$$e^{-i\alpha' k^2\tau}[L_m,V] = [L_m,V_-]:V_0:V_+ + V_-[L_m,:V_0:]V_+ + V_-:V_0:[L_m,V_+]\,.$$

[11] Suppose you forgot an exact coefficient in front of the commutator term. Write the operator identity in the form $e^Ae^B=e^{A+B+\alpha[A,B]}$ with the forgotten coefficient α. Rescale A and B with a small parameter ϵ to get $e^{\epsilon A}e^{\epsilon B}=e^{\epsilon A+\epsilon B+\epsilon^2\alpha[A,B]}$. Now expand both sides up to order ϵ^2 keeping the order of operators. You will find that fulfilment of the operator relation at order ϵ^2 will require to fix $\alpha=\frac{1}{2}$.

Assuming for definiteness that $m > 0$ it is not difficult to find[12]

$$
\begin{aligned}
&[L_m, V_-] = \frac{1}{2\sqrt{\pi T}} \sum_{p=1}^{\infty} e^{ip\tau} \Big[V_-(k\alpha_{m-p}) + (k\alpha_{m-p}) V_- \Big] \\
&[L_m, :V_0:] =: V_0 : \frac{(\alpha_m k)}{\sqrt{\pi T}} \\
&[L_m, V_+] = \frac{1}{2\sqrt{\pi T}} \sum_{p=-\infty}^{-1} e^{ip\tau} \Big[V_+(k\alpha_{m-p}) + (k\alpha_{m-p}) V_+ \Big] = \sum_{p=1}^{\infty} \frac{e^{-ip\tau}}{\sqrt{\pi T}} : V_+(k\alpha_{m+p}) :
\end{aligned}
$$

Here the first commutator is of particular importance because it still contains the terms which are not normal-ordered. Indeed, it can be written in the form

$$
\begin{aligned}
[L_m, V_-] = & \frac{1}{\sqrt{\pi T}} \sum_{p=1, p\neq m}^{\infty} e^{ip\tau} : (k\alpha_{m-p}) V_- : \\
& + \frac{1}{\sqrt{\pi T}} e^{im\tau} V_-(k\alpha_0) + \frac{1}{2\sqrt{\pi T}} \sum_{p=1}^{m-1} e^{ip\tau} [k\alpha_{m-p}, V_-] \,.
\end{aligned}
$$

Further we get

$$
[k\alpha_{m-p}, V_-] = \frac{k^2}{\sqrt{\pi T}} e^{i(m-p)\tau} V_-
$$

This leads to

$$
\begin{aligned}
[L_m, V_-] = & \frac{1}{\sqrt{\pi T}} \sum_{p=1, p\neq m}^{\infty} e^{ip\tau} : (k\alpha_{m-p}) V_- : \\
& + \frac{1}{\sqrt{\pi T}} e^{im\tau} V_-(k\alpha_0) + \frac{k^2}{2\pi T} \underbrace{\sum_{p=1}^{m-1} e^{im\tau} V_-}_{(m-1) e^{im\tau} V_-} \,.
\end{aligned}
$$

---

[12] The calculation is as follows:

$$
[L_m, V_-] = \frac{1}{2} \sum_{n=-\infty}^{+\infty} [\alpha^{\mu}_{m-n} \alpha_{n,\mu}, V_-] = \frac{1}{2} \sum_{n=-\infty}^{+\infty} [\alpha^{\mu}_{m-n}, V_-] \alpha_{n,\mu} + \alpha^{\mu}_{m-n} [\alpha_{n,\mu}, V_-] \,.
$$

The two terms on the r.h.s. are computed separately, for instance

$$
\begin{aligned}
\frac{1}{2} \sum_{n=-\infty}^{+\infty} [\alpha^{\mu}_{m-n}, V_-] \alpha_{n,\mu} &= \frac{1}{2} \sum_{n=-\infty}^{m-1} [\alpha^{\mu}_{m-n}, e^{\frac{1}{\sqrt{\pi T}} \sum_{p=1}^{\infty} \frac{k_\nu \alpha^{\nu}_{-p}}{p} e^{ip\tau}}] \alpha_{n,\mu} = \frac{1}{2\sqrt{\pi T}} \sum_{p=1}^{\infty} \sum_{n=-\infty}^{m-1} \underbrace{[\alpha^{\mu}_{m-n}, \alpha^{\nu}_{-p}]}_{n=m-p} \frac{k_\nu}{p} e^{ip\tau} V_- \alpha_{n,\mu} \\
&= \frac{1}{2\sqrt{\pi T}} \sum_{p=1}^{\infty} e^{ip\tau} V_- (k^{\mu} \alpha_{m-p,\mu}) \,.
\end{aligned}
$$

The other terms are computed in a similar way.

Plugging everything together we find

$$e^{-i\alpha' k^2\tau}[L_m,V] = \underbrace{\sum_{p=1,p\neq m}^{\infty}\frac{e^{ip\tau}}{\sqrt{\pi T}}:(k\alpha_{m-p})V_-V_0V_+:}$$
$$+\underbrace{\frac{e^{im\tau}}{\sqrt{\pi T}}:V_-V_0(k\alpha_0)V_+:}+\frac{e^{im\tau}}{\sqrt{\pi T}}:V_-[k\alpha_0,V_0]V_+:+\alpha' k^2(m-1)e^{im\tau}:V_-V_0V_+:$$
$$+\underbrace{\frac{1}{\sqrt{\pi T}}:V_-V_0V_+(k\alpha_m):}+\underbrace{\sum_{p=1}^{\infty}\frac{e^{-ip\tau}}{\sqrt{\pi T}}:V_-V_0V_+(k\alpha_{m+p}):}$$

Here the underlined terms are nicely combined with a single sum[13] with the range of summation variable p form $-\infty$ to $+\infty$ and if we further take into account that

$$[k\alpha_0,V_0]=\frac{k^2}{\sqrt{\pi T}}V_0$$

we will get

$$\begin{aligned}[L_m,V] &= e^{im\tau}e^{i\alpha' k^2\tau}\sum_{p=-\infty}^{\infty}\frac{e^{-ip\tau}}{\sqrt{\pi T}}:V_-V_0V_+(k\alpha_p): \\ &+\alpha' k^2(m+1)e^{im\tau}\underbrace{e^{i\alpha' k^2\tau}:V_-V_0V_+:}_{V}\end{aligned}$$

It remains to note that

$$-i\partial_\tau V=\alpha' k^2 V+e^{i\alpha' k^2\tau}\sum_{p=-\infty}^{\infty}\frac{e^{-ip\tau}}{\sqrt{\pi T}}:V_-V_0V_+(k\alpha_p):$$

With the account of this formula we obtain

$$[L_m,V]=e^{im\tau}\Big(-i\partial_\tau+\alpha' k^2 m\Big)V$$

and, therefore, we conclude that the operator V has the following conformal dimension Δ:

$$\Delta=\alpha' k^2\,.$$

In particular, for $k^2=\frac{1}{\alpha'}$ the conformal dimension $\Delta=1$ and the vertex operator we discuss corresponds to emission of the tachyon with the mass $m^2=-\frac{1}{\alpha'}$.

We also see that on the zero-momentum ground state

$$V(k,0)|0\rangle = V_-e^{ik_\mu x^\mu}|0\rangle\,.$$

[13]It is convenient to shift the summation variable p for $p\to p-m$.

As we will discuss later on, one can perform an analytic continuation $\tau \to -i\tau$ under which the exponent entering the vertex operator V_- will transform as $e^{i\tau} \to e^{\tau}$. Then in the limit $\tau \to -\infty$ we see that $V_- \to 1$ and in the *Euclidean* picture

$$\lim_{\tau\to-\infty} V(k,0)|0\rangle = e^{ik_\mu x^\mu}|0\rangle\,,$$

i.e. at $\tau \to -\infty$ this vertex operator creates a particle with the momentum k_μ.

Normal-ordering the product of exponents

Consider the normal product $:e^X::e^Y:$, where X and Y are two operators with the propagator $\langle XY\rangle$. Clearly one gets

$$:e^X::e^Y:=\sum_{n,m=0}^{\infty}\frac{:X^n::Y^m:}{n!\quad m!}\,.$$

To apply the Wick theorem we first calculate the number of ways we can pick up k X's from X^n, which is obviously $\frac{n!}{k!(n-k)!}$. Analogously, the number of ways to pick up k Y's from Y^m is $\frac{m!}{k!(m-k)!}$. Now we have to pair (i.e. to form propagators) k fields X with k fields Y

$$:\underbrace{X\ldots X}_{k}::\underbrace{Y\ldots Y}_{k}:\,.$$

The are $k!$ ways to pair all the terms in the last expression. Thus, application of the Wick theorem gives

$$:e^X::e^Y:=\sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(n,m)}:\frac{X^{n-k}}{n!}\frac{Y^{m-k}}{m!}:\frac{n!}{k!(n-k)!}\frac{m!}{k!(m-k)!}k!\,\langle XY\rangle^k=$$

$$=\sum_{n,m=0}^{\infty}\sum_{k=0}^{\min(n,m)}:\frac{X^{n-k}}{(n-k)!}\frac{Y^{m-k}}{(m-k)!}:\frac{\langle XY\rangle^k}{k!}=\sum_{k=0}^{\infty}\frac{\langle XY\rangle^k}{k!}\sum_{n,m=k}^{\infty}:\frac{X^{n-k}}{(n-k)!}\frac{Y^{m-k}}{(m-k)!}:$$

Thus, we find

$$:e^X::e^Y:=:e^{\langle XY\rangle+X+Y}:$$

It is now easy to see that the last formula can be generalized for the case of several vertex operators as follows

$$\prod_i :e^{X_i}:=e^{\sum_{i<j}\langle X_iX_j\rangle}:e^{\sum_i X_i}: \tag{4.12}$$

Two-point function of tachyon vertex operators

Consider the two-point correlation function

$$\begin{aligned}\langle V(k_1,\tau_1)V(k_2,\tau_2)\rangle &\equiv \langle 0|V(k_1,\tau_1)V(k_2,\tau_2)|0\rangle\\ &= e^{i\alpha' k_1^2\tau_1+i\alpha' k_2^2\tau_2}\langle 0|:V(k_1,\tau_1)::V(k_2,\tau_2):|0\rangle\,,\end{aligned}$$

where we assume that $\tau_1 > \tau_2$. By using the formula (4.12) we find

$$\langle V(k_1,\tau_1)V(k_2,\tau_2)\rangle = e^{i\alpha' k_1^2\tau_1 + i\alpha' k_2^2\tau_2} e^{\frac{k_1 k_2}{\pi T}\log(e^{i\tau_1}-e^{i\tau_2})}\langle 0| : V(k_1,\tau_1)V(k_2,\tau_2) : |0\rangle \,.$$

The vacuum expectation value of the normal-ordered expression on the right hand side reduces to the contribution of the zero modes only and we get

$$\langle V(k_1,\tau_1)V(k_2,\tau_2)\rangle = e^{i\alpha' k_1^2\tau_1 + i\alpha' k_2^2\tau_2} e^{2\alpha' k_1 k_2 \log(e^{i\tau_1}-e^{i\tau_2})}\underbrace{\langle 0|e^{i(k_1^\mu + k_2^\mu)x_\mu}|0\rangle}_{\delta(k_1+k_2)} \,.$$

Thus, the two-point function is non-zero only if $k_1 = k = -k_2$. In this case we get

$$\langle V(k,\tau_1)V(-k,\tau_2)\rangle = \frac{e^{i\alpha' k^2(\tau_1+\tau_2)}}{(e^{i\tau_1}-e^{i\tau_2})^{2\alpha' k^2}} \,.$$

We thus find that

$$\langle V(k,\tau_1)V(-k,\tau_2)\rangle = \frac{e^{i\Delta(\tau_1+\tau_2)}}{(e^{i\tau_1}-e^{i\tau_2})^{2\Delta}} \,.$$

Four-tachyon scattering amplitude

Here we compute the scattering amplitude of four tachyonic particles. This is the famous Veneziano amplitude which subsequently led to discovery of string theory.

The amplitude is defines as

$$A = \int_0^\infty \mathrm{d}\tau \; \langle k_4|V(k_3,\tau)V(k_2,0)|k_1\rangle$$

Here $\langle k_4|$ is understood in an unusual way $\langle k_4| = \langle 0|e^{ik_4^\mu x_\mu}$. Using the definition of the tachyonic vertex operators and the formula (4.12) one finds

$$\begin{aligned} V(k_3,\tau)V(k_2,0) &= e^{i\alpha' k_3^2\tau} : V(k_3,\tau) :: V(k_2,0) : \\ &= e^{i\alpha' k_3^2\tau} e^{-k_3^\mu k_2^\nu \langle X_\mu(\tau)X_\nu(0)\rangle} : V(k_3,\tau)V(k_2,0) : \end{aligned}$$

Recalling the open string propagator at $\sigma = \sigma' = 0$:

$$\langle X(\tau)X(0)\rangle = -\frac{\eta^{\mu\nu}}{\pi T}\log\Big(e^{i\tau}-1\Big) \,.$$

Thus we find

$$A = \int_0^\infty \mathrm{d}\tau \; e^{i\alpha' k_3^2\tau} e^{2\alpha'(k_2k_3)\log(e^{i\tau}-1)}\langle k_4|e^{i(k_2^\mu+k_3^\mu)x_\mu} e^{i2\alpha' k_3^\mu p_\mu \tau}|k_1\rangle$$

Further simplification gives

$$\begin{aligned} A &= \int_0^\infty \mathrm{d}\tau \; e^{i\alpha' k_3^2\tau}(e^{i\tau}-1)^{2\alpha'(k_2k_3)} e^{2i\alpha'(k_3k_1)\tau}\delta\Big(\sum_{i=1}^4 k_i\Big) \\ &= \int_0^\infty \mathrm{d}\tau \; e^{i\alpha' k_3^2\tau} e^{2i\alpha'(k_1+k_2)k_3\tau}(1-e^{-i\tau})^{2\alpha'(k_2k_3)} \; \delta\Big(\sum_{i=1}^4 k_i\Big) \\ &= \int_0^\infty \mathrm{d}\tau \; e^{-i\alpha' k_3^2\tau} e^{-2i\alpha'(k_3k_4)\tau}(1-e^{-i\tau})^{2\alpha'(k_2k_3)} \; \delta\Big(\sum_{i=1}^4 k_i\Big) \,. \end{aligned}$$

Recall that for tachyons on shell we have $k_i^2 = \frac{1}{\alpha'}$. Performing now the Wick rotation $\tau \to -i\tau$ and changing the integration variable τ for $x = e^{-\tau}$ we find

$$A = \int_0^1 \mathrm{d}x \; x^{2\alpha' k_3 k_4}(1-x)^{2\alpha' k_2 k_3} \,,$$

where for the sake of simplicity we omitted the δ-function which encodes the conservation law of the momenta. Introducing the Mandelstam variables $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ the last formula can be cast in the form

$$A = \int_0^1 \mathrm{d}x \; x^{\alpha' s - 2}(1-x)^{\alpha' t - 2} = \frac{\Gamma(\alpha' s - 1)\Gamma(\alpha' t - 1)}{\Gamma(\alpha'(s+t) - 2)} \,.$$

In fact this function is known as the Euler beta-function. One of the interesting properties of the representation of A in terms of the Euler beta-function is that the latter is explicitly symmetric under the interchange of s and t. Search of amplitudes with this symmetry property led Veneziano in 1960's to this amplitude which was the starting point of modern string theory.

It is interesting to analyze the Veneziano formula in more detail. The Γ-function has poles at non-positive integers with residues

$$\Gamma(x) \to \frac{(-1)^n}{n!}\frac{1}{x+n} \qquad \text{as} \quad x \to -n \,, \quad n \geq 0 \,.$$

Thus, when $\alpha' s \to 1 - n$, $n = 0, 1, \ldots$ the amplitude behaves as

$$A(s,t) \to \frac{(-1)^n}{n!}\frac{1}{\alpha' s - 1 + n}\frac{\Gamma(\alpha' t - 1)}{\Gamma(\alpha' t - 1 - n)}$$

Here the dependents of the variable t is polynomial because for $n > 0$ we have

$$\begin{aligned} \frac{\Gamma(\alpha' t - 1)}{\Gamma(\underbrace{\alpha' t - 1 - n}_{w})} &= \frac{\Gamma(w + n)}{\Gamma(w)} = (w + n - 1)\cdots(w+1)w \\ &= (\alpha' t - 2)(\alpha' t - 3)\cdots(\alpha' t - n - 1) \equiv \mathcal{P}_n(\alpha' t) \,, \end{aligned}$$

i.e. the r.h.s. is a polynomial of degree n. Thus, the scattering amplitude can be essentially written as

$$A(s,t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{\mathcal{P}_n(\alpha' t)}{n - 1 + \alpha' s} \,, \qquad \mathcal{P}_0(\alpha\; t) = 1 \,.$$

In scattering theory resonances (or simply poles) of the scattering amplitude are interpreted as an exchange by intermediate particles whose masses are obtained from the condition of having poles. In our case we see that the poles arise due to exchange by hypothetical particles whose masses are *quantized* as

$$M_n^2 = -s = \frac{1}{\alpha'}(n-1) \,.$$

It also follows from the scattering theory that appearance of a polynomial of degree n as the residue of the amplitude signals that the exchanged particles of the mass M_n^2 carry spin up to the maximal value $J_{\max} = n$. Our later analysis of string in the physical gauge will reveal that the exchanged particles delivering the resonances of the Veneziano amplitude are those from the spectrum of open string!

Operator Product Expansion

Consideration of the Veneziano amplitude shows that the object of primary interest in string perturbation theory is a correletation function of local operators

$$\langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\ldots\mathcal{O}_{i_n}(x_n)\rangle \,,$$

where $\mathcal{O}_i(x)$ is a local operator. Here $x \equiv (\tau,\sigma)$ is a point on the two-dimensional world-sheet. It is important to understand the behavior of the correlation function when two operators are taken to approach each other. The technique to describe this limit in known as the *Operator product Expansion* or OPE for short. The Operator Product Expansion states that a product of two local operators can be approximated to arbitrary accuracy by a sum of local operators

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k C_{ij}^k(x-y,\partial_y)\mathcal{O}_k(y)\,.$$

Let us take as a local operator the stress tensor and try to work out the corresponding OPE. Introduce the short-hand notation $T \equiv T_{++}$ and $X^\mu \equiv X_L^\mu$. Consider the component T_{--} of the stress tensor normalized as

$$T \equiv T_{--} = \frac{1}{\alpha'} : \partial_- X^\mu \partial_- X_\mu := \frac{1}{\alpha'} : \partial_- X_L^\nu \partial_- X_{L\nu} : \,.$$

We will also use the concise notation $z = e^{i(\tau-\sigma)}$ and $w = e^{i(\tau'-\sigma')}$.

In what follows we consider the product of two stress tensors evaluated at two different points

$$T(\tau,\sigma)T(\tau',\sigma') = \frac{1}{\alpha'^2} : \partial_- X^\mu(z)\partial_+ X_\mu(z) :: \partial_- X^\nu(w)\partial_+ X_\nu(w) :$$

and try to expand it over a basis of local operators. By using the Wick theorem, we get

$$\begin{aligned} T(\tau,\sigma)T(\tau',\sigma') &= \frac{1}{\alpha'^2} : \partial_- X^\mu(z)\partial_- X_\mu(z)\partial_- X^\nu(w)\partial_- X_\nu(w) : \\ &+ \frac{4}{\alpha'^2}\langle \partial_- X^\mu(z)\partial_- X^\nu(w)\rangle\ : \partial_- X_\mu(z)\partial_- X_\nu(w) : \\ &+ \frac{2}{\alpha'^2}\langle \partial_- X^\mu(z)\partial_- X^\nu(w)\rangle \langle \partial_- X_\mu(z)\partial_- X_\nu(w)\rangle \,. \end{aligned}$$

We can now rewrite the r.h.s. by using the propagators introduces above

$$
\begin{aligned}
T(\tau,\sigma)T(\tau',\sigma') &= \frac{1}{\alpha'^2} :\partial_- X^\mu(z)\partial_- X_\mu(z)\partial_- X^\nu(w)\partial_- X_\nu(w): \\
&+ \frac{4}{\alpha'^2}\partial_-^x\partial_-^y\langle X^\mu(z)X^\nu(w)\rangle :\partial_- X_\mu(z)\partial_- X_\nu(w): \\
&+ \frac{2}{\alpha'^2}\partial_-^z\partial_-^w\langle X^\mu(z)X^\nu(w)\rangle\,\partial_-^z\partial_-^w\langle X_\mu(z)X_\nu(w)\rangle\,.
\end{aligned}
$$

A little computation gives

$$
\begin{aligned}
T(\tau,\sigma)T(\tau',\sigma') &= \frac{1}{\alpha'^2} :\partial_- X^\mu(z)\partial_- X_\mu(z)\partial_- X^\nu(w)\partial_- X_\nu(w): \\
&- \frac{2}{\alpha'}\frac{zw}{(z-w)^2} :\partial_- X^\mu(z)\partial_- X_\mu(w): \\
&+ \frac{\eta^{\mu\nu}\eta_{\mu\nu}}{2}\frac{z^2w^2}{(z-w)^4}\,.
\end{aligned}
$$

It is further convenient to redefine the stress tensor as follows

$$
\mathscr{T}(z) = \frac{T(\tau,\sigma)}{z^2}
$$

so that

$$
\begin{aligned}
\mathscr{T}(z)\mathscr{T}(w) = &= \frac{1}{\alpha'^2\, z^2w^2} :\partial_- X^\mu(z)\partial_- X_\mu(z)\partial_- X^\nu(w)\partial_- X_\nu(w): \\
&- \frac{2}{\alpha' zw}\frac{1}{(z-w)^2} :\partial_- X^\mu(z)\partial_- X_\mu(w): \\
&+ \frac{\eta^{\mu\nu}\eta_{\mu\nu}}{2}\frac{1}{(z-w)^4}\,.
\end{aligned}
$$

Since

$$
\partial_- = \frac{1}{2}\Big(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\Big) = \frac{1}{2}\Big(\frac{\partial z}{\partial\tau} - \frac{\partial z}{\partial\sigma}\Big)\frac{\partial}{\partial z} = iz\frac{\partial}{\partial z}\,,
$$

we have $\frac{\partial}{\partial z} = \frac{1}{iz}\frac{\partial}{\partial z}$ and, therefore, the last formula can be written as

$$
\begin{aligned}
\mathscr{T}(z)\mathscr{T}(w) = &= \frac{1}{\alpha'^2} :\partial_z X^\mu(z)\partial_z X_\mu(z)\partial_w X^\nu(w)\partial_w X_\nu(w): \\
&+ \frac{2}{\alpha'}\frac{1}{(z-w)^2} :\partial_z X^\mu(z)\partial_w X_\mu(w): + \frac{\eta^{\mu\nu}\eta_{\mu\nu}}{2}\frac{1}{(z-w)^4}\,.
\end{aligned}
$$

Expanding the r.h.s. around the point $w=z$, we will find the following most singular $z\to w$ contribution

$$
\mathscr{T}(z)\mathscr{T}(w) = \frac{d/2}{(z-w)^4} + \frac{2}{(z-w)^2}\mathscr{T}(w) + \frac{1}{z-w}\partial_w\mathscr{T}(w) + \dots \qquad (4.13)
$$

The first term here reflects the appearance of the conformal anomaly (a purely quantum mechanical effect). In the general setting the coefficient of this term is $c/2$, where c is the central charge. The coefficients 2 of the second term coincides with the conformal dimension of $\mathscr{T}$.

4.2 Quantization in the physical gauge

Quantization of strings in the physical (light-cone) gauge is perhaps the most straightforward way to obtain a restriction on the space-time dimension as well as to understand the spectrum of physical excitations.

Using the Poisson brackets and the basic quantization rules we can easily get the table of the basic commutator relations of the light-cone string theory.

$[\ ,\ ]$	p^+	p^-	p^j	x^j	x^-	α_m^j	α_m^-
p^+	0	0	0	0	i	0	0
p^-	0	0	0	$-i\frac{p^i}{p^+}$	$-i\frac{p^-}{p^+}$	$-\frac{2\pi T}{p^+}m\alpha_m^j$	$-\frac{2\pi T}{p^+}m\alpha_m^-$
p^i	0	0	0	$-i\delta^{ij}$	0	0	0
x^i	0	$i\frac{p^i}{p^+}$	$i\delta^{ij}$	0	0	$i\frac{\delta^{ij}\delta_m}{4\pi T}$	$i\frac{\alpha_m^i}{p^+}$
x^-	$-i$	$i\frac{p^-}{p^+}$	0	0	0	0	$i\frac{\alpha_m^-}{p^+}$
α_n^i	0	$\frac{2\pi T}{p^+}n\alpha_n^i$	0	$-i\frac{\delta^{ij}\delta_n}{4\pi T}$	0	$n\delta^{ij}\delta_{n+m}$	$\frac{\sqrt{4\pi T}}{p^+}n\alpha_{n+m}^i$
α_n^-	0	$\frac{2\pi T}{p^+}n\alpha_n^-$	0	$-i\frac{\alpha_n^i}{p^+}$	$-i\frac{\alpha_n^-}{p^+}$	$-\frac{\sqrt{4\pi T}}{p^+}m\alpha_{n+m}^i$	$[\alpha_n^-, \alpha_m^-]$

Tab. 2. Canonical structure of the light-cone modes. The variable p^- is essentially the Hamiltonian: $p^- = 2\pi T\frac{\mathrm{H}}{p^+}$. The commutators involving $\bar{\alpha}$ variables are the same.

We would like to point out the following commutator

$$[\alpha_n^i, \alpha_m^-] = \frac{\sqrt{4\pi T}}{p^+}n\alpha_{n+m}^i \,.$$

One of the most important commutators of the light-cone theory is $[\alpha_m^-, \alpha_n^-]$. It can be computed precisely in the same way as $[L_m, L_n]$ of the previous section. We find the same result as before except we have now only $d-2$ transversal fields which contribute to the central charge term with the factor $d-2$ instead of d

$$[\alpha_m^-, \alpha_n^-] = \frac{\sqrt{4\pi T}}{p^+}(m-n)\alpha_{m+n}^- + \frac{4\pi T}{(p^+)^2}\frac{d-2}{12}m(m^2-1)\delta_{m+n} \,. \tag{4.14}$$

The normal ordering ambiguity

$$\alpha_n^- \to \alpha_n^- - \frac{\sqrt{4\pi T}}{p^+}a\delta_{n,0}$$

leads to the change as

$$[\alpha_m^-, \alpha_n^-] = \frac{\sqrt{4\pi T}}{p^+}(m-n)\alpha_{m+n}^- + \frac{4\pi T}{(p^+)^2}\Big(\frac{d-2}{12}m^3 + 2am - \frac{d-2}{12}m\Big)\delta_{m+n} \,.$$

4.2.1 Lorentz symmetry and critical dimension

Studying the classical string in the light-cone gauge we realized that the generators J^{i-} of the Lorentz symmetry become rather complicated functions of transversal oscillators

$$J^{i-} = x^i p^- - x^- p^i - i\sum_{n=1}^{\infty} \frac{1}{n}\big(\alpha^i_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^i_n\big) - i\sum_{n=1}^{\infty} \frac{1}{n}\big(\bar{\alpha}^i_{-n}\bar{\alpha}^-_n - \bar{\alpha}^-_{-n}\bar{\alpha}^i_n\big)\,.$$

We would like to ask a question whether we can use this expression in quantum theory regarding α_n and $\bar{\alpha}_n$ as operators to define the quantum Lorentz generators? Due to the unusual canonical structure of the light-cone theory it is not obvious that consistent Lorentz generators should exist.

In quantum theory we want to realize a unitary representation of the Poincaré group and therefore, we require the Lorentz generators to be *hermitian*, i.e.

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}\,,$$

where we treat $J^{\mu\nu}$ as an operator acting in the Hilbert space. Also the Lorentz generators must be normal-ordered to have the well-defined action on the vacuum state. Consider the following ansatz for the Lorentz generators J^{i-}, which are the most intricate generators to be defined in quantum theory,

$$J^{i-} = \underbrace{\frac{1}{2}(x^i p^- + p^- x^i) - x^- p^i}_{\ell^{i-}} - i\sum_{n=1}^{\infty} \frac{1}{n}\big(\alpha^i_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^i_n\big) - i\sum_{n=1}^{\infty} \frac{1}{n}\big(\bar{\alpha}^i_{-n}\bar{\alpha}^-_n - \bar{\alpha}^-_{-n}\bar{\alpha}^i_n\big)\,.$$

One can see that these generators are hermitian and normal-ordered so they can be considered as candidates to realize the Lorentz algebra symmetry. The latter requirement is equivalent to

$$[J^{i-}, J^{j-}] = 0\,.$$

This is an equation we would like to prove.

First we discuss the orbital part. We have

$$\begin{aligned}
[\ell^{i-}, \ell^{j-}] &= [\tfrac{1}{2}(x^i p^- + p^- x^i) - x^- p^i, \tfrac{1}{2}(x^j p^- + p^- x^j) - x^- p^j] \\
&= \tfrac{1}{4}[x^i p^-, x^j p^-] \to \tfrac{i}{4}(x^j p^i - x^i p^j)\tfrac{p^-}{p^+} \\
&+ \tfrac{1}{4}[x^i p^-, p^- x^j] \to \tfrac{i}{4}(p^i x^j - x^i p^j)\tfrac{p^-}{p^+} \\
&- \tfrac{1}{2}[x^i p^-, x^- p^j] \to -\tfrac{i}{2}(x^- p^- \delta^{ij} - x^i p^j \tfrac{p^-}{p^+}) \\
&+ \tfrac{1}{4}[p^- x^i, x^j p^-] \to -\tfrac{i}{4}\tfrac{p^-}{p^+}(p^j x^i - x^j p^i) \\
&+ \tfrac{1}{4}[p^- x^i, x^- p^j] \to -\tfrac{i}{4}\tfrac{p^-}{p^+}(p^j x^i - p^i x^j) \\
&- \tfrac{1}{2}[p^- x^i, x^- p^j] \to \tfrac{i}{2}(\tfrac{p^-}{p^+} p^j x^i - p^- x^- \delta^{ij}) \\
&- \tfrac{1}{2}[x^- p^i, x^j p^-] \to -\tfrac{i}{2}(x^j p^i \tfrac{p^-}{p^+} - x^- p^- \delta^{ij}) \\
&- \tfrac{1}{2}[x^- p^i, p^- x^j] \to -\tfrac{i}{2}(\tfrac{p^-}{p^+} x^j p^i - x^- p^- \delta^{ij})\,.
\end{aligned}$$

Here by arrow we indicated the explicit expression for the corresponding commutator. Reducing similar terms we get

$$[\ell^{i-},\ell^{j-}] \;=\; \tfrac{i}{4}[p^i,x^j]\tfrac{p^-}{p^+} + \tfrac{i}{4}\tfrac{p^-}{p^+}[p^i,x^j] + \tfrac{i}{2}[x^-,p^-]\delta^{ij} = 0\,.$$

Thus, the generators of the orbital part of the total momentum have vanishing commutator. Next we compute

$$\begin{aligned}
[\ell^{i-},S^{j-}]+[S^{i-},\ell^{j-}] \;&=\; -2\frac{p^-}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{[i}_{-n}\alpha^{j]}_{n} + \\
&+\;\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\Big[-(\alpha^j_{-n}\alpha^-_n-\alpha^-_{-n}\alpha^j_n)p^i+(\alpha^i_{-n}\alpha^-_n-\alpha^-_{-n}\alpha^i_n)p^j\Big]\,.
\end{aligned} \tag{4.15}$$

Here and below we use the concise notation $\alpha^{[i}_{-n}\alpha^{j]}_{n}=\alpha^i_{-n}\alpha^j_n-\alpha^j_{-n}\alpha^i_n$.

Now we are in a position to study the most difficult commutator $[S^{i-},S^{j-}]$. We will do further analysis in several steps.

1. First we consider the following commutator

$$\begin{aligned}
[S^{i-},\alpha^-_m] \;&=\; -i\sum_{n}^{\infty}\frac{1}{n}[\alpha^i_{-n}\alpha^-_n-\alpha^-_{-n}\alpha^i_n,\alpha^-_m] \\
&=\; -i\sum_{n=1}^{\infty}\frac{1}{n}\Big(\underbrace{-\frac{\sqrt{4\pi T}}{p^+}n\alpha^i_{m-n}\alpha^-_n+\alpha^i_{-n}[\alpha^-_n,\alpha^-_m]}_{\text{A}}-[\alpha^-_{-n},\alpha^-_m]\alpha^i_n-\frac{\sqrt{4\pi T}}{p^+}n\alpha^-_{-n}\alpha^i_{n+m}\Big)\,.
\end{aligned} \tag{4.16}$$

Therefore,

$$\begin{aligned}
[S^{i-},\alpha^-_m] \;&=\; -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\Big(-n\alpha^i_{m-n}\alpha^-_n+\alpha^i_{-n}(n-m)\alpha^-_{m+n}+(n+m)\alpha^-_{m-n}\alpha^i_n-n\alpha^-_{-n}\alpha^i_{n+m}\Big) \\
&-\;i\frac{f(m)}{m}\alpha^i_m\,.
\end{aligned}$$

This further gives

$$\begin{aligned}
[S^{i-},\alpha^-_m] \;&=\; -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\Big(\alpha^i_{-n}\alpha^-_{m+n}-\underbrace{\alpha^i_{m-n}\alpha^-_n}_{\text{A}}+\underbrace{\alpha^-_{m-n}\alpha^i_n}_{\text{B}}-\alpha^-_{-n}\alpha^i_{n+m}\Big) \\
&+\;i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\Big(\alpha^i_{-n}\alpha^-_{m+n}-\alpha^-_{m-n}\alpha^i_n\Big)-i\frac{f(m)}{m}\alpha^i_m\,.
\end{aligned}$$

The terms in the first line of the last equation can be partially cancelled upon changing the summation index and we find that

$$\begin{aligned}
[S^{i-},\alpha^-_m] \;&=\; -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{m}\Big(-\underbrace{\alpha^i_{m-n}\alpha^-_n}_{\text{A}}+\underbrace{\alpha^-_{m-n}\alpha^i_n}_{\text{B}}\Big) \\
&+\;i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\Big(\alpha^i_{-n}\alpha^-_{m+n}-\alpha^-_{m-n}\alpha^i_n\Big)-i\frac{f(m)}{m}\alpha^i_m\,.
\end{aligned}$$

Since we have

$$\sum_{n=1}^{m}\alpha^-_{m-n}\alpha^i_n=\sum_{k=m-1}^{0}\alpha^-_k\alpha^i_{m-k}\equiv\sum_{n=0}^{m-1}\alpha^-_n\alpha^i_{m-n}$$

we see that

$$\begin{aligned}
\sum_{n=1}^{m}\Big(-\alpha^i_{m-n}\alpha^-_n+\alpha^-_{m-n}\alpha^i_n\Big) \;&=\; -\sum_{n=1}^{m}\alpha^i_{m-n}\alpha^-_n+\sum_{n=0}^{m-1}\alpha^-_n\alpha^i_{m-n} \\
&=\; \alpha^-_0\alpha^i_n-\alpha^i_0\alpha^-_m+\sum_{n=1}^{m-1}[\alpha^-_n,\alpha^i_{m-n}]
\end{aligned}$$

Using the fact that $\sum_{n=1}^{m-1}(m-n)=\frac{1}{2}m(m-1)$ we obtain

$$\begin{aligned}
[S^{i-},\alpha^-_m] \;&=\; i\frac{\sqrt{4\pi T}}{p^+}(\alpha^i_0\alpha^-_m-\alpha^-_0\alpha^i_m)+i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\Big(\alpha^i_{-n}\alpha^-_{m+n}-\alpha^-_{m-n}\alpha^i_n\Big) \\
&+\;i\left(\frac{4\pi T}{(p^+)^2}\frac{m(m-1)}{2}-\frac{f(m)}{m}\right)\alpha^i_m\,.
\end{aligned} \tag{4.17}$$

Thus, under the action of the Virasoro operators α^-_m the spin components S^{i-} transform in a nontrivial manner. Note that the r.h.s. of eq.(4.17) is normal-ordered.

2. Analogously, we compute ($m>0$)

$$
\begin{aligned}
[S^{i-},\alpha^-_{-m}\,] \;&=\; i\frac{\sqrt{4\pi T}}{p^+}(\alpha^-_{-m}\alpha^i_0-\alpha^i_{-m}\alpha^-_0)-i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\Big(\alpha^i_{-n}\alpha^-_{n-m}-\alpha^-_{-m-n}\alpha^i_n\Big)\\
&+\; i\,\Big(\frac{4\pi T}{(p^+)^2}\frac{m(m-1)}{2}-\frac{f(m)}{m}\Big)\alpha^i_{-m}\,.
\end{aligned}
$$

In fact these formula is also obtained from eq.(4.17) by simply substituting $m\to -m$.

3. The next step consists in finding the commutator

$$
\begin{aligned}
[S^{i-},\alpha^j_{-m}\,] \;&=\; -i\sum_{n=1}^{\infty}\frac{1}{n}\Big(\alpha^i_{-n}[\alpha^-_n,\alpha^j_{-m}]-[\alpha^-_{-n},\alpha^j_{-m}]\alpha^i_n-\underbrace{\alpha^-_{-n}\delta^{ij}\delta_{n-m}}_{0\text{ as } i\neq j}\Big)\\
&=\; -i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\Big(\alpha^i_{-n}\alpha^j_{n-m}-\alpha^j_{-n-m}\alpha^i_n\Big)\,.
\end{aligned}
$$

4. Finally we compute the commutator

$$
[S^{i-},\alpha^j_m\,] \;=\; i\frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{m}{n}\Big(\alpha^i_{-n}\alpha^j_{m+n}-\alpha^j_{m-n}\alpha^i_n\Big)\,.
$$

Substituting all our findings into the commutator $[S^{i-},S^{j-}]$ we obtain

$$
\begin{aligned}
[S^{i-},S^{j-}] \;&=\; \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\Big(-\alpha^j_{-n}\alpha^-_0\,\alpha^i_n+\alpha^j_{-n}\alpha^i_0\alpha^-_n+\alpha^i_{-n}\alpha^-_0\,\alpha^j_n-\alpha^-_{-n}\alpha^i_0\alpha^j_n\Big)\\
&-\;\sum_{n=1}^{\infty}\Big(\frac{4\pi T}{(p^+)^2}\frac{n-1}{2}-\frac{f(n)}{n^2}\Big)\alpha^{[i}_{-n}\alpha^{j]}_n\\
&+\;\frac{\sqrt{4\pi T}}{p^+}\sum_{m,n=1}^{\infty}\frac{1}{n}\Big(\alpha^j_{-m}(\alpha^i_{-n}\alpha^-_{m+n}-\alpha^-_{m-n}\alpha^i_n)-(\alpha^i_{-n}\alpha^j_{n-m}-\alpha^j_{-m-n}\alpha^i_n)\alpha^-_m\\
&+\;(\alpha^i_{-n}\alpha^-_{n-m}-\alpha^-_{-n-m}\alpha^i_n)\alpha^j_m-\alpha^-_{-m}(\alpha^i_{-n}\alpha^j_{m+n}-\alpha^j_{m-n}\alpha^i_n)\Big)\,.
\end{aligned}
$$

We first analyze the last two lines of the equation above, which we write as follows

$$
\begin{aligned}
I^{ij} \;&=\; \frac{\sqrt{4\pi T}}{p^+}\sum_{m,n=1}^{\infty}\frac{1}{n}\Big(\underbrace{\alpha^j_{-m}\alpha^i_{-n}\alpha^-_{m+n}}_{\text{A}}-(\underbrace{\alpha^i_{-n}\alpha^j_{n-m}}_{\text{A}}-\alpha^j_{-m-n}\alpha^i_n)\alpha^-_m\\
&+\;\alpha^i_{-n}\alpha^-_{n-m}\alpha^j_m-\alpha^j_{-m}\alpha^-_{m-n}\alpha^i_n-\underbrace{\alpha^-_{-n-m}\alpha^i_n\alpha^j_m}_{\text{B}}-\alpha^-_{-m}(\underbrace{\alpha^i_{-n}\alpha^j_{m+n}}_{\text{B}}-\alpha^j_{m-n}\alpha^i_n)\Big)\,.
\end{aligned}
$$

Upon changing the summation variables the A-terms can be partially cancelled, the same is for the B-terms. We therefore obtain

$$
\begin{aligned}
I^{ij} \;&=\; \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\Big(-\sum_{m=1}^{n}\frac{1}{n}\alpha^i_{-n}\alpha^j_{n-m}\alpha^-_m+\sum_{m=1}^{n}\frac{1}{n}\alpha^-_{-m}\alpha^j_{m-n}\alpha^i_n\Big)\\
&+\;\frac{\sqrt{4\pi T}}{p^+}\sum_{n,m=1}^{\infty}\Big(\alpha^j_{-m-n}\alpha^i_n\alpha^-_m+\alpha^i_{-n}\alpha^-_{n-m}\alpha^j_m-\alpha^j_{-m}\alpha^-_{m-n}\alpha^i_n-\alpha^-_{-m}\alpha^i_{-n}\alpha^j_{m+n}\Big)\,.
\end{aligned}
$$

One can recognize that in the second line of the equation above the first and the last terms are not normal-ordered. We consider the first sum, which is not normal-ordered, and try to bring it to the normal-ordered form:

$$
\begin{aligned}
\sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^j_{-m-n}\alpha^i_n\alpha^-_m \;&=\; \sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^j_{-m-n}\alpha^-_m\alpha^i_n+\sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^j_{-m-n}[\alpha^i_n,\alpha^-_m]\\
&=\; \sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^j_{-m-n}\alpha^-_m\alpha^i_n+\frac{\sqrt{4\pi T}}{p^+}\sum_{n,m=1}^{\infty}\alpha^j_{-m-n}\alpha^i_{n+m}\\
&=\; \sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^j_{-m-n}\alpha^-_m\alpha^i_n+\frac{\sqrt{4\pi T}}{p^+}\sum_{k=2}^{\infty}(k-1)\alpha^j_{-k}\alpha^i_k\,,
\end{aligned}
$$

where in the last sum we made a substitution $k=m+n$ and then summed over m,n with the condition $m+n=k$ kept fixed; this resulted in the factor $k-1$. Analogously, we achieve the normal-ordering of the second sum

$$
\sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^-_{-m}\alpha^i_{-n}\alpha^j_{m+n}=\sum_{n,m=1}^{\infty}\frac{1}{n}\alpha^i_{-n}\alpha^-_{-m}\alpha^j_{m+n}+\frac{\sqrt{4\pi T}}{p^+}\sum_{k=2}^{\infty}(k-1)\alpha^i_{-k}\alpha^j_k\,.
$$

Thus, our commutator takes the form

$$
\begin{aligned}
I^{ij} &= \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\Big(-\sum_{m=1}^{n}\frac{1}{n}\alpha^i_{-n}\alpha^j_{n-m}\alpha^-_m + \sum_{m=1}^{n}\frac{1}{n}\alpha^-_{-m}\alpha^j_{m-n}\alpha^i_n\Big) \\
&+ \frac{\sqrt{4\pi T}}{p^+}\sum_{n,m=1}^{\infty}\frac{1}{n}\Big(\underbrace{\alpha^j_{-m-n}\alpha^-_m\alpha^i_n}_{\mathrm{A}} + \underbrace{\alpha^i_{-n}\alpha^-_{n-m}\alpha^j_m}_{\mathrm{B}} - \underbrace{\alpha^j_{-m}\alpha^-_{m-n}\alpha^i_n}_{\mathrm{A}} - \underbrace{\alpha^i_{-n}\alpha^-_{-m}\alpha^j_{m+n}}_{\mathrm{B}}\Big) \\
&- \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}(n-1)\alpha^{[i}_{-n}\alpha^{j]}_n\,.
\end{aligned}
$$

Again we see that upon change of the summation index the A-terms partially cancel (the same is for the B-terms) and we arrive at

$$
\begin{aligned}
I^{ij} &= \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{n}\Big(-\alpha^i_{-n}\alpha^j_{n-m}\alpha^-_m + \alpha^-_{-m}\alpha^j_{m-n}\alpha^i_n\Big) \\
&+ \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{1}{n}\Big(-\alpha^j_{m-n}\alpha^-_{-m}\alpha^i_n + \alpha^i_{-n}\alpha^-_m\alpha^j_{n-m}\Big) \\
&- \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}(n-1)\alpha^{[i}_{-n}\alpha^{j]}_n\,.
\end{aligned}
$$

From here we find

$$
\begin{aligned}
I^{ij} &= \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{1}{n}\Big(-\alpha^i_{-n}\alpha^j_0\alpha^-_n + \alpha^-_{-n}\alpha^j_0\alpha^i_n - \alpha^j_{-n}\alpha^-_0\alpha^i_n + \alpha^i_{-n}\alpha^-_0\alpha^j_n\Big) \\
&- \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}\underbrace{\left(\sum_{m=1}^{n-1}\frac{n-m}{n}\right)}_{\frac{1}{2}(n-1)}\alpha^{[i}_{-n}\alpha^{j]}_n - \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}(n-1)\alpha^{[i}_{-n}\alpha^{j]}_n\,.
\end{aligned}
$$

Thus, the commutator of the internal spin components we are interested in acquires the form

$$
\begin{aligned}
[S^{i-},S^{j-}] &= \frac{\sqrt{4\pi T}}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\Big(-\alpha^j_{-n}\alpha^-_0\alpha^i_n + \alpha^j_{-n}\alpha^i_0\alpha^-_n + \alpha^i_{-n}\alpha^-_0\alpha^j_n - \alpha^-_{-n}\alpha^i_0\alpha^j_n \\
&\qquad - \alpha^i_{-n}\alpha^j_0\alpha^-_n + \alpha^-_{-n}\alpha^j_0\alpha^i_n - \alpha^j_{-n}\alpha^-_0\alpha^i_n + \alpha^i_{-n}\alpha^-_0\alpha^j_n\Big) \\
&- \sum_{n=1}^{\infty}\left(\frac{4\pi T}{(p^+)^2}2(n-1) - \frac{f(n)}{n^2}\right)\alpha^{[i}_{-n}\alpha^{j]}_n\,.
\end{aligned}
$$

The final step consists in commuting the factor α^-_0 on the left to compare with eq.(4.15).

We thus find

$$
\begin{aligned}
[S^{i-},S^{j-}] = 2\frac{2\sqrt{\pi T}\alpha^-_0}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{[i}_{-n}\alpha^{j]}_n + \\
+\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\Big[(\alpha^j_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^j_n)p^i - (\alpha^i_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^i_n)p^j\Big] \\
-\sum_{n=1}^{\infty}\left(\frac{4\pi T}{(p^+)^2}2n - \frac{f(n)}{n^2}\right)\alpha^{[i}_{-n}\alpha^{j]}_n\,.
\end{aligned}
$$

Finally, adding this expression to eq.(4.15) we arrive at

$$
\begin{aligned}
[J^{i-},J^{j-}] = 2\Big(p^- - 2\sqrt{\pi T}\alpha^-_0\Big)\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{[i}_{-n}\alpha^{j]}_n + \\
+\frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}\left(\Big[\frac{d-2}{12}-2\Big]n + \frac{1}{n}\Big[2a - \frac{d-2}{12}\Big]\right)\alpha^{[i}_{-n}\alpha^{j]}_n + (\alpha_n\to\bar{\alpha}_n).
\end{aligned}
$$

As for the case of classical string the first term here vanishes due to the level matching condition $\alpha_0 = \bar{\alpha}_0$, while the second sum is an anomaly which appears due to non-commutativity of the oscillators in quantum theory. Thus, for general values of a and d the theory is not Lorentz invariant: the quantum effects destroy the Lorentz invariance which was present at the classical level. However, for special values

$$d = 26\,, \qquad\qquad a = 1$$

the anomaly term vanishes and the Lorentz invariance is restored!

4.2.2 The spectrum

In the light-cone gauge the spectrum is generated by acting with transversal oscillators on the vacuum state. We first discuss the spectrum of open strings.

The mass operator is the light-cone gauge for open strings is

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\Big(\alpha^i_{-n}\alpha^i_n - a\Big)\,,$$

where, as was discussed in the previous chapter, the normal-ordering constant a should be equal to 1 in order to guarantee the Lorentz invariance of the light-cone theory.

The ground state $|p^i\rangle$ carries no oscillators and it has a mass

$$\alpha' M^2|p^i\rangle = -|p^i\rangle \quad \Longrightarrow \quad \alpha' M^2 = -1.$$

This is a tachyon.

The first excited state is $\alpha^i_{-1}|p^j\rangle$. It is a $d-2$ component vector which transforms irreducibly under the transverse group SO(24). We see that

$$\alpha' M^2(\alpha^i_{-1}|p^i\rangle) = (1-a)\alpha^i_{-1}|p^i\rangle = 0\,,$$

i.e. this vector is massless.

level	$\alpha'\text{mass}^2$	rep of SO(24)	little group	rep of little group
0	-1	$\underbrace{\vert 0\rangle}_{\mathbf{1}}$	SO(1, 24)	$\mathbf{1}$
1	0	$\underbrace{\alpha^i_{-1}\vert 0\rangle}_{\mathbf{24}}$	SO(24)	$\mathbf{24}$
2	$+1$	$\underbrace{\alpha^i_{-2}\vert 0\rangle}_{\mathbf{24}} \quad \underbrace{\alpha^i_{-1}\alpha^j_{-1}\vert 0\rangle}_{\mathbf{299}_{\rm s}+\mathbf{1}}$	SO(25)	$\mathbf{324}_{\rm s}$
3	$+2$	$\underbrace{\alpha^i_{-3}\vert 0\rangle}_{\mathbf{24}} \quad \underbrace{\alpha^i_{-2}\alpha^j_{-1}\vert 0\rangle}_{\mathbf{276}_{\rm a}+\mathbf{299}_{\rm s}+\mathbf{1}} \quad \underbrace{\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\vert 0\rangle}_{\mathbf{2576}_{\rm s}+\mathbf{24}}$	SO(25)	$\mathbf{2900}_{\rm s}+\mathbf{300}_{\rm a}$

Tab. 3. The spectrum of open bosonic string up to level 3.

In general, the Lorentz invariance requires that physical states transform irreducibly under the *little Lorentz group* which is

- $\mathrm{SO}(d-2)$ for massless particles
- $\mathrm{SO}(d-1)$ for massive particles (for tachyon $\mathrm{SO}(1, d-2)$)

For tachyon the little Lorentz group is non-compact. Unitary representations of non-compact groups are either trivial (i.e. one-dimensional) or infinite-dimensional. Tachyon realizes the one-dimensional representation.

Further analysis reveals that *all states corresponding to higher levels are massive and that being the tensors of SO(24) they combine at any given mass level to representations of SO(25), the latter is the little Lorentz group for massive states.* This is highly non-trivial implication of the Lorentz invariance and it occurs only in the critical dimension and for $a = 1$!

At level n the mass of the corresponding states is $\alpha' M^2 = n - 1$. Among them there is always a symmetric traceless tensor of rank n. This is a state with maximal spin $J_{\max} = n$ and, therefore, we have $J_{\max} = n = \alpha' M^2 + 1$. In general states obey the inequality

$$J \leq \alpha' M^2 + 1\,.$$

The values of J and M^2 are quantized and the last inequality implies that the states lie on the Regge trajectories.

level	$\alpha'\text{mass}^2$	rep of SO(24)	little group	rep of little group
0	-4	$\underbrace{\vert 0\rangle}_{\mathbf{1}}$	$\mathrm{SO}(1,24)$	$\mathbf{1}$
1	0	$\underbrace{\alpha^i_{-1}\bar{\alpha}^j_{-1}\vert 0\rangle}_{\mathbf{299}_\mathrm{s}+\mathbf{276}_\mathrm{a}+\mathbf{1}}$	SO(24)	$\mathbf{299}_\mathrm{s}+\mathbf{276}_\mathrm{a}+\mathbf{1}$
2	$+4$	$\underbrace{\alpha^i_{-2}\bar{\alpha}^j_{-2}\vert 0\rangle}_{\mathbf{299}_\mathrm{s}+\mathbf{276}_\mathrm{a}+\mathbf{1}}$ $\underbrace{\alpha^i_{-1}\alpha^j_{-1}\bar{\alpha}^k_{-1}\bar{\alpha}^l_{-1}\vert 0\rangle}_{\mathbf{299}_\mathrm{s}+\mathbf{1}+\mathbf{299}_\mathrm{s}+\mathbf{1}}$ $\underbrace{\alpha^i_{-2}\bar{\alpha}^j_{-1}\bar{\alpha}^k_{1}\vert 0\rangle}_{(\mathbf{24})\times(\mathbf{299}+\mathbf{1})}$ $\underbrace{\alpha^i_{-1}\alpha^j_{-1}\bar{\alpha}^k_{-2}\vert 0\rangle}_{(\mathbf{299}+\mathbf{1})\times(\mathbf{24})}$	SO(25)	$\mathbf{20150}_\mathrm{s}+\mathbf{32175}$ $\mathbf{52026}+\mathbf{324}_\mathrm{s}+\mathbf{300}_\mathrm{a}+\mathbf{1}$

Tab. 4. The spectrum of closed bosonic string up to level 2.

Now we discuss the spectrum of closed strings. The mass operator for closed strings is

$$M^2 = \frac{2}{\alpha'}\Big(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty}\bar{\alpha}^i_{-n}\bar{\alpha}^i_n - 2a\Big)$$

In addition one has to impose the level-matching condition

$$\mathcal{V} = \sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n - \sum_{n=1}^{\infty}\bar{\alpha}^i_{-n}\bar{\alpha}^i_n = 0\,,$$

which simply means that the excitation (level) number of α-oscillators should be equal to the excitation number of $\bar{\alpha}$-oscillators.

The ground state is a tachyon which is scalar particle with $\alpha' M^2 = -4$. The first excited state $\alpha^i_{-1}\bar{\alpha}^j_{-1}|0\rangle$ is massless. It can be decomposed into irreducible representations of the transversal (and simultaneously little) Lorentz group SO(24) as follows

$$\alpha^i_{-1}\bar{\alpha}^j_{-1}|0\rangle = \underbrace{\alpha^{[i}_{-1}\bar{\alpha}^{j]}_{-1}|0\rangle}_{\mathbf{276}} + \underbrace{\Big(\alpha^{(i}_{-1}\bar{\alpha}^{j)}_{-1} - \frac{1}{24}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}\Big)|0\rangle}_{\mathbf{299}} + \underbrace{\frac{1}{24}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}|0\rangle}_{\text{singlet}}\,.$$

The massless excitation of spin two transforming in representation **299** of SO(24) was proposed to be identified with a *graviton*, the quantum of the gravitational interaction. To make this identification one has to relate the string scale α' with the Planck scale $G = M_{\rm P}^{-2}$, where G is the Newton constant and $M_{\rm P}$ is the Planck mass:

$$\alpha' = M_{\rm P}^{-2}\,.$$

Since the masses of the massive string modes are proportional to $1/\alpha' = M_{\rm P}^2$, these string excitations are extremely heavy due to the large value of $M_{\rm P}^2$ and, by this reason, they do not show up at the energy scales of the Standard Model.

As in the opens string case the higher massive states of closed string are combined at a given mass level into representations of the little Lorentz group SO(25). The relation between maximal spin and the mass is now

$$J_{\rm max} = \frac{\alpha'}{2}M^2 + 2\,.$$

4.3 BRST quantization

The path integral approach proved to be a very useful tool for quantizing the theories with local (gauge) symmetries. The starting point is the Polyakov action and a new BRST (Becchi-Rouet-Stora-Tyutin) symmetry. We know that the induced metric $\Gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$ and the intrinsic metric $h_{\alpha\beta}$ are related classically through the condition $T_{\alpha\beta} = 0$. However, quantum-mechanically this is not the case.

The basic idea is to define the path integral using the Polykov action and integrate over

$$h_{\alpha\beta}, \quad X^\mu$$

being considered as independent variables:

$$Z = \int \mathcal{D}h_{\alpha\beta}(\sigma,\tau)\mathcal{D}X^\mu(\sigma,\tau)e^{iS_p[X,h]}$$

Due to the gauge invariance the last integral is ill-defined. This occurs because we integrate infinitely many times over physically equivalent, i.e. related to each other by gauge transformations, configurations. This can be understood looking at a much simpler example of the two-dimensional integral

$$Z = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} {\rm d}x{\rm d}y\; e^{-(x-y)^2}\,.$$

It is divergent since the integrand depends on $x-y$ only. Also one sees that the group of translations

$$x \longrightarrow x+a, \qquad y \longrightarrow y+a$$

leaves the measure invariant.

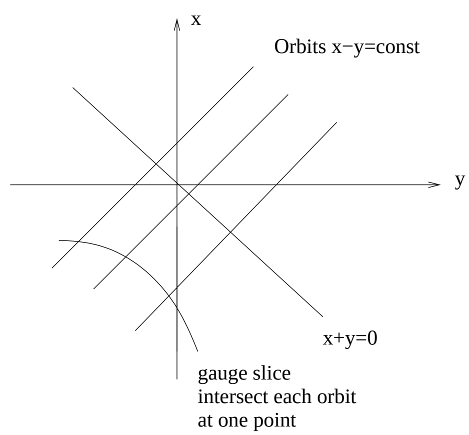


Fig. 4. Divergence caused by a presence of the translational symmetry. One integrates over the orbits of the gauge group while both the measure and the integrand are translation invariant.

Let us split the coordinates (x, y) as

$$(x,y)=\underbrace{\left(\frac{x-y}{2},\frac{y-x}{2}\right)}_{\frac{x-y}{2}+\frac{y-x}{2}=0}+\underbrace{\left(\frac{x+y}{2},\frac{x+y}{2}\right)}_{\text{shift by }(a,a),\quad a=\frac{x+y}{2}}$$

This suggests to introduce new coordinates "along" the gauge orbit and "orthogonal" to it:

$$(x,y)\longrightarrow(u,v) \qquad u=x-y, \qquad v=x+y,$$

i.e. $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$. Then the integral takes the form

$$Z=\frac{1}{2}\int_{-\infty}^{+\infty}\underbrace{\left(\int_{-\infty}^{+\infty}e^{-u^2}\mathrm{d}u\right)}_{\sqrt{\pi}}\mathrm{d}v=\frac{\sqrt{\pi}}{2}\int_{-\infty}^{+\infty}\mathrm{d}(x+y)=\sqrt{\pi}\underbrace{\int_{-\infty}^{+\infty}\mathrm{d}a}_{\text{volume}}\ .$$

This example illustrates the basic idea to define the path integral – *one has to divide the original Z by the infinite volume of a symmetry group.* Thus, our discussion suggests that a proper definition of the path integral in string theory should be

$$Z=\frac{1}{V_{\text{Diff}}V_{\text{Weyl}}}\int\mathcal{D}h_{\alpha\beta}(\sigma,\tau)\mathcal{D}X^{\mu}(\sigma,\tau)e^{iS_p[X,h]}\,,$$

where we divided over the infinite volumes of the reparametrization and Weyl groups.

Let us illustrate this procedure on our simplified example. The integral can be rewritten by using the δ-function:

$$Z=\int_{-\infty}^{+\infty}\mathrm{d}a\int_{-\infty}^{+\infty}e^{-(x-y)^2}\mathrm{d}x\mathrm{d}y\,\Big(2\delta(x+y)\Big)=\int_{-\infty}^{+\infty}\mathrm{d}a\int_{-\infty}^{+\infty}e^{-z^2}\mathrm{d}z\,.$$

Here insertion of the δ-function can be regarded as the gauge-fixing condition; it selects a single representative from each gauge orbit. What happens if we change the gauge fixing condition? Suppose we take another slice "orthogonal" to the gauge orbits. Let s will be a free one-dimensional parameter on this slice

$$s:\quad f(x,y)=0\quad\Longrightarrow\quad(x(s),y(s))$$

Chose now new coordinates

$$(x,y)=\underbrace{(x',y')}_{\text{along } s}+(a,a)$$

Then

$$Z=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x-y)^2}\mathrm{d}x\mathrm{d}y=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x'-y')^2}\mathrm{d}s\mathrm{d}a|J|\,,$$

where $x-y=x'-y'$ and

$$J=\frac{\partial(x,y)}{\partial(s,a)}$$

is the Jacobian. Since $x=x'+a$ and $y=y'+a$ we have

$$J=\begin{pmatrix}\frac{\partial x}{\partial s}&\frac{\partial x}{\partial a}\\ \frac{\partial y}{\partial s}&\frac{\partial y}{\partial a}\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial s}&1\\ \frac{\partial y}{\partial s}&1\end{pmatrix}=\frac{\partial x}{\partial s}-\frac{\partial y}{\partial s}=\frac{\partial}{\partial s}(x'-y')\,.$$

Along s

$$0=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x'}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y'}{\partial s}\,,$$

which allows for a solution $\frac{\partial y'}{\partial s}=-\frac{\partial f}{\partial x}$ and $\frac{\partial x'}{\partial s}=\frac{\partial f}{\partial y'}$ and, therefore,

$$J=\frac{\partial}{\partial s}(x'-y')=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\,.$$

The integral can be now written as

$$Z=\int_{-\infty}^{+\infty}\mathrm{d}a\int_{-\infty}^{+\infty}\mathrm{d}s\;e^{-(x'-y')^2}\Big|\frac{\partial f}{\partial x'}+\frac{\partial f}{\partial y'}\Big|,$$

Here the integral over s is one-dimensional, the functions x' and y' are the functions of s and it is independent of a. One can convert this one-dimensional integral into a two-dimensional one by substituting the δ-function with the gauge condition

$$\int_{-\infty}^{+\infty}\mathrm{d}a\;e^{-(x'-y')^2}\Big|\frac{\partial f}{\partial x'}+\frac{\partial f}{\partial y'}\Big|=\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{d}y\;\delta(f(x,y))e^{-(x-y)^2}\Big|\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\Big|.$$

The infinite volume arising upon integrating over a can be factored out and taking into account that

$$\frac{\partial f}{\partial a}|_{a=0}=\Big(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\Big)|_{a=0}$$

we obtain the final and finite expression

$$Z_{\text{finite}}=\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{d}y\;\delta(f(x,y))\Big|\frac{\partial f}{\partial a}\Big|_{a=0}e^{-(x-y)^2}$$

Here $\Big|\frac{\partial f}{\partial a}\Big|_{a=0}$ is known as the Faddeev-Popov determinant.

The first problem in realizing this approach for string theory is to find a measure for functional integration that preserves all symmetries of the classical theory (reparametrizations + Weyl symmetry).

$$(\delta h,\delta h)=\int\mathrm{d}^2\sigma\sqrt{h}h^{\alpha\beta}h^{\gamma\delta}\delta h_{\alpha\gamma}\delta h_{\beta\delta}$$
$$(\delta X,\delta X)=\int\mathrm{d}^2\sigma\sqrt{h}h^{\alpha\beta}\delta X^{\mu}\delta X_{\mu}$$

These scalar products define natural reparametrization-invariant and Poincaré invariant measures, however none of them is Weyl invariant.

Let us *first* assume the simplifying situation when all metric on a world-sheet $\mathcal{M}$ with a given topology are conformally equivalent (i.e. they are related to each other by diffeomorphism and Weyl rescalings; this is the case when operator $P^\dagger$ does not have zero modes). In this case by using reparametrizations we can bring the metric to the form

$$h_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}\,,$$

where $g_{\alpha\beta}$ is a fiducial (reference) metric. Under reparametrizations and the Weyl rescalings the variation of the metric can be decomposed as

$$\delta h_{\alpha\beta} = (P\xi)_{\alpha\beta} + 2\tilde{\Lambda} h_{\alpha\beta}\,, \quad \tilde{\Lambda} = \Lambda + \frac{1}{2}\nabla_\gamma \xi^\gamma\,,$$

where P is the following operator

$$(P\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - \nabla_\gamma \xi^\gamma h_{\alpha\beta}\,,$$

which maps vectors into traceless symmetric tensors. Then the integration measure can be written as follows

$$\mathcal{D}h = \mathcal{D}(P\xi)\mathcal{D}(\tilde{\Lambda}) = \mathcal{D}(\xi)\mathcal{D}(\Lambda) \underbrace{\left| \frac{\partial(P\xi, \tilde{\Lambda})}{\partial(\xi, \Lambda)} \right|}_{\text{Jacobian}}\,,$$

where in the last formula we changed the variables

$$(P\xi, \tilde{\Lambda}) \to (\xi, \Lambda)$$

for the price of getting a non-trivial Jacobian. Here $\mathcal{D}(\xi)$ is the measure which gives upon integration an *infinite* volume of the diffeomorphism group and

$$\left| \frac{\partial(P\xi, \tilde{\Lambda})}{\partial(\xi, \Lambda)} \right| = \begin{vmatrix} \frac{\partial(P\xi)}{\partial\xi} & \frac{\partial(P\xi)}{\partial\Lambda} \\ \frac{\partial\tilde{\Lambda}}{\partial\xi} & \frac{\partial\tilde{\Lambda}}{\partial\Lambda} \end{vmatrix} = \begin{vmatrix} \frac{\partial(P\xi)}{\partial\xi} & 0 \\ \frac{\partial\tilde{\Lambda}}{\partial\xi} & 1 \end{vmatrix} = |\mathrm{det}P|$$

In fact, we have

$$\frac{\delta(P\xi)_{\alpha\beta}(\sigma)}{\delta\xi^\gamma(\sigma')} = \Big(\delta^\gamma_\beta \nabla_\alpha + \delta^\gamma_\alpha \nabla_\beta - h_{\alpha\beta}\nabla^\gamma\Big)\delta(\sigma - \sigma')$$

Thus,

$$|\mathrm{det}P| = \int \mathcal{D}b\mathcal{D}c \; e^{-i\frac{T}{2}2\int \mathrm{d}^2\sigma \mathrm{d}^2\sigma' \sqrt{h}\; b^{\alpha\beta}(\sigma)\left(\delta^\gamma_\beta \nabla_\alpha + \delta^\gamma_\alpha \nabla_\beta - h_{\alpha\beta}\nabla^\gamma\right)_\sigma \delta(\sigma-\sigma')c_\gamma(\sigma')}\,.$$

Here c^α is called a *ghost* field, while $b_{\alpha\beta}$ is traceless and symmetric, it is called *antighost* field. The ghost c^α corresponds to infinitezimal reparametrizations while $b_{a\beta}$ corresponds to variations perpendicular to the gauge orbits. Both the ghost and the antighost fields are real. The last formula can be now written as

$$|\mathrm{det}P| = \int \mathcal{D}b\mathcal{D}c\; e^{-\frac{i}{\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{h}\; b_{\alpha\beta}\nabla^\alpha c^\beta}\,.$$

Thus, the total action is given now by the sum of the Polyakov action and the ghost action:

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\gamma^{\alpha\beta}\Big(\partial_\alpha X^\mu\partial_\beta X_\mu + 4ib_{\beta\gamma}\nabla_\alpha c^\gamma\Big)$$

There are several subtle issues we have not touched so far

- Conformal anomaly, i.e. possible dependence of the thrown away volume of the diffeomorphism group on the Weyl (scale) degree of freedom ϕ.
- Reparametrizations which satisfy $P\xi = 0$, i.e. conformal Killing vectors. We see that equations of motion for c^α are just conformal Killing equations. Therefore, in order not to overcount the configurations which are related by a conformal transformation one has to exclude integration over the zero modes of c^α ghosts.
- So far we *assumed* that all symmetric traceless deformations of the metric can be generated by reparametrizations. This is however not the case if $P^\dagger$ has zero modes. These zero modes correspond to zero modes of the b ghosts.

We can define the stress-energy tensor of the ghost fields

$$\delta S_{\mathrm{gh}} = T\int \mathrm{d}^2\sigma\sqrt{-h}\; T^{\alpha\beta}\delta h_{\alpha\beta}\,.$$

Performing the variation one finds

$$T^{\mathrm{gh}}_{\alpha\beta} = i(b_{\alpha\gamma}\nabla_\beta c^\gamma + b_{\beta\gamma}\nabla_\alpha c^\gamma - c^\gamma\nabla_\gamma b_{\alpha\beta} - h_{\alpha\beta}b_{\gamma\delta}\nabla^\gamma c^\delta)\,.$$

Here the last term vanishes on shell. In the deriving this expression we also used the tracelessness of $b_{\alpha\beta}$. One can verify that this tensor is covariantly conserved $\nabla^\alpha T_{\alpha\beta} = 0$.

In the world-sheet light-cone coordinates $\sigma^\pm$ the non-vanishing components of the stress-tensor are

$$\begin{aligned} T_{++} &= i\big(2b_{++}\partial_+ c^+ + (\partial_+ b_{++})c^+\big)\,, \\ T_{--} &= i\big(2b_{--}\partial_- c^- + (\partial_- b_{--})c^-\big)\,. \end{aligned}$$

Equations of motion are

$$\partial_- b_{++} = \partial_+ b_{--} = 0\,,$$
$$\partial_+ c^- = \partial_- c^+ = 0\,.$$

This equations are supplemented by

- by periodicity condition for the closed closed string case
$$b(\sigma + 2\pi) = b(\sigma)\,, \qquad c(\sigma + 2\pi) = c(\sigma)\,;$$
- by boundary conditions for the open string case
$$b_{++}(\sigma) = b_{--}(\sigma)\,, \qquad c^+(\sigma) = c^-(\sigma) \qquad \text{for} \qquad \sigma = 0, \pi\,,$$
which follow by requiring the vanishing of the boundary terms arising upon deriving equations of motion.

Note that for the closed string case b_{++} and c^+ are left-moving waves, while b_{--} and c^- are the right-moving ones. The canonical *anti*-commutation relations are

$$\{b_{++}(\sigma,\tau), c^+(\sigma',\tau)\} = 2\pi\delta(\sigma - \sigma')\,,$$
$$\{b_{--}(\sigma,\tau), c^-(\sigma',\tau)\} = 2\pi\delta(\sigma - \sigma')\,.$$

For the closed string case the Fourier mode expansions look as

$$c^+(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} \bar{c}_n e^{-in(\tau+\sigma)}\,,$$
$$c^-(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} c_n e^{-in(\tau-\sigma)}\,,$$
$$b_{++}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} \bar{b}_n e^{-in(\tau+\sigma)}\,,$$
$$b_{--}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau-\sigma)}\,.$$

For the anti-commutation relations this becomes

$$\{b_m, c_n\} = \delta_{m+n}\,,$$
$$\{b_m, b_n\} = \{c_m, c_n\} = 0$$

and the same for the barred oscillators. The Virasoro generators are

$$L_m^{\rm gh} = \sum_{n=-\infty}^{\infty} (m-n) : b_{m+n} c_{-n} :$$
$$\bar{L}_m^{\rm gh} = \sum_{n=-\infty}^{\infty} (m-n) : \bar{b}_{m+n} \bar{c}_{-n} :$$

The ghosts and anti-ghosts are *conformal fields*. Indeed, it is easy to compute

$$[L_m^{\rm gh}, b_n] = (m-n)b_{m+n}\,, \qquad [L_m^{\rm gh}, c_n] = -(2m+n)c_{m+n}\,.$$

Comparing this with the transformation rule of the modes of a conformal operator of dimension Δ

$$[L_m, A_n] = \big(m(\Delta-1)-n\big)A_{m+n}$$

we conclude that b and c are indeed the conformal fields of the conformal dimension $\Delta = 2$ and $\Delta = -1$ respectively.

Using the explicit expressions for the ghost generators $L^{\rm gh}$ it is not difficult to compute the algebra

$$[L_m^{\rm gh}, L_n^{\rm gh}] = (m-n)L_{m+n}^{\rm gh} + c^{\rm gh}(m)\delta_{m+n}\,,$$

where the central charge appears to be

$$c^{\rm gh}(m) = \frac{1}{12}(2m - 26m^3)\,.$$

Now if we introduce the total Virasoro generator as

$$L_m = L_m^X + L_m^{\rm gh} - a\delta_{m,0}$$

then it will satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + c(m)\delta_{m+n}$$

with

$$c = \frac{d}{12}(m^3-m) + \frac{1}{12}(2m-26m^3) + 2am\,.$$

We see that the total central charge vanishes for $d = 26$ and $a = 1$. We again found the same values for the critical dimension and the normal-ordering constant as followed from the light-cone approach! Here these conditions on the theory follow from the requirement of vanishing of the total central charge.

BRST operator

The concept of the BRST operator is very general. In fact, the BRST operator can be associated to any Lie algebra and it is a useful tool to compute the Lie algebra cohomologies.

Consider a Lie algebra with generators K_i satisfying the relations

$$[K_i, K_j] = f_{ij}^k K_k\,.$$

Introduce ghost and anti-ghost fields c^i and b_i satisfying the anti-commutation relations

$$\{c^i, b_j\} = \delta^i_j\,, \qquad i = 1,\ldots,{\rm dim}K$$

Introduce the ghost number operator U:

$$U = \sum_i c^i b_i \,.$$

The eigenvalues of this operator are integers ranging from 0 up to $\dim K$.

The BRST operator is defined as

$$Q = c^i K_i - \tfrac{1}{2} f_{ij}^k c^i c^j b_k \,. \tag{4.18}$$

First we compute a commutator

$$\begin{aligned}
[Q, U] &= [c^i K_i - \tfrac{1}{2} f_{ij}^k c^i c^j b_k, c^m b_m] \\
&= -c^m \{c^i, b_m\} K_i - c^m \tfrac{1}{2} f_{ij}^k \{c^i c^j, b_m\} b_k - \tfrac{1}{2} f_{ij}^k c^i c^j \{b_k, c^m\} b_m \\
&= -c^i K_i + f_{ij}^k c^i c^j b_k - \tfrac{1}{2} f_{ij}^k c^i c^j b_k = -Q \,.
\end{aligned}$$

Thus, the BRST operator has the following commutator with U:

$$[U, Q] = Q$$

and as the result it increases the ghost number by one:

$$UQ|\chi\rangle = (QU + Q)|\chi\rangle = (N_{\rm gh} + 1) Q|\chi\rangle \,.$$

Second, compute the anticommutator

$$\begin{aligned}
\{Q, Q\} &= \{c^i K_i - \tfrac{1}{2} f_{ij}^k c^i c^j b_k, c^s K_s - \tfrac{1}{2} f_{mn}^p c^m c^n b_p\} \\
&= c^i c^s K_i K_s - c^s c^i K_s K_i - \tfrac{1}{2} f_{ij}^k c^i c^j \{b_k, c^s\} K_s - \tfrac{1}{2} f_{mn}^p c^m c^n \{c^i, b_p\} K_i \\
&+ \tfrac{1}{4} f_{ij}^k f_{mn}^p \{c^i c^j b_k, c^m c^n b_p\} \,.
\end{aligned}$$

It is easy to find

$$f_{ij}^k f_{mn}^p \{c^i c^j b_k, c^m c^n b_p\} = 4 f_{ij}^k f_{km}^p c^i c^j c^m b_p \,.$$

Therefore, the expression we are interested in reduces to

$$\{Q, Q\} = c^i c^j [K_i, K_j] - \tfrac{1}{2} f_{ij}^k c^i c^j K_k - \tfrac{1}{2} f_{ij}^k c^i c^j K_k + f_{ij}^k f_{km}^p c^i c^j c^m b_p \,.$$

Due to the algebra relation $[K_i, K_j] = f_{ij}^k K_k$ the first three terms in the last expression cancel out and we are left with

$$\{Q, Q\} = f_{ij}^k f_{km}^p c^i c^j c^m b_p \,.$$

Due to the anti-commuting property of the ghosts the last expression can be rewritten as

$$\{Q, Q\} = \tfrac{1}{3} (f_{ij}^k f_{km}^p + f_{mi}^k f_{kj}^p + f_{jm}^k f_{ki}^p) c^i c^j c^m b_p \,.$$

The expression in the bracket vanishes because this is the Jacobi identity written for structure constants of the Lie algebra

$$f_{ij}^k f_{km}^p + f_{mi}^k f_{kj}^p + f_{jm}^k f_{ki}^p = 0\,.$$

Thus, we found that the BRST operator is *nilpotent*, i.e. it its square is zero

$$Q^2 = \tfrac{1}{2}\{Q,Q\} = 0\,.$$

This is the fundamental property of the BRST operator. We also assume that Q is hermitian, i.e. $Q^\dagger = Q$.

Let $\mathcal{H}^k$ will be a Hilbert space of states with fixed ghost number $U = k$. An element $|\chi\rangle \in \mathcal{H}^k$ is called *BRST-invariant* if

$$Q\chi = 0 \tag{4.19}$$

Clearly, any state of the form $Q|\lambda\rangle$, where $|\lambda\rangle$ is any state with the ghost number[14] $k-1$, is BRST-invariant because

$$Q(Q|\lambda\rangle) = Q^2|\lambda\rangle = 0\,.$$

The state $Q|\lambda\rangle$ has zero norm because

$$\langle\lambda|Q^\dagger Q|\lambda\rangle = \langle\lambda|Q^2|\lambda\rangle = 0\,.$$

The most important BRST-invariant states are those which can not be written in the form $|\chi\rangle = Q|\lambda\rangle$. We will regard two solutions of equation (4.19) equivalent if

$$|\chi'\rangle - |\chi\rangle = Q|\lambda\rangle$$

for some λ.

In fact, we recognize that the BRST-operator mimics all the properties of the de-Rahm operator d which acts on the space of external (differential) forms on a manifold $\mathcal{M}$. Indeed, it has a property that $d^2 = 0$. A differential form ω is called *closed* if $d\omega = 0$ and it is called *exact* if there is another form θ such that $\omega = d\theta$. The factor-space of all closed forms over all exact forms of a given degree n

$$\mathrm{H}^n(\mathcal{M}) = \frac{\text{closed forms}}{\text{exact forms}}$$

is called n-th cohomology group of the manifold $\mathcal{M}$. In our present case the operator Q takes values in the Lie algebra and it defines cohomologies with values in an given representation of the Lie algebra.

Furthermore, the states with zero ghost charge are of special importance. Such a state must be annihilated by all b_k. For such states the BRST operator reduces to

$$Q|\chi\rangle = c^i K_i|\chi\rangle = 0\,.$$

[14] As we found the BRST-operator increases the ghost number by one.

Thus, $Q|\chi\rangle = 0$ is equivalent to the condition that $K_i|\chi\rangle = 0$, i.e. it is invariant under the action of the Lie algebra. On the other hand, this state cannot be represented as $|\chi\rangle = Q|\lambda\rangle$ for some $|\lambda\rangle$ because the ghost number of $|\lambda\rangle$ should be equal -1 which is impossible.

Let us apply this general construction to the string case. The Lie algebra in this case is the Virasoro algebra and we supply it with the ghosts c_m and ant-ghosts b_m. The BRST operator is now

$$Q = \sum_{-\infty}^{+\infty} L^X_{-m} c_m - \frac{1}{2}\sum_{-\infty}^{\infty}(m-n) : c_{-m}c_{-n}b_{m+n} : - a c_0 \,,$$

where a is the normal-ordering ambiguity constant for L_0. It turns out that this expression can be written as

$$Q = \sum_{-\infty}^{+\infty} : \Big(L^X_{-m} + \tfrac{1}{2}L^{\rm gh}_{-m} - a\delta_{m,0}\Big)c_m :$$

The ghost-number operator is

$$U = \sum_{-\infty}^{+\infty} : c_{-m}b_m :$$

We would like to investigate the fulfilment of the relation $Q^2 = 0$ in quantum theory. We find

$$Q^2 = \frac{1}{2}\{Q,Q\} = \sum_{n,m=-\infty}^{+\infty} \Big([L_m, L_n] - (m-n)L_{m+n}\Big)c_{-m}c_{-n} \,.$$

Here $L_m = L^X_m + L^{\rm gh}_m - a\delta_{m,0}$ is a total Virasoro operator. Thus, $Q^2 = 0$ for $d = 26$ and $a = 1$ as the consequence of vanishing of the total central charge!

Inverse statement is also true: from $Q^2 = 0$ it follows that the central charge of the Virasoro algebra vanishes. Indeed, we first note that

$$L_m = \{Q, b_m\}$$

From here

$$[L_m, Q] = [\{Q, b_m\}, Q] = (Qb_m + b_mQ)Q - Q(Qb_m + b_mQ) = [b_m, Q^2] = 0$$

as $Q^2 = 0$. Therefore, we see that

$$[L_m, L_n] = [L_m, \{Q, b_n\}] = \underbrace{\{[L_m, Q], b_n\}}_{=0} + \{Q, [L_m, b_n]\} = (m-n)\{Q, b_{m+n}\} = (m-n)b_{m+n} \,.$$

One can check that these objects are charges which correspond to new conserved currents

$$
\begin{aligned}
J_+^{\mathrm{B}} &= 2c^+(T_{++}^X + \tfrac{1}{2}T_{++}^{\mathrm{gh}}) \\
J_+ &= c^+ b_{++} \,.
\end{aligned}
$$

There is new and more fundamental *fermionic* symmetry present – it is BRST symmetry. Let λ be a grassman (anticomuting) parameter. The BRST transformation is defined as

$$
\delta Y = [\lambda Q, Y]
$$

It is given by

$$
\begin{aligned}
\delta X^\mu &= \lambda c^+ \partial_+ X^\mu + \lambda c^- \partial_- X^\mu \,, \\
\delta c^+ &= \lambda c^+ \partial_+ c^+ \,, \\
\delta b_{++} &= 2i\lambda T_{++} \,, \\
\delta T_{++} &= 0 \,.
\end{aligned}
$$

The ghost number operator

$$
U = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n} b_n - b_{-n} c_n)
$$

Here c_n, b_n for $n > 0$ are *annihilation* operators.

Zero-modes require special treatment. We have

$$
c_0^2 = b_0^2 = 0 \,, \qquad \{c_0, b_0\} = 1
$$

There is a two-dimensional representation of this relations:

$$
\begin{aligned}
c_0|\downarrow\rangle &= |\uparrow\rangle \,, \qquad & b_0|\uparrow\rangle &= |\downarrow\rangle \\
c_0|\uparrow\rangle &= 0 \,, & b_0|\downarrow\rangle &= 0 \,.
\end{aligned}
$$

The ghost numbers are $U_\downarrow = -1/2$ and $U_\uparrow = 1/2$.

Physical states should have the ghost number $-1/2$. They are annihilated by b_0. Indeed, consider

$$
c_n|\chi\rangle = b_n|\chi\rangle = 0 \,, \quad n > 0 \;\text{ and }\; b_0|\chi\rangle = 0 \,.
$$

The condition of the BRST invariance reduces to

$$
0 = Q|\chi\rangle = \Big(c_0(L_0 - 1) + \sum_{n>0} c_{-n} L_n\Big)|\chi\rangle \,.
$$

We thus reproduced the conditions for a physical state obtained in the old covariant quantization approach. *Physical states of bosonic string are cohomology classes of the BRST operator with the ghost number* $-1/2$.

5. Geometry and topology of string world-sheet

5.1 From Lorentzian to Euclidean world-sheets

String world-sheets and the target space both have Lorentzian signature. The connection to the theory of Riemann surfaces can be made by performing the Wick rotation of both world-sheet and the target space metrics. In particular, for the world-sheet time τ this means $\tau \to -i\tau$. Thus, on the string-world sheet this Euclidean continuation corresponds to

$$\sigma^{\pm} = \tau \pm \sigma \to -i(\tau \pm i\sigma)\,. \tag{5.1}$$

A word of caution is needed here: One cannot really prove that the theory of the Lorentzian world-sheets is equivalent to the theory of the Euclidean ones. However, treating the world-sheets as Euclidean provides by itself a consistent theory of interacting strings. The Euclidean version can be then used to learn as much as possible about its Lorentzian counterpart.

We start with considering a single closed string whose world-sheet has topology of a cylinder. Formula (5.1) suggests to introduce the complex coordinates

$$w = \tau + i\sigma\,, \qquad \bar{w} = \tau - i\sigma\,.$$

The Euclidean closed string covers only a finite interval of σ on the complex plane $0 \le \sigma \le 2\pi$ and, therefore, only a strip of the 2dim plane.

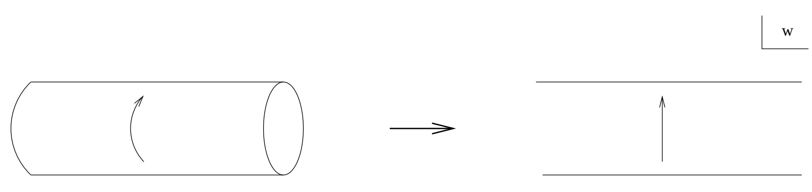


Fig. 5. Mapping the cylinder (τ, σ) to a strip $(w, \bar{w})$ on the complex w-plane.

One can further map the strip to the whole complex plane by using the conformal map

$$z = e^{w} = e^{\tau + i\sigma}\,.$$

Lines of constant τ are mapped into circles on the z plane and the operation of time translation $\tau \to \tau + a$ becomes the dilatation

$$z \to e^{a} z\,.$$

A procedure of identifying dilatations with the Hamiltonian and circles about the origin with equal-time surfaces is called sometimes *radial quantization*. Mapping from the cylinder to the plane cannot change the physical content of the theory if the theory is conformally invariant, which is the case of string theory in critical dimension.

We note that the plane is a non-compact manifold. However, one can compactify it by adding a point at infinity. The corresponding compact surface arising in this way is the Riemann sphere. A metric on a plane can be transformed to a metric on a sphere by a suitable choice of the conformal prefactor. For instance one can pick up the metric

$$\mathrm{d}s^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}$$

The formula $z = \cot\frac{\theta}{2}e^{i\phi}$ defines a stereographic projection of the sphere onto the plane and under this projection the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\,,$$

i.e. it is the standard round metric on a sphere.

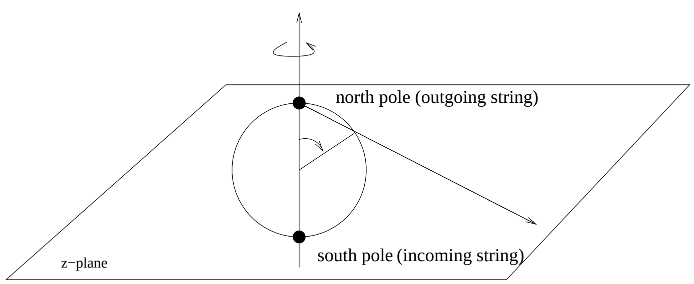


Fig. 6. Stereographic projection of a sphere onto z-plane. Asymptotic incoming and outgoing strings are mapped to the south and the north poles of the sphere respectively.

Since $\cot\frac{\pi}{2} = 0$ and $\cot(0) = \infty$ the incoming and outgoing strings are mapped to the south and the north poles of the sphere respectively.

The example above can be generalized to general world-sheets corresponding to interacting strings. *The crucial observation is that conformal invariance allows to consider compact world-sheets instead of surfaces with boundaries corresponding incoming and outgoing strings. The string boundaries are mapped to punctures on a compact Riemann surface.*

Under the Euclidean continuation the basic equation $\Box X = 0$ transforms into

$$\partial_z\partial_{\bar{z}}X = 0$$

with a general solution

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z})\,,$$

i.e. the left and right-moving excitation correspond now to analytic and anti-analytic fields on the complex z-plane.

5.2 Riemann surfaces

On a two-dimensional real manifold $\mathcal{M}$ the metric can be locally (i.e. in a given coordinate chart) defined by the line element

$$\mathrm{d}s^2 = h_{11}dx^2 + 2h_{12}\mathrm{d}x\mathrm{d}y + h_{22}\mathrm{d}y^2$$

Introducing the complex coordinates

$$z = x + iy\,, \qquad \bar{z} = x - iy$$

the line element can be written as

$$\mathrm{d}s^2 = 2e^{\phi}|dz + \mu d\bar{z}|^2$$

with the following identifications

$$e^{\phi} = \tfrac{1}{8}\Big(h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}\Big)\,, \qquad \mu = \frac{h_{11} - h_{22} + 2ih_{12}}{h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}}\,.$$

If $h_{11} = h_{12}$ and $h_{12} = 0$ then $\mu = 0$ and the metric takes in this coordinate chart the form

$$\mathrm{d}s^2 = 2e^{\phi}|dz|^2 = 2e^{\phi}(dx^2 + dy^2)\,.$$

The corresponding coordinate system is called *isothermal or conformal* and the coordinates (x, y) define a conformal map of a coordinate chart of a manifold to the Euclidean plane.

A theorem of Gauss

For any real two-dimensional orientable surface with a positive definite metric there always exists a system of isothermal coordinates (the theorem of Gauss). It is unique up to conformal transformations. First, assume that we have already found a system of isothermal coordinates, i.e. the metric is locally in the form

$$\mathrm{d}s^2 = 2e^{\phi}|dz|^2\,.$$

Performing the coordinate transformation with an analytic function of z:

$$z \longrightarrow f(z)$$

we get

$$\mathrm{d}s^2 \to \mathrm{d}s^2 = 2e^{\phi}|f(z)|^2|dz|^2\,,$$

i.e. we get a conformally equivalent metric and, therefore, a new system of the isothermal coordinates.

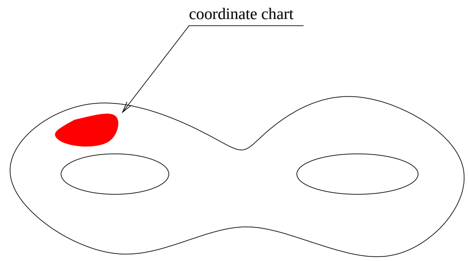


Fig. 6. Covering the Riemann surface with coordinate patches. Every patch is homeomorphic to an open domain of the Euclidean plane.

Second, in order to prove that isothermal coordinates exist, consider the so-called *Beltrami equation*

$$\frac{\partial w}{\partial \bar{z}} = \mu(z, \bar{z})\frac{\partial w}{\partial z}\,.$$

Suppose we solve this equation, then

$$|\mathrm{d}w|^2 = |\partial_z w + \partial_{\bar{z}} w|^2 = |\partial_z w|^2 |\mathrm{d}z + \mu \mathrm{d}\bar{z}|^2 = \frac{|\partial_z w|^2}{2e^{\phi}}\mathrm{d}s^2\,.$$

Thus,

$$\mathrm{d}s^2 = \frac{2e^{\phi}}{|\partial_z w|^2}|\mathrm{d}w|^2 \equiv \lambda |\mathrm{d}w|^2\,,$$

i.e. w defines a system of isothermal coordinates. It is a mathematical theorem that for a sufficiently small coordinate patch and a differentiable metric a solution of the Beltrami equation with $\partial_z w \neq 0$ always exists.

If two metrics are related by a diffeomorphism and a Weyl rescaling they are said to define the same *conformal structure*. If a manifold $\mathcal{M}$ is covered by a system of conformal (isothermal) coordinate patches U_α, then on the overlaps the metrics are conformally related, i.e. the transition functions on the overlaps $U_\alpha \cap U_\beta$ are analytic and the complex coordinates are globally defined. A system of analytic coordinate patches is called *a complex structure* and it is the same as a conformal structure.

A two-dimensional topological manifold endowed with a complex structure is called a Riemann surface. Thus, a Riemann surface is a complex manifold.

Another way to understand that Riemann surface is a complex manifold is to note that in two dimensions the metric provides a globally-defined integrable complex structure

$$I^{\alpha}{}_{\beta} = \sqrt{h} h^{\alpha\gamma}\epsilon_{\gamma\beta}$$

such that $I^2 = -1$. The conformal structure is conformally-invariant and globally well-defined. The existence of an integrable complex structure is necessary and sufficient for an even-dimensional manifold to be complex.

Recall the fundamental result from the theory of two-dimensional (real) manifolds. *Any compact orientable connected two-dimensional manifold is homeomorphic to a sphere with handles. The number of handles, g, is called the genus, a topological invariant. Thus, every compact Riemann surface, being a compact orientable connected one-dimensional manifold, has an associated genus.*

Gauss-Bonnet theorem

Let $\mathcal{M}$ is a compact orientable two-dimensional manifold of genus g with a metric $h_{\alpha\beta}$. Then

$$\frac{1}{4\pi}\int_{\mathcal{M}}\sqrt{h}R = \chi(\mathcal{M}) = 2-2g \tag{5.2}$$

is a topological invariant and it coincides with the Euler characteristic

$$\chi(\mathcal{M}) = 2-2g\,.$$

Here we briefly discuss one way to prove the Gauss-Bonnet theorem. Recall that in two dimensions the Riemann tensor has only one independent component which is the scalar curvature:

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}(h_{\alpha\gamma}h_{\beta\delta}-h_{\alpha\delta}h_{\beta\gamma})\,.$$

Let us choose a system of isothermal coordinates. In this coordinate system the line element is $\mathrm{d}s^2 = 2e^{\phi}\mathrm{d}z\mathrm{d}\bar{z}$ and, therefore, the metric has two components $h_{z\bar{z}} = h_{\bar{z}z} = e^{\phi}$. The Riemann tensor simplifies to

$$R_{z\bar{z}z\bar{z}} = -h_{z\bar{z}}R_{z\bar{z}} = -\frac{1}{2}(h_{z\bar{z}})^2R = e^{\phi}\partial\bar{\partial}\phi\,.$$

The scalar curvature is

$$R = -2e^{-\phi}\partial\bar{\partial}\phi \qquad \Longrightarrow \qquad \sqrt{h}R = -4\partial\bar{\partial}\phi = -\triangle\phi\,,$$

where $\triangle$ is two-dimensional Laplacian (the Euclidean analogue of the $\Box$ operator).

It is also useful to rewrite the integration measure in the complex coordinates

$$\mathrm{d}x\wedge\mathrm{d}y = \frac{i}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}\,,$$

i.e. we get

$$\sqrt{h}R\ \mathrm{d}^2x = -4\frac{\partial^2\phi}{\partial z\partial\bar{z}}\ \mathrm{d}x\wedge\mathrm{d}y = -2i\frac{\partial^2\phi}{\partial z\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z} = 2i\left(\mathrm{d}\bar{z}\frac{\partial}{\partial\bar{z}}\right)\mathrm{d}z\frac{\partial\phi}{\partial z}.$$

The last formula can be understood in the sense of calculus of exterior differential forms $\partial f = \frac{\partial f}{\partial z}$ and $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}}$. In particular, the de Rahm operator $\mathrm{d} = \mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}$ can be written as

$$\mathrm{d} = \partial + \bar{\partial} \equiv \mathrm{d}z\frac{\partial}{\partial z} + \mathrm{d}\bar{z}\frac{\partial}{\partial \bar{z}}\,.$$

Also one has $\partial\partial = \bar{\partial}\bar{\partial} = 0$. Therefore, we can rewrite our formula as

$$\sqrt{h}R = 2i\mathrm{d}(\partial\phi)\,.$$

This shows that $\sqrt{h}R$ is locally a *total derivative* and therefore, we see again that eq.(5.2) cannot change under smooth variations of the metric, i.e. it is a topological invariant.

On a compact Riemann surface $\mathcal{M}$ we consider an abelian differential Ω, which is a meromorphic differential form. It means that in a given coordinate patch (U_α, z_α) it can be written in the form $\Omega = f_\alpha \mathrm{d}z_\alpha$, where f_α is a meromorphic function[15]. We can also suppose that the coordinate patches are chosen in such a way that every U_α contains at most one pole or one zero of Ω. In a patch U_α the metric is $\mathrm{d}s^2 = 2e^{\phi_\alpha}|\mathrm{d}z_\alpha|^2$. Thus, on the intersections of the patches $U_\alpha \cap U_\beta$ we have

$$\frac{e^{\phi_\alpha}}{e^{\phi_\beta}} = \Big|\frac{f_\alpha}{f_\beta}\Big|^2\,.$$

Thus, there exists a globally defined function

$$\varphi = \frac{e^{\phi_\alpha}}{|f_\alpha|^2} \quad \text{on} \;\; U_\alpha \;\; \text{for all } \alpha\,.$$

This function is smooth except for singularities at zeros and poles of Ω. Since $\log|f_\alpha|^2$ is harmonic outside zeros and poles of Ω we have

$$\sqrt{h}R = 2i\mathrm{d}(\partial\log\varphi)\,.$$

Let $\mathcal{M}_\epsilon = \mathcal{M} - \cup D_{k,\epsilon}$, where $D_{k,\epsilon}$ are small disks around the singularities of Ω. Then, by Stokes's theorem we have

$$\int_{\mathcal{M}} \sqrt{h}R = 2i\lim_{\epsilon\to 0}\int_{\mathcal{M}_\epsilon} \mathrm{d}(\partial\log\varphi) = -2i\sum_k \lim_{\epsilon\to 0}\int_{\partial D_{k,\epsilon}} \partial\log\varphi\,.$$

To evaluate the integrals over the circles we note that at a zero or pole of Ω the function φ is of the form $\varphi = \psi/|z|^{2m}$ with a smooth function ψ without zeros and m is the order of zero or pole ($m < 0$ in the latter case). Therefore,

$$\lim_{\epsilon\to 0}\int_{\partial D_{k,\epsilon}} \partial\log\varphi = \lim_{\epsilon\to 0}\int_{|z|=\epsilon} \partial\log|z|^{-2m} = -m\lim_{\epsilon\to 0}\int_{|z|=\epsilon}\frac{\mathrm{d}z}{z} = -2\pi i m\,.$$

[15] A function $f(z)$ is called meromorphic if it does not have any other singularities except poles.

Summing up we obtain

$$\int_{\mathcal{M}} \sqrt{h}R = -4\pi \deg \Omega \,,$$

where $\deg \Omega$ is defined as the difference of the number of zeros and the number of poles of Ω:

$$\deg \Omega = \#\text{ zeros} - \#\text{ poles}\,.$$

It is now the Poincaré-Hopf theorem that states that $\deg \Omega = 2g-2$, where g is the genus of the Riemann surface. Thus, the Gauss-Bonnet theorem follows from the Poincaré-Hopf theorem.

Finally we note that due to the identity

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} = h^{\alpha\gamma}h^{\beta\delta} - h^{\alpha\delta}h^{\beta\gamma}$$

the Euler characteristic can be rewritten in the form[16]

$$\chi(\mathcal{M}) = \frac{1}{8\pi}\int_{\mathcal{M}} \epsilon^{\alpha\beta}\epsilon^{\gamma\delta}R_{\alpha\beta\gamma\delta}\,.$$

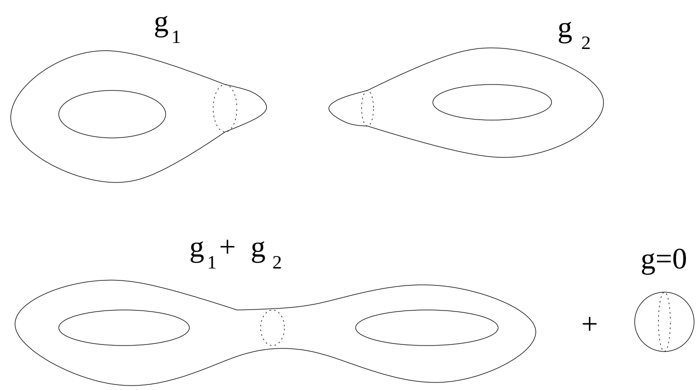


Fig. 7. If there are two surfaces of genera g_1 and g_2 then by removing from each surface a half-sphere we can glue the resulting surfaces into a surface of genus g_1+g_2 and the Riemann sphere of genus zero.

To illustrate the Gauss-Bonnet theorem, we compute the topological invariant eq.(5.2) for a sphere. We will take a model of a sphere which represent it as the complex plane (including the point at infinity) with the metric

$$\mathrm{d}s^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2} = 2e^{\phi}\mathrm{d}z\mathrm{d}\bar{z}\,,$$

[16]This is a non-trivial characteristic class of the tangent bundle to $\mathcal{M}$ known as the Euler class.

i.e the conformal factor ϕ and the action of Laplacian on it are

$$\phi = \ln 2 - 2\ln(1+|z|^2) \qquad \Longrightarrow \qquad -\triangle\phi = \frac{8}{(1+|z|^2)^2}\,.$$

Therefore,

$$\frac{1}{4\pi}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\sqrt{h}R = \frac{1}{4\pi}\int_0^\infty 2\pi r\mathrm{d}r\frac{8}{(1+r^2)^2} = 2\,,$$

which is indeed the Euler characteristic for sphere, a compact orientable manifold with genus $g=0$. It is also known that on a torus, a manifold of genus $g=1$, there exists the globally defined flat metric. Therefore, the Euler characteristic of torus is $\chi=0$. In fact, the Euler characteristic $\chi(g)$ for a Riemann surface of arbitrary genus g can be found by using the recurrent formula

$$\chi(g_1+g_2) = \chi(g_1)+\chi(g_2)-\chi(0) = \chi(g_1)+\chi(g_2)-2$$

and the fact that $\chi(1)=0$. This again leads to the formula $\chi(g)=2-2g$.

5.3 Moduli space

The moduli space of all the metrics is the same as the moduli space of Riemann surfaces and it is defined as the space of all metrics devided by diffeomorphisms and Weyl rescalings

$$\mathcal{M}_g = \frac{\text{all metrics}}{\text{diffeomorphisms} \times \text{Weyl rescalings}}\,.$$

The moduli space is finite-dimensional and it is parametrized by a finite number of complex parameters τ_i called moduli. The dimension of the moduli space is another topological invariant and it depends on the genus g only.

Complex geometry

Since we have a system of well-defined complex coordinates on a Riemann surface we can consider general tensors

$$V^{z\ldots z\bar{z}\ldots\bar{z}}{}_{z\ldots z\bar{z}\ldots\bar{z}}(z,\bar{z})\,,$$

in particular, $V^z\partial_z$ and $V^{\bar{z}}\partial_{\bar{z}}$ are vector fields and $V_z\mathrm{d}z$ and $V_{\bar{z}}\mathrm{d}\bar{z}$ are one-forms. All these tensors are one component objects. The metric $h_{z\bar{z}}$ and $h^{z\bar{z}}$ can be used to convert all $\bar{z}$ indices into z-indices. Tensors with one type of indices (for example, z indices) are called *holomorphic*. Holomorphic tensors which depend on z variable only are called *analytic*. A holomorphic tensor with p lower and q upper indices has, by definition, the rank $n = p - q$:

$$
\begin{aligned}
&V^{\overbrace{z\ldots z}^{q}}{}_{\underbrace{z\ldots z}_{p}}(z,\bar z) \qquad \Leftarrow \qquad \text{holomorphic tensor of rank } n = p - q; \\
&V^{z\ldots z}{}_{z\ldots z}(z) \qquad \Leftarrow \qquad \text{analytic tensor;}
\end{aligned}
$$

Under analytic coordinate transformations $z \to f(z)$ a holomorphic tensor of rank n transforms as follows

$$
V(z,\bar z) \to \left(\frac{\partial f(z)}{\partial z}\right)^n V(f(z), \bar f(\bar z))\,.
$$

Denote by $V^{(n)}$ the space of all holomorphic tensors of rank n. This space can be supplied with the scalar product

$$
(V_1^{(n)}|V_2^{(n)}) = \int \mathrm{d}^2 z\sqrt{h}\ (h^{z\bar z})^n \left(V_1^{(n)}\right)^* V_2^{(n)}\,, \qquad V_1^{(n)}, V_2^{(n)} \in V^{(n)}
$$

and the associated norm $||V^{(n)}||^2 = (V^n|V^{(n)})$. This scalar product is Weyl-invariant for $n = 1$ only.

In the analytical coordinate system the Christoffel connection has only two non-vanishing components

$$
\Gamma^{\,z}_{zz} = \partial\phi\,, \qquad \Gamma^{\,\bar z}_{\bar z\bar z} = \bar\partial\phi\,.
$$

This connection allows to define two covariant derivatives

$$
\begin{aligned}
\nabla_z^{(n)}: &\quad V^{(n)} \to V^{(n+1)}, \qquad \nabla_z^{(n)} T^{(n)}(z,\bar z) = (\partial - n\partial\phi) T^{(n)}(z,\bar z) \\
\nabla^z_{(n)}: &\quad V^{(n)} \to V^{(n-1)}, \qquad \nabla^z_{(n)} T^{(n)}(z,\bar z) = h^{z\bar z}\nabla_{\bar z} T^{(n)}(z,\bar z) = h^{z\bar z}\bar\partial T^{(n)}(z,\bar z)\,.
\end{aligned}
$$

These two differential operators defined on holomorphic tensors of a fixed rank n commute with the analytic coordinate transformations $z \to f(z)$. One can compute an adjoint of $\nabla_z^{(n)}$ and find that

$$
(\nabla_z^{(n)})^\dagger = -\nabla^z_{(n+1)}\,.
$$

The elements of the complex geometry we introduced above allows one to obtain some information about the moduli space. Consider an arbitrary infinitesimal change of the metric

$$
\delta h_{\alpha\beta} = \underbrace{\Lambda h_{\alpha\beta}}_{\text{Weyl}} + \underbrace{\nabla_\alpha V_\beta + \nabla_\beta V_\alpha}_{\text{diff}} + \underbrace{\sum_i \delta\tau_i \frac{\partial}{\partial\tau_i} h_{\alpha\beta}}_{\text{moduli}}\,.
$$

The last term here reflects the dependence of the metric on moduli which cannot be compensated by diffeomorphisms and Weyl rescalings. We can split this variation into trace and traceless parts

$$
\begin{aligned}
\delta h^{\rm trace}_{\alpha\beta} &= \Big(\Lambda+\nabla^\gamma V_\gamma+\frac{1}{2}h^{\gamma\delta}\sum_i \delta\tau_i\frac{\partial}{\partial\tau_i}h_{\gamma\delta}\Big)h_{\alpha\beta}\\
\delta h^{\rm traceless}_{\alpha\beta} &= \underbrace{\nabla_\alpha V_\beta+\nabla_\beta V_\alpha-h_{\alpha\beta}\nabla^\gamma V_\gamma}_{\text{Operator P}}+\sum_i\delta\tau_i\Big(\frac{\partial}{\partial\tau_i}h_{\alpha\beta}-\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\frac{\partial}{\partial\tau_i}h_{\gamma\delta}\Big)\,.
\end{aligned}
$$

We see that we can always shift the Weyl rescaling parameter Λ as

$$
\Lambda\to\Lambda-\nabla^\gamma V_\gamma-\frac{1}{2}h^{\gamma\delta}\sum_i\delta\tau_i\frac{\partial}{\partial\tau_i}h_{\gamma\delta}
$$

so that the trace part of the variation will transform as

$$
\delta h^{\rm trace}_{\alpha\beta}=\Lambda h_{\alpha\beta}\,.
$$

In the complex coordinates we have $h_{zz}=h_{\bar z\bar z}=0$ and therefore rewriting the variation formulae in these coordinates we find

$$
\begin{aligned}
\delta h_{z\bar z} &= \Lambda h_{z\bar z}\,,\\
\delta h_{zz} &= \underbrace{2\nabla^{(1)}_z V_z}_{\text{Operator P}}+\sum_i\delta\tau^i\mu^i{}_{zz}\,,
\end{aligned}
$$

where $\mu^i{}_{zz}=\partial_{\tau_i}h_{zz}=h_{z\bar z}\mu^{i\bar z}_z$. We see, in particular, that an infinitesimal change of $h_{z\bar z}$ can always be written as a Weyl rescaling. Also the covariant derivative $\nabla^{(1)}_z$ introduced above should be naturally identified with the operator P. Finally, we note that decomposition into sum of two terms

$$
\delta h_{zz}=2\nabla^{(1)}_z V_z+\sum_i\delta\tau^i\mu^i{}_{zz}
$$

is not orthogonal w.r.t. to the scalar product we introduced above. Denote by ϕ^i_{zz} a basis of the orthogonal complement of $\nabla^{(1)}_z$:

$$
(\phi^i_{zz}|\nabla^{(1)}_z V_z)=-(\nabla^z_{(2)}\phi^i_{zz}|V_z)\quad\text{ for any }\ V_z\in V^{(1)}.
$$

The last equation is equivalent to

$$
\nabla^z_{(2)}\phi^i_{zz}=0\qquad\Longrightarrow\qquad\bar\partial\phi^i_{zz}=0\,.
$$

Thus, the kernel (or, in other words, the space of zero modes) of the operator $P^\dagger=\nabla^z_{(2)}$ consists of global analytic tensors of the second rank. Such tensors are of special importance and they are called *quadratic differentials*. Thus, the dimension of the moduli space is equal to the number of lineally independent quadratic differentials on a given Riemann surface of genus g. The theory of quadratic differentials was developed by Kurt Strebel (I will add more on Strebel theory in due course).

The kernel of the operator $\nabla_z^{(1)}$ is spanned by vectors from $V^{(1)}$:

$$\nabla_z^{(1)} V_z = h_{z\bar{z}} \partial V^{\bar{z}} = 0 \qquad \Longrightarrow \qquad \partial V^{\bar{z}} = 0\,.$$

The globally defined vector fields which span a kernel of $\nabla_z^{(1)}$ are called *conformal Killing vectors*. They generate conformal Killing group (or the group of conformal isometries, i.e. globally defined diffeomorphisms which can be completely absorbed by the Weyl rescalings).

Riemann-Roch theorem

An important question of how many moduli for a Riemann surface of genus g exists is answered by *the Riemann-Roch theorem*. Define the index of $\nabla_z^{(n)}$ as the number of its zero modes minus the number of zero modes of its adjoint $\nabla^z_{(n+1)}$. The Riemann-Roch theorem states that

$$\mathrm{ind}\nabla_z^{(n)} = \dim\,\ker\nabla_z^{(n)} - \dim\,\ker\nabla^z_{(n+1)} = -(2n+1)(g-1) = \frac{1}{2}(2n+1)\chi_g\,.$$

For $n=1$ we therefore have

$$\#\text{complex moduli} - \#\text{conformal Killing vectors} = 3g-3$$

One can find the number of conformal Killing vectors for a compact Riemann surface in an independent way. These are globally defined analytic vector fields whose norm is finite

$$||V||^2 = \int_{\mathcal{M}_g} \sqrt{h} h_{z\bar{z}} V^z V^{\bar{z}} < \infty\,.$$

Here $V^z = V_n z^n$. For the case of sphere with metric $\mathrm{d}s^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}$ one finds that there exists three independent conformal Killing vectors

$$\partial_z\,, \qquad z\partial_z\,, \qquad z^2\partial_z\,.$$

Indeed, for the norm we have

$$||V||^2 = 2\pi\int_0^\infty r\mathrm{d}r\frac{8}{(1+r^2)^4}r^{2k} = \begin{cases} \frac{8\pi}{3} & k=0 \\ \frac{4\pi}{3} & k=1 \\ \frac{8\pi}{3} & k=2 \end{cases}$$

where $k=0,1,2$ for the three vector fields in question. We see that if $k\geq 3$ the integral becomes divergent, therefore, for instance, the field $z^3\partial_z$ has an infinite norm, i.e. it is not normalizable. The three conformal Killing vectors are well behaved at ∞ as can be seen by making conformal transformation $w = 1/z$ under which

$$-w^2\partial_w\,, \qquad -w\partial_w \qquad -\partial_w\,.$$

The limit $z \to \infty$ corresponds now to $w \to 0$ and we see that these fields are well-behaved at this point[17]. The three conformal Killing vectors we found correspond to the Virasoro generators L_0 and $L_{\pm 1}$ and they span (over the complex field) the Lie algebra $sl(2,\mathbb{C})$. Therefore, $sl(2,\mathbb{C})$ is the Lie algebra of analytic globally defined maps of the Riemann sphere onto itself.

Thus, the Riemann sphere has three conformal Killing vectors and, according to the Riemann-Roch theorem, no moduli. That means that all metrics on the sphere are conformally equivalent, or, in other words, there is a unique Riemann surface of genus zero.

One can count the number of conformal Killing vectors for higher genus Riemann surfaces as well. To this end one has to use the Ricci identity

$$\nabla^z_{(n+1)}\nabla^{(n)}_z - \nabla^{(n-1)}_z\nabla^z_{(n)} = \frac{1}{2}nR\,.$$

Let $V^{(n)} \in \ker\nabla^{(n)}_z$. Then we have

$$\begin{aligned}
0 &= (\nabla^{(n)}_z V^{(n)}|\nabla^{(n)}_z V^{(n)}) = -(V^{(n)}|\nabla^z_{(n+1)}\nabla^{(n)}_z V^{(n)}) = \\
&= -\frac{1}{2}(V^{(n)}|\nabla^z_{(n+1)}\nabla^{(n)}_z V^{(n)}) - \frac{1}{2}(V^{(n)}|\nabla^z_{(n+1)}\nabla^{(n)}_z V^{(n)}) \\
&= -\frac{1}{2}(V^{(n)}|\nabla^{(n-1)}_z\nabla^z_{(n)} + \frac{1}{2}nR) - \frac{1}{2}(V^{(n)}|\nabla^z_{(n+1)}\nabla^{(n)}_z V^{(n)}) \\
&= \frac{1}{2}\Big[(\nabla^{(n)}_z V^{(n)}|\nabla^{(n)}_z V^{(n)}) + (\nabla^z_{(n)} V^{(n)}|\nabla^z_{(n)} V^{(n)}) - \frac{1}{2}nR(V^{(n)}|V^{(n)})\Big]\,.
\end{aligned}$$

Therefore, for any vector from the kernel of $\nabla^{(n)}_z$ the following equality is valid

$$\underbrace{(\nabla^{(n)}_z V^{(n)}|\nabla^{(n)}_z V^{(n)})}_{\text{non}-\text{negative}} + \underbrace{(\nabla^z_{(n)} V^{(n)}|\nabla^z_{(n)} V^{(n)})}_{\text{non}-\text{negative}} - \frac{1}{2}nR(V^{(n)}|V^{(n)}) = 0\,.$$

Consider the case of a torus $g = 1$. On a torus there is a globally defined flat metric $\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}$ which gives $R = 0$. Therefore, the equality above leads to two equations

$$\partial V^{(n)} = \bar{\partial} V^{(n)} = 0\,,$$

i.e. $V^{(n)} = \text{const}$ and, therefore, $\dim\ker\nabla^{(n)}_z = 1$. Thus, there is a unique generator of conformal isometries, it corresponds to the rigid $\mathrm{U}(1)\times\mathrm{U}(1)$ rotations of the torus. The Riemann-Roch theorem gives for $g = 1$

$$\#\text{complex moduli} - \underbrace{\#\text{conformal Killing vectors}}_{=1} = 3 - 3 = 0\,,$$

[17]According to our general discussion of Riemann surfaces the sphere requires at least two coordinate patches to make an atlas. Transformation from one patch to another is analytic and is given by $w = 1/z$.

i.e. the torus is characterized by one complex modulus τ.

For $g \geq 2$ there is theorem that states that the corresponding manifold always admits a metric with constant negative curvature. For $n \neq 0$ this means that $-\frac{1}{2}nR > 0$ and, therefore, dim ker$\nabla^z_{(n)} = 0$. The Riemann surfaces with $g \geq 2$ have no conformal isometries and by the Riemann-Roch theorem that means that the number of complex moduli $n = 1$ is $3g - 3$.

For $n = 0$ we have dim ker$\nabla_z^{(0)} = 1$, because the corresponding kernel is spanned by constants. The Riemann-Roch theorem implies then that

$$\text{dim ker}\nabla^z_{(1)} - \underbrace{\text{dim ker}\nabla_z^{(0)}}_{=1} = \underbrace{(2n+1)}_{n=0}(g-1) = g - 1\,,$$

i.e. dim ker$\nabla^z_{(1)} = g$. The kernel of $\nabla^z_{(1)}$ is spanned by one forms

$$\nabla^z_{(1)}\omega_z = h^{z\bar{z}}\bar{\partial}\omega_z = 0 \qquad \Longrightarrow \qquad \bar{\partial}\omega_z = 0\,.$$

Thus we arrive at another important consequence of the Riemann-Roch theorem: *on a Riemann surface of the genus g there exists precisely g linearly independent (globally defined) analytic one-forms.* These analytic differential forms are called *abelian differentials of the first kind.*

The information we obtained by using the Riemann-Roch theorem is summarized in the Table below.

g	dim ker$\nabla_z^{(n)}$	dim ker$\nabla^z_{(n+1)}$
0	$2n+1$	0
1	1	1
>1	1 for $n = 0$ 0 for $n > 0$	g $(2n+1)(g-1)$

Moduli space of tori

Here we would like to look more closely at the moduli space which describes conformally non-equivalent tori – the Riemann surfaces of the genus $g = 1$. The torus can be obtained by performing the following identification on the complex plane

$$z \equiv z + n\lambda_1 + m\lambda_2\,, \qquad n, m \in \mathbb{Z}\,, \qquad \lambda_1, \lambda_2 \in \mathbb{C}\,.$$

The parameters $\lambda_{1,2}$ are subject to conformal transformations $z \to \lambda z$ and, therefore, only their ratio $\tau = \frac{\lambda_2}{\lambda_1}$ is scale-invariant. By using this freedom (the U(1)-rotation + real rescaling) one can always bring the parallelogram defining the torus upon gluing the opposite sides to the canonical form depicted on Fig.9, which corresponds to $\text{Im}\tau > 0$.

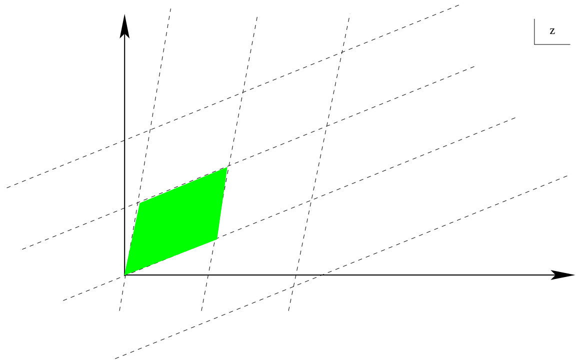


Fig. 8. Defining the torus by factorizing the complex z-plane.

The parameter τ takes values in the upper half-plane which is called Teichmüller space. The parameter τ itself is named the modular or Teichmüller parameter. The Teichmüller parameter is not yet a parameter describing the moduli space.

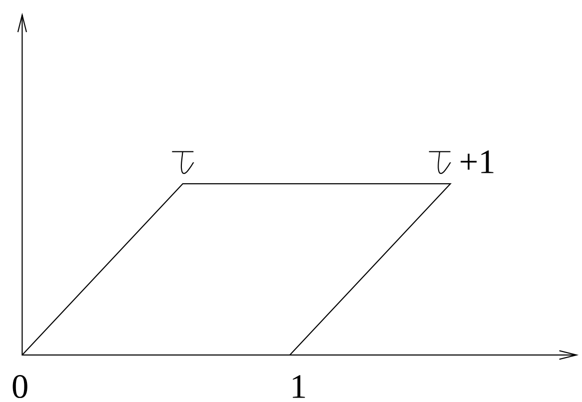


Fig. 9. Canonical representation of the torus by parameter τ taking values in the Techmüller space which is identified with the upper half-plane.

The reason is that there are global diffeomorphisms which are not smoothly connected to the identity; they leave the torus invariant but they act non-trivially on the Teichmüller parameter. They correspond to the so-called Dehn twists

- $\lambda_1 \to \lambda_1$, $\lambda_2 \to \lambda_1 + \lambda_2$, which gives $\tau \to \tau + 1$;
- $\lambda_1 \to \lambda_1 + \lambda_2$, $\lambda_2 \to \lambda_2$, which gives $\tau \to \frac{\tau}{\tau+1}$;

It turns out that these two transformations generate the group $\text{SL}(2, \mathbb{Z})$. It is a group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The action on the modular parameter τ is in the form of the fractional-linear transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}.$$

One can check that these transformations preserve the area of the parallelogram. Since two $\mathrm{SL}(2, \mathbb{Z})$-matrices

$$+\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

act on τ in the same way, *the modular group* of the torus is $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$.

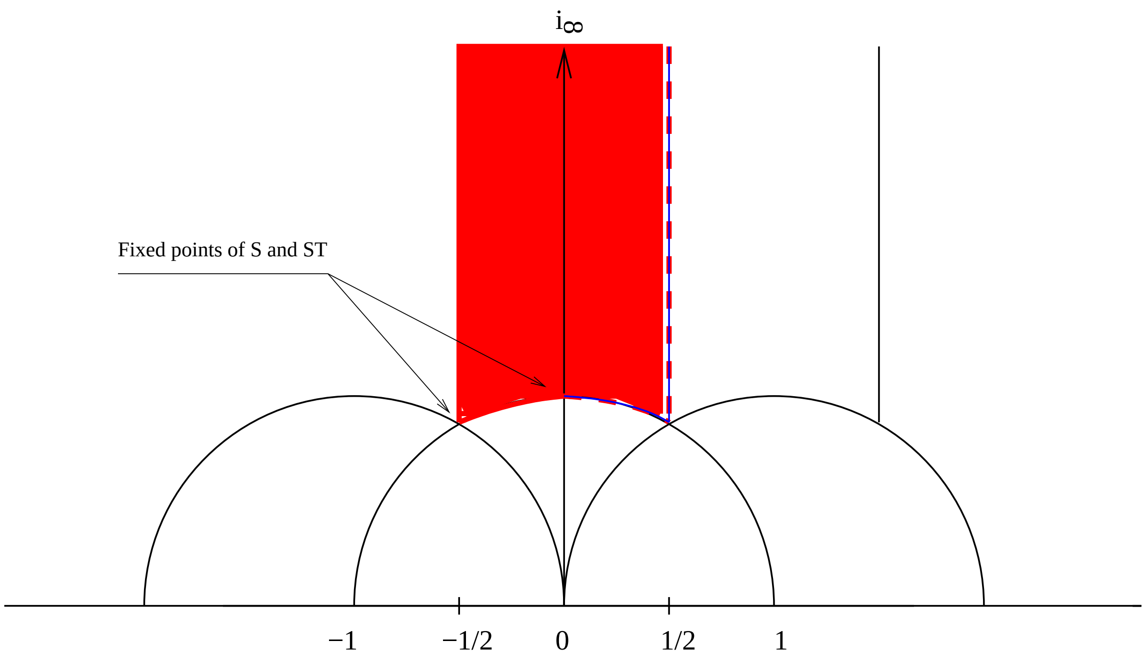


Fig. 8. Fundamental domain $\mathcal{M}_{g=1}$ of the Teichimüller space describing the moduli space of conformally non-equivalent tori.

The moduli space of conformally non-equivalent tori is then the quotient of the Teichmüller space of the modular group

$$\mathcal{M}_{g=1} = \frac{\text{Teichmüller space}}{\text{modular group}}.$$

One usually uses the following generators of the modular group

$$T: \quad \tau \to \tau + 1, \qquad\qquad S: \quad \tau \to -\frac{1}{\tau}.$$

Any element of $\mathrm{SL}(2, \mathbb{Z})$ is a composition of a certain number of S and T generators:

$$SSTSTTTST....SST$$

Any point in the upper-half plane is related by a modular transformation to a point in the so-called fundamental domain $\mathcal{F} \equiv \mathcal{M}_{g=1}$ of the modular group. It can be chosen as

$$\mathcal{M}_{g=1} = \Big\{ -\frac{1}{2} \leq \text{Re}\tau \leq 0, |\tau|^2 \geq 1 \cup 0 < \text{Re}\tau < \frac{1}{2}, |\tau|^2 > 1 \Big\} .$$

The modular group does not act freely on the upper-half plane because some of the modular transformations have fixed points. The point $\tau = i$ is the fixed point of S: $\tau \to -\frac{1}{\tau}$ and $\tau = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is the fixed point of ST. The existence of the fixed points implies that $\mathcal{M}_{g=1}$ is not a smooth manifold, rather it has singularities of the orbifold type.

Since it does not matter which fundamental region to choose in order to integrate over in the one-loop path integral the integrand the corresponding string amplitude should be invariant under modular transformations. The requirement of the modular invariance is one of the most important principles of string theory. In particular, it leads to strong restrictions on the possible gauge groups for the heterotic string.

6. Classical fermionic superstring

Introduction of the world-sheet fermionic degrees of freedom requires understanding of how spinors on curved manifolds (world-sheets) are defined. The discussion in the next paragraph is general and can be applied to a manifold of an arbitrary dimension d.

6.1 Spinors in General Relativity

It is not straightforward to introduce spinors in General Relativity. If we have a tensor field $T^{i_1 \dots i_p}_{j_1 \dots j_q}$ of rank (p, q) on a manifold $\mathcal{M}$ then under general coordinate transformations of the coordinates x^i on $\mathcal{M}$: $x^i \to x'^i(x^j)$, this field transforms as follows

$$T'^{k_1 \dots k_p}_{l_1 \dots l_q}(x') = \frac{\partial x'^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{k_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x'^{l_1}} \cdots \frac{\partial x^{j_q}}{\partial x'^{l_q}} \, T^{i_1 \dots i_p}_{j_1 \dots j_q}(x) \, .$$

Here tensor indices are acted with the matrices $\frac{\partial x'^i}{\partial x^j}$ which form a group $\text{GL}(d, \mathbb{R})$. This is a group of all invertible real $d \times d$ matrices. This group *does not* have spinor representations.

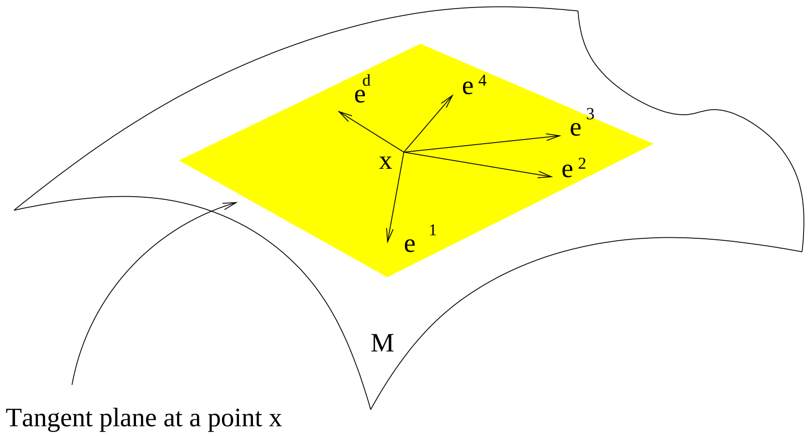


Fig. 9. Vielbien e^a_α is a collection of d orthogonal vectors forming a basis of the tangent space at any point x on the d-dimenional manifold. In string theory $\mathcal{M}$ with $d=2$ is the string world-sheet and $x \equiv (\sigma, \tau)$.

On the other hand, spinors are objects which transform under the spinor representations of the Lorentz group. The Lorentz group in d-dimensions is $\mathrm{SO}(d-1,1)$ and it is not $\mathrm{GL}(d,\mathbb{R})$. At each point of the manifold there is an inertial frame. In this inertial frame the Lorentz transformations are well defined. One can think that the Lorentz transformations act in the flat Minkowski space tangent to the manifold $\mathcal{M}$ at any given point x. In the tangent plane we introduce a basis $e^a_\alpha(x)$, $a=1,\ldots,d$ of orthonormal vectors:

$$(e^a, e^b) \equiv h^{\alpha\beta} e^a_\alpha e^b_\beta = \eta^{ab} \,,$$

where $\eta^{\alpha\beta}$ is the flat Minkowski metric. In fact, e^a_α is an invertible $d \times d$ x-dependent matrix which is called *vielbein*. The index α is called "curved" and its acted by the general coordinate transformations (diffeomorphisms) as the usual vector index, while the index a is called "flat" and its acted by the local (i.e. x-dependent) Lorentz transformations as we will see in a moment. The inverse matrix is e^α_a and it obeys $e^a_\alpha e^\alpha_b = \delta^a_b$. Because of this relation, we also have that

$$\eta_{ab} e^a_\alpha e^b_\beta = h_{\alpha\beta} \,.$$

There is no a preferred choice of the basis in the tangent space and one orthonormal set of tangent vectors can be transformed into the other by means of local Lorentz transformations

$$e^a_\alpha \to \Lambda^a{}_b e^b_\alpha \,.$$

Introduction of the vielbein in favour of the metric introduces additional degrees of freedom. Indeed, the vielbein being $d \times d$-matrix has d^2 components, while the metric $h_{\alpha\beta}$ has only $\frac{d(d+1)}{2}$ components in d dimensions. On the other hand, if we require that the theory we consider has the local Lorentz symmetry, then there are

$\frac{d(d-1)}{2}$ local Lorentz transformations which should allow to remove the additional (unphysical) components of the vielbein:

$$\underbrace{d^2}_{\text{vielbein}} - \underbrace{\frac{d(d-1)}{2}}_{\text{local Lorentz}} = \underbrace{\frac{d(d+1)}{2}}_{\text{metric}}$$

Thus, the vielbein brings new degrees of freedom, but they can be removed by a new symmetry which is the local Lorentz transformations. On the other hand, the vielbein allows one to introduce coupling of the gravitational degrees of freedom with spinors.

Spinor representations

(Pseudo-)orthogonal groups ($\mathrm{SO}(d-1,1)$) $\mathrm{SO}(d)$ in addition to the usual tensor representations have also spinor representations. These are not single-valued but rather double-valued representations of $\mathrm{SO}(d)$.

Consider, for instance, the group SO(3). This group has the so-called universal covering group which is SU(2). The relation between them is as follows

$$\mathrm{SO}(3) \to \mathrm{SU}(2)/\mathbb{Z}_2\,.$$

The group SO(3) is not simply connected, while SU(2) is. Here

$$\mathbb{Z}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

is the *center* of SU(2), i.e. a set of matrices from SU(2) which commute with all other SU(2)-matrices. Due to existence of the discrete center $\mathbb{Z}_2$ representations of SU(2) split into two different classes of integer and half-integer spin. Only representations with integer spin are those of (single-valid) SO(3). *Representations of* SU(2) *with half-integer spin are spinor (double-valid) representations of* SO(3). Indeed, rotation by the angle ϕ around the axes given by a fixed 3dim unit vector $n = (n_1, n_2, n_3)$, $n_i^2 = 1$, corresponds the transformation

$$g(\phi, n) = \exp\Big(-\frac{i}{2}\phi\, n_i \sigma_i\Big) = \begin{pmatrix} \cos\frac{\phi}{2} - in_3 \sin\frac{\phi}{2} & -(in_1 + n_2)\sin\frac{\phi}{2} \\ (-in_1 + n_2)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + in_3 \sin\frac{\phi}{2} \end{pmatrix},$$

where σ_i are three Pauli matrices. One can easily verify that $g(\phi, n) \in \mathrm{SU}(2)$. Rotation by the angle $\phi = 2\pi$ is an identity transformation in SO(3) but it is not the identity in SU(2). Indeed, one can see that

$$g(\phi + 2\pi, n) = -g(\phi, n)\,,$$
$$g(\phi + 4\pi, n) = g(\phi, n)\,.$$

Another example is provided by the Lorentz group of the 4dim Minkowski space-time, which is $\mathrm{O}(3,1)$. The spinor representation of $\mathrm{O}(3,1)$ is realized on the space $\mathbb{C}^4$, which is called a space of 4-component spinors.[18] This representation looks as

$$g(\omega) = \exp\left(\frac{1}{2}\gamma^{ab}\omega_{ab}\right),$$

where γ^{ab} is the anti-symmetric product of the 4dim γ-matrices and $\omega_{ab} = -\omega_{ba}$ are parameters of the Lorentz transformation. It is important to note that in general dimension d the spinor has $2^{\left[\frac{d}{2}\right]}$ complex components.

The spinor $\bar{\psi} = \psi^\dagger\gamma^0$ is called the Dirac conjugate of ψ. Its importance is explained by the fact that the quantity $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is an invariant of $\mathrm{O}(3,1)$.

In any dimension one can define the charge conjugation matrix C. Indeed, the Clifford algebra of γ-matrices transforms into itself under operation of transposition

$$\{\gamma^a, \gamma^b\}^t = \{(\gamma^a)^t, (\gamma^b)^t\} = 2\eta^{ab},$$

therefore by irreducibility of the corresponding representation of the Clifford algebra there should exists a matrix C which intertwines the original and the transposed representation of the algebra, namely:

$$(\gamma^a)^t = -C\gamma^a C^{-1}.$$

Matrix C is called the *charge conjugation matrix*.

Sometimes (depending on the dimension and signature od space-time) it is possible to define the notion of *Majorana* spinor. Majorana conjugate spinor is, by definition, $\psi^t C$. The Majorana spinor is then the spinor for which the Dirac conjugate is equal to the Majorana conjugate:

$$\psi^\dagger\gamma^0 = \psi^t C.$$

Spinor algebra in two dimensions

[18]The group $\mathrm{O}(3,1)$ is not connected and it has four connected components, which however are not simply connected. The component which contains an identity coincides with $\mathrm{SO}(3,1)$, which are the transformations preserving orientation of the vierbein. The transformations which preserve the direction of time are called orthochronous and they form the subgroup $\mathrm{SO}^+(3,1)$. The quotient group $\mathrm{O}(3,1)/\mathrm{SO}^+(3,1)$ is the Klein four-group $\mathbb{Z}_2\times\mathbb{Z}_2$, which is the semidirect product of $\mathrm{SO}^+(3,1)$ with an element of the discrete group $\{1, P, T, PT\}$, where P and T are the space inversion and time reversal operators

$$P = \mathrm{diag}(1,-1,-1,-1), \qquad T = \mathrm{diag}(-1,1,1,1).$$

The covering (or spin) group of $\mathrm{SO}^+(3,1)$ coincides with $\mathrm{SL}(2,\mathbb{C})$. One can show that $\mathrm{SO}^+(3,1) = \mathrm{SL}(2,\mathbb{C})/\{I,-I\} \equiv \mathrm{PSL}(2,\mathbb{C})$. Thus, $\mathrm{SL}(2,\mathbb{C})$ is the double-cover of $\mathrm{SO}^+(3,1)$.

The two-dimensional Dirac matrices ρ^a, $a = 0, 1$ obey the algebra

$$\{\rho^a, \rho^b\} = 2\eta^{ab}\,, \qquad \eta^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}.$$

A particular basis foe the Clifford algebra is given by

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We also define a matrix $\bar\rho$

$$\bar\rho = \rho^0\rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

being a 2dim analogue of the 4dim matrix γ^5. The charge conjugation matrix can be taken to be $C = \rho^0$. The Majorana spinor is then $\psi^\dagger\rho^0 = \psi^t C = \psi^t\rho^0$, i.e. $\psi = \psi^*$ which simple means that the spinor is real. Thus, in 2dim Majorana spinor is just a spinor with real components.

Finally, with the help of the vielbein ("zweibein" in 2dim) one can define the "curved" ρ-matrices:

$$\rho^\alpha = e^\alpha_a\rho^a\,.$$

They satisfy the following algebra

$$\{\rho^\alpha, \rho^\beta\} = 2h^{\alpha\beta}\,.$$

Spinors in 2dim have various interesting properties. One of them is the so-called spin-flip identity. If we have two Majorana spinors ψ_1 and ψ_2, then the following identity is valid

$$\bar\psi_1\rho^{\alpha_1}\cdots\rho^{\alpha_n}\psi_2 = (-1)^n\bar\psi_2\rho^{\alpha_n}\cdots\rho^{\alpha_1}\psi_1\,.$$

It is proved as follows.

$$\begin{aligned}\bar\psi_1\rho^{\alpha_1}\cdots\rho^{\alpha_n}\psi_2 &= (\bar\psi_1\rho^{\alpha_1}\cdots\rho^{\alpha_n}\psi_2)^t = -\psi_2^t(\rho^{\alpha_n})^t\cdots(\rho^{\alpha_1})^t(\rho^0)^t\psi_1 \\ &= (-1)^n\psi_2^t C\rho^{\alpha_n}C^{-1}\cdots C\rho^{\alpha_1}C^{-1}C\psi_1 = (-1)^n\bar\psi_2\rho^{\alpha_n}\cdots\rho^{\alpha_1}\psi_1\,.\end{aligned}$$

Another identity is

$$\rho^\alpha\rho_\beta\rho_\alpha = 0\,. \tag{6.1}$$

Indeed, one has from the Clifford algebra that $\rho_a\rho^\alpha = 2$ and, therefore

$$\rho^\alpha\rho_\beta\rho_\alpha = -\rho^\alpha\rho_\alpha\rho_\beta + 2\rho_\beta = -2\rho_\beta + 2\rho_\beta = 0\,.$$

Finally, we discuss the completeness condition for the ρ-matrices. The matrices ρ^a, $\bar{\rho}$ and the identity matrix $\mathbb{I}$ form a basis in the space of all 2×2-matrices. Any 2×2-matrix M can be expanded as

$$M = q_a\rho^a + \bar{q}\bar{\rho} + q\mathbb{I}\,.$$

The coefficients of this expansion are found as follows

$$q^a = \tfrac{1}{2}\mathrm{Tr}(M\rho^a)\,, \qquad \bar{q} = \tfrac{1}{2}\mathrm{Tr}(M\bar{\rho})\,, \qquad q = \tfrac{1}{2}\mathrm{Tr}(M)\,.$$

Plugging this back we obtain

$$2M = \mathrm{Tr}(M\rho_a)\rho^a + \mathrm{Tr}(M\bar{\rho})\bar{\rho} + \mathrm{Tr}(M)\mathbb{I}$$

or, more explicitly,

$$2M_{\alpha\beta} = M_{\gamma\delta}\Big[(\rho_a)_{\delta\gamma}\rho^a_{\alpha\beta} + \bar{\rho}_{\delta\gamma}\bar{\rho}_{\alpha\beta} + \delta_{\delta\gamma}\delta_{\alpha\beta}\Big]\,.$$

Since M is an arbitrary matrix from here we derive the following completeness relation

$$(\rho_a)_{\delta\gamma}\rho^a_{\alpha\beta} + \bar{\rho}_{\delta\gamma}\bar{\rho}_{\alpha\beta} + \delta_{\delta\gamma}\delta_{\alpha\beta} = 2\delta_{\alpha\gamma}\delta_{\beta\delta}\,.$$

Spin connection

We would like to introduce a new local symmetry which is the Lorentz symmetry. However, we have to guarantee that the the theory we are after should be invariant w.r.t. these local transformations. As for the case of any local gauge invariance, the local Lorentz invariance can be achieved by introducing q gauge field $\omega_{\alpha\ b}^{\ a}(x)$ for $\mathrm{SO}(d-1,d)$. Here a,b are $\mathrm{SO}(d-1,d)$-indices and α is the "curved" vector index. Under the local Lorentz transformations with the matrix Λ this field transforms as follows

$$\omega_\alpha \to \Lambda\omega_\alpha\Lambda^{-1} - \partial_\alpha\Lambda\Lambda^{-1}\,.$$

The gauge field of the local Lorentz symmetry is usually called the *spin connection*. The spin connection plays the same role for the "flat" indices as the Christoffel connection plays for the "curved" ones. We have the following substitution of the basic objects in the theory

$$\Big(h_{\alpha\beta}(x), \Gamma^\delta_{\alpha\beta}(x)\Big) \to \Big(e^a_\alpha(x), \omega_{\alpha\ b}^{\ a}(x)\Big)\,.$$

Introduction of the spin connection should not change the gravitational content of the theory. This means that the spin connection should not be a new independent field, rather it should be determined in terms of vielbein. The simplest and elegant way to do it is to notice that we have the covariant derivatives

$$D_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\delta}V^\delta$$
$$D_\alpha V^a = \partial_\alpha V^a + \omega_{\alpha\ b}^{\ a}V^b$$

The "flat" and "curved" indices of the vector are related as $V^a = e^a_\alpha V^\alpha$. This way of transforming "flat" to "curved" indices should be valid for any tensor, in particular, one has to have

$$D_\alpha V^a = e^a_\beta D_\alpha V^\beta \,.$$

This is possible only if

$$D_\alpha e^a_\beta = \partial_\alpha e^a_\beta - \Gamma^\lambda_{\alpha\beta} e^a_\lambda + \omega_\alpha{}^a{}_b e^b_\beta = 0 \,.$$

This equation can be solved for ω_α expressing it through the vielbein:

$$\omega^{ab}_\alpha = \frac{1}{2} e^{\beta a}(\partial_\alpha e^b_\beta - \partial_\beta e^b_\alpha) - \frac{1}{2} e^{\beta b}(\partial_\alpha e^a_\beta - \partial_\beta e^a_\alpha) - \frac{1}{2} e^{\lambda a} e^{\gamma b}(\partial_\lambda e_{\gamma c} - \partial_\gamma e_{\lambda c}) e^c_\alpha \,.$$

Since the connection is completely expressed via the dynamical vielbein and, by this means, is not an independent field, it is called sometimes *composite*.

Spin manifolds

On which manifolds one can introduce spinors? This is rather non-trivial question. Vielbein can always be introduces locally in a coordinate patch U_α. It is quite rare that the vielbein can be also globally defined. In the latter case such manifolds are called *parallelizable*. Examples of parallelizable manifolds are Lie groups. On the other hand, two-sphere is not parallelizable because there is no globally defined vector field (not talking about the vielbein) which vanishes nowhere. Thus, the vielbein is defined locally and in the intersection of the coordinate patches $U_\alpha \cap U_\beta$ one has

$$e_{(\alpha)}(x) = \Lambda_{(\alpha\beta)}(x) e_{(\beta)}(x) \,.$$

Here $\Lambda_{(\alpha\beta)}(x)$ is the local Lorentz transformation (i.e. a matrix from $\mathrm{SO}(d-1,1)$) which is called the *transition function*. In the region of triple intersection $U_\alpha \cap U_\beta \cap U_\gamma = U_{\alpha\beta\gamma}$ the transition functions should satisfy the following condition

$$\Lambda_{(\alpha\beta)}\Lambda_{(\beta\gamma)}\Lambda_{(\gamma\alpha)} = 1 \,.$$

Now if we introduce locally a spinor field $\psi_{(\alpha)}$ then passing from one coordinate patch to another one the field must transform according to

$$\psi_{(\alpha)}(x) = \bar{\Lambda}_{(\alpha\beta)} \psi_{(\beta)}(x) \,.$$

Here $\bar{\Lambda}_{(\alpha\beta)}$ is the $\mathrm{SO}(d-1,1)$ matrix in *the spinor representation*. For spinorial transition functions $\bar{\Lambda}$ it also makes sense to require that in the triple intersection region the following relation is satisfied

$$\bar{\Lambda}_{(\alpha\beta)}\bar{\Lambda}_{(\beta\gamma)}\bar{\Lambda}_{(\gamma\alpha)} = \pm 1 \,. \tag{6.2}$$

However, since the spinor representation is double-valued, instead of $\bar{\Lambda}$ one can equally use $-\bar{\Lambda}$. Thus, to define spinors on a non-parallelizable manifold one has to pick up the signs for $\bar{\Lambda}_{(\alpha\beta)}$ such that relation (6.2) is satisfied. When it is possible, the corresponding manifold $\mathcal{M}$ is said to admit the *spin structure* and it is called the *spin manifold*. Note that $\mathcal{M}$ may admit several inequivalent spin structures.

String theory contains world-sheet fermions and therefore it can be defined on spin-manifolds. It turns out that in 2 and 3dim any orientable manifold is the spin manifold. It is not so in 4dim and higher. Finally, we state without proof that on a Riemann surface of genus g there are 2^{2g} inequivalent spin structures.

6.2 Superstring action and its symmetrices

The superstring action is based on two multiplest of 2dim supersymmetry. The first one is the matter multiplet

$$\Big(X^\mu, \psi^\mu, F^\mu\Big)\,.$$

Here X^μ is a bosonic field of the Polyakov string, ψ^μ is the Majorana spinor in 2dim and F^μ is the real scalar, which is an auxiliary field to guarantee the equality of the bososnic and fermionic degrees of freedom off-shell (i.e. without usage of the equations of motion). Also the target space-time index μ is just a label so that for $\mu = 0, \ldots, d-1$ we have d matter multiplets. the second multiplet is the supergravity multiplet

$$\Big(e^a_\alpha, \chi_\alpha, A\Big)\,.$$

Here χ_α is the *gravitino*, i.e. the Majorana spinor which is also a vector of the 2dim world-sheet. The field A is an auxiliary scalar field which is needed to guarantee the equality of the bososnic and fermionic degrees of freedom off-shell. We note that the kinetic term for the gravitino in any dimension is $\bar{\chi}_\alpha \gamma^{\alpha\beta\gamma} D_\beta \chi_\gamma$ and it is absent in two dimensions (because there are only two ρ -matrices while the anti-symmetrized product $\gamma^{\alpha\beta\gamma}$ requires at least three to exist). Finally we introduce $e \equiv |\mathrm{det} e^a_\alpha| = \sqrt{h}$.

The superstring action is a generalization of the bosonic Polyakov action to include the include the world-sheet fermionic degrees of freedom in the supersymmetric way. It has the following structure

$$S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma e\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial^\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \Big(\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta \psi_\mu\Big)\right)\,.$$

This action has five local symmetries

1. *Local supersymmetry.* Let ϵ is the Majorana spinor. We consider it as the infinitezimal parameter of local supersymmetry transformations

$$\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu\,, \qquad \delta_\epsilon e^a_\alpha = \frac{i}{2}\bar{\epsilon}\rho^a \chi_\alpha\,,$$
$$\delta_\epsilon \psi^\mu = \frac{1}{2}\rho^\alpha\Big(\partial_\alpha X^\mu - \frac{i}{2}\bar{\chi}_\alpha \psi^\mu\Big)\epsilon\,, \qquad \delta_\epsilon \chi_\alpha = 2D_\alpha \epsilon\,.$$

Here $D_\alpha\epsilon = \partial_\alpha\epsilon - \frac{1}{2}\omega_\alpha\bar{\rho}\epsilon$ and ω_α is the connection with torsion:

$$\omega_\alpha = \omega_\alpha(e) + \frac{i}{4}\bar{\chi}_\alpha\bar{\rho}\rho^\beta\chi_\beta \ ,$$

where $\omega_\alpha(e) = -\frac{1}{e}e_{\alpha a}\epsilon^{\beta\gamma}\partial_\beta e^a_\gamma$ is the standard composite spin connection (it has only one-non-trivial component in 2dim).

2. *Weyl invariance.* Let Λ be a bosonic local parameter $\Lambda = \Lambda(\sigma,\tau)$. The Weyl transformations are

$$\begin{aligned} \delta_\Lambda X^\mu &= 0\,, & \delta_\Lambda e^a_\alpha &= \Lambda e^a_\alpha\,,\\ \delta_\Lambda\psi^\mu &= -\frac{1}{2}\Lambda\psi^\mu\,, & \delta_\Lambda\chi_\alpha &= \frac{1}{2}\Lambda\chi_\alpha\,. \end{aligned}$$

3. *Super Weyl invariance.* Let η be the Majorana spinor. Under the super Weyl transformations only the gravitino transforms as

$$\delta_\eta\chi_\alpha = \rho_\alpha\eta\,.$$

The invarince of the action easily follows from the identity $\rho^\alpha\rho_\beta\rho_\alpha = 0$.

4. *Local Lorentz symmetry.* Let ℓ be a bosonic local parameter $\ell = \ell(\sigma,\tau)$. The local Lorentz transformations are

$$\begin{aligned} \delta_\ell X^\mu &= 0\,, & \delta_\ell e^a_\alpha &= \ell\epsilon^a{}_b e^b_\alpha\,,\\ \delta_\ell\psi^\mu &= \frac{1}{2}\ell\bar{\rho}\psi^\mu\,, & \delta_\ell\chi_\alpha &= \frac{1}{2}\ell\bar{\rho}\chi_\alpha\,. \end{aligned}$$

5. *Reparametrizations.* Let ξ^α be a bosonic vector parameter $\xi^\alpha = \xi^\alpha(\sigma,\tau)$. The reparametrizations (diffeomorphisms) are

$$\begin{aligned} \delta_\xi X^\mu &= \xi^\beta\partial_\beta X^\mu\,, & \delta_\xi e^a_\alpha &= \xi^\beta\partial_\beta e^b_\alpha + e^a_\beta\partial_\alpha\xi^\beta\,,\\ \delta_\xi\psi^\mu &= \xi^\beta\partial_\beta\psi^\mu\,, & \delta_\xi\chi_\alpha &= \xi^\beta\partial_\beta\chi_\alpha + \chi_\beta\partial_\alpha\xi^\beta\,. \end{aligned}$$

6.3 Superconformal gauge and supermoduli

The gravitino field is the reducible representation of the Lorentz group. To decompose it into irreducible representations one can use the following trick:

$$\chi_\alpha = \delta^\beta_\alpha\chi_\beta = \left(\delta^\beta_\alpha - \frac{1}{2}\rho_\alpha\rho^\beta\right)\chi_\beta + \frac{1}{2}\rho_\alpha\rho^\beta\chi_\beta = \underbrace{\frac{1}{2}\rho^\beta\rho_\alpha\chi_\beta}_{\tilde{\chi}_\alpha} + \frac{1}{2}\rho_\alpha\rho^\beta\chi_\beta$$

Here $\tilde{\chi}_\alpha$ part is called ρ–traceless because, due to the identity $\rho^\alpha\rho_\beta\rho_\alpha = 0$ we get $\rho^\alpha\tilde{\chi}_\alpha = \rho^a e^\alpha_a\tilde{\chi}_\alpha = \rho^a\tilde{\chi}_a = 0$. Indeed the gravitino χ_a transforms under local Lorentz transformations as the spin-vector:[19]

$$\delta_\ell \chi_a = -\ell \epsilon_a{}^b \chi_b + \frac{1}{2}\ell\bar{\rho}\chi_a \,.$$

Condition $\rho^a\chi_a = 0$ remains invariant under these transformations, i.e. $\rho^a\delta_\ell\chi_a = 0$. Decomposition of the gravitino into the ρ-trace and ρ-traceless part is decomposition into two irreducible representations of the Lorentz group corresponding to helicities $\pm 3/2$ and $\pm 1/2$ respectively. This decomposition is orthogonal w.r.t. the scalar product $(\phi|\psi) = \int \mathrm{d}^2\sigma \bar{\phi}^\alpha \psi_\alpha$.

The local supersymmetry transformation for the gravitino filed can be also decomposed into the traceless- and the trace-parts:

$$\delta_\epsilon \chi_\alpha = 2D_\alpha \epsilon = 2(\Pi\epsilon)_\alpha + \underbrace{\rho_\alpha \rho^\beta D_\beta \epsilon}_{\text{trace part}}$$

Here we defined the operator

$$(\Pi\epsilon)_\alpha = \frac{1}{2}\rho^\beta\rho_\alpha D_\beta \epsilon \,, \qquad \Longrightarrow \qquad \rho^\alpha(\Pi\epsilon)_\alpha = 0 \,.$$

Locally one can show that there always exists a spinor κ such that $\tilde{\chi}_\alpha = \rho^\beta\rho_\alpha D_\beta \kappa$. Comparing this with the supersymmetry transformation for χ_α we conclude that κ can always be eliminated (locally!) by a supersymmetry variation. The possibility to eliminate κ globally depends on the existence of a globally defined spinor ϵ which solves the equation

$$(\Pi\epsilon)_\alpha = \tau_\alpha$$

for arbitrary τ_α satisfying the condition $\rho^\alpha\tau_\alpha$=0. Global solvability of the last expression relies on the absence of zero modes of the operator $\Pi^\dagger$: $(\Pi^\dagger\tau) = -2D^\alpha\tau_\alpha$. This equation is the supercousin of the bosonic equation

$$(P\xi)_{\alpha\beta} = t_{\alpha\beta}$$

whose global solvability relies on the absence of zero modes of $P^\dagger$. According to our discussion of the bosonic case it makes sense to call

$$\begin{aligned} \dim \ker P^\dagger &= \text{moduli} \\ \dim \ker \Pi^\dagger &= \text{supermoduli} \end{aligned}$$

and also

$$\begin{aligned} \dim \ker P &= \text{conformal Killing vectors} \\ \dim \ker \Pi &= \text{conformal Killing spinors} \,. \end{aligned}$$

[19]The element $\epsilon_a{}^b = \eta_{ac}\epsilon^{cb}$ is the following matrix $\epsilon_a{}^b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

By using reparametrizations and local Lorentz transformations the zweibein can be brought to the form $e^a_\alpha = e^\phi \delta^a_\alpha$ which is locally always possible. The gauge

$$e^a_\alpha = e^\phi \delta^a_\alpha \,, \qquad \chi_\alpha = \rho_\alpha \lambda$$

is called *superconformal gauge*. In classical theory the Weyl symmetry and super Weyl symmetry can be used to eliminate the remaining gravitational degrees of freedom ϕ and λ. In quantum theory it will be possible in critical dimension only.

6.4 Action in the superconformal gauge

In the superconformal gauge the action becomes rather simple

$$S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma \Big(\partial_\alpha X^\mu \partial^\alpha X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\Big) \,. \tag{6.3}$$

The world-sheet indices are now raised and lowered with the help of the flat world-sheet metric $\eta^{a\beta}$ and $\rho^\alpha = \delta^\alpha_a \rho^a$. This action is invariant w.r.t. local repramétrizations and supersymmetry transformations which satisfy the requirement

$$P\xi = 0 \,, \qquad \Pi\epsilon = 0 \,.$$

We would like to check directly that the action (6.3) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu \,, \\ \delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^\alpha \partial_\alpha X^\mu \epsilon \\ \delta_\epsilon \bar{\psi}^\mu &= -\frac{1}{2}\bar{\epsilon}\rho^\alpha \partial_\alpha X^\mu \end{aligned}$$

provided the parameter ϵ satisfies the following equation

$$\rho^\beta \rho_\alpha \partial_\beta \epsilon = 0 \,. \tag{6.4}$$

To check the invariance we perform the variation

$$\delta_\epsilon S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma \Big(2\partial_\alpha X^\mu \partial^\alpha (i\bar{\epsilon}\psi^\mu) + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha(\rho^\beta \partial_\beta X_\mu \epsilon) - \bar{\epsilon}\rho^\alpha \partial_\alpha X^\mu \rho^\beta \partial_\beta \psi_\mu\Big) \,.$$

Now we integrate by parts the first term and write out the second term more explicitly

$$\begin{aligned} \delta_\epsilon S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma \Big(&- 2i\Box X^\mu \, \bar{\epsilon}\psi_\mu + i\bar{\psi}^\mu \underbrace{\rho^\alpha \rho^\beta}_{\eta^{\alpha\beta}} \partial_\alpha \partial_\beta X_\mu \epsilon \\ &+ i\bar{\psi}^\mu \rho^\alpha \rho^\beta \partial_\beta X_\mu \partial_\alpha \epsilon - \bar{\epsilon}\rho^\alpha \partial_\alpha X^\mu \rho^\beta \partial_\beta \psi_\mu \Big) \,. \end{aligned}$$

Now we apply the spin-flip identity in the second and the third term and get

$$\delta_\epsilon S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma\Big(- 2i\Box X^\mu\ \bar{\epsilon}\psi_\mu + i\bar{\epsilon}\psi_\mu\Box X^\mu$$
$$+ i\partial_\alpha\bar{\epsilon}\rho^\beta\rho^\alpha\psi^\mu\partial_\beta X_\mu - \bar{\epsilon}\rho^\beta\rho^\alpha\partial_\beta X^\mu\partial_\alpha\psi_\mu\Big)\,.$$

Finally, we integrate the last term by parts and get

$$\delta_\epsilon S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma\ \ 2i\partial_\alpha\bar{\epsilon}\rho^\beta\rho^\alpha\psi^\mu\partial_\beta X_\mu\,.$$

This vanishes as the consequence of eq.(6.4).

We can write equation (6.4) more explicitly

$$\rho^\beta\rho_0\partial_\beta\epsilon = \Big(\rho^0\rho_0\partial_0 + \rho^1\rho_0\partial_1\Big)\epsilon = \Big(-\rho^0\rho^0\partial_0 + \rho^0\rho^1\partial_1\Big)\epsilon = \Big(\partial_0 + \bar{\rho}\partial_1\Big)\epsilon = 0\,,$$
$$\rho^\beta\rho_1\partial_\beta\epsilon = \Big(\rho^0\rho_1\partial_0 + \rho^1\rho_1\partial_1\Big)\epsilon = \Big(\rho^0\rho^1\partial_0 + \rho^1\rho^1\partial_1\Big)\epsilon = \Big(\bar{\rho}\partial_0 + \partial_1\Big)\epsilon = 0\,.$$

Since $\bar{\rho}^2 = 1$ the second equation is obtained from the first by multiplying with $\bar{\rho}$ and by this reason it is redundant. To analyze the first equation it is convenient to denote the components of any spinor as follows

$$\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \qquad \text{and} \qquad \epsilon = \begin{pmatrix}\epsilon_+\\ \epsilon_-\end{pmatrix}\,.$$

We thus see that the first equation reduces to

$$(\partial_0 + \partial_1)\epsilon_+ = \partial_+\epsilon_+ = 0\,, \qquad (\partial_0 - \partial_1)\epsilon_- = \partial_-\epsilon_- = 0$$

One cab define the spinors with upper indices by using the following convention

$$\psi^- = \psi_+\,, \quad \psi^+ = -\psi_-\,.$$

With this convention we obtain that components of the Majorana spinor which is a parameter of the supersymmetry transformations satisfy the equations

$$\partial_+\epsilon^- = \partial_-\epsilon^+ = 0 \qquad \Longrightarrow \qquad \epsilon^\pm \equiv \epsilon^\pm(\sigma^\pm)\,.$$

This equations should be contracted with the equations defining the conformal Killing vectors, i.e. reparametrizations which do not destroy the conformal gauge choice:

$$\partial_+\xi^- = \partial_-\xi^+ = 0 \qquad \Longrightarrow \qquad \xi^\pm \equiv \xi^\pm(\sigma^\pm)\,.$$

On-shell closer of the supersymmetry algebra

Consider first the commutator of two supersymmetry variations applied to a bosonic field X^μ:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]X^\mu = i\bar{\epsilon}_1\delta_{\epsilon_2}\psi^\mu - i\bar{\epsilon}_1\delta_{\epsilon_2}\psi^\mu = \frac{i}{2}\left(\bar{\epsilon}_1\rho^\alpha\epsilon_2 - \bar{\epsilon}_2\rho^\alpha\epsilon_1\right)\partial_\alpha X^\mu = i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha X^\mu\,,$$

where the last formula stems from the spin-flip property. We see that the commutator of two super-symmetry transformations generates a diffeomorphism transformation

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]X^\mu = \xi^\alpha\partial_\alpha X^\mu$$

with the parameter $\xi^\alpha = i\bar{\epsilon}_1\rho^\alpha\epsilon_2$. In fact, this is not an arbitrary diffeomorohism, rather it is a conformal transformation, because ξ^α is nothing else as a conformal Killing vector! Thus, bilinear combinations made of conformal spinors

Consider now the commutator of two supersymmetry variations applied to a fermion ψ^μ:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi^\mu = \frac{1}{2}\partial_\alpha(\delta_{\epsilon_1}X^\mu)\rho^\alpha\epsilon_2 - \frac{1}{2}\partial_\alpha(\delta_{\epsilon_2}X^\mu)\rho^\alpha\epsilon_1 = \frac{i}{2}\partial_\alpha(\bar{\epsilon}_1\psi^\mu)\rho^\alpha\epsilon_2 - \frac{i}{2}\partial_\alpha(\bar{\epsilon}_2\psi^\mu)\rho^\alpha\epsilon_1$$

Therefore,

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi^\mu = &\frac{i}{2}(\partial_\alpha\bar{\epsilon}_1\psi^\mu)\rho^\alpha\epsilon_2 - \frac{i}{2}(\partial_\alpha\bar{\epsilon}_2\psi^\mu)\rho^\alpha\epsilon_1 + \\ &+\frac{i}{2}(\bar{\epsilon}_1\partial_\alpha\psi^\mu)\rho^\alpha\epsilon_2 - \frac{i}{2}(\bar{\epsilon}_2\partial_\alpha\psi^\mu)\rho^\alpha\epsilon_1\end{aligned} \tag{6.5}$$

Constraints

In the original (gauge-unfixed) theory we have a world-sheet metric (vielbein) and the gravitino field. They are removed upon imposition of the superconformal gauge. However, before we fix the gauge, the metric and the gravitino have their equations of motion which become the constraints on the other fields of the theory after fixing the gauge. The stress tensor in now defined as

$$T_{\alpha\beta} = -\frac{2\pi}{e}\frac{\delta S}{\delta e_a^\beta}e_{\alpha a}\,.$$

We can also define the supercurrent as response of the action for variation of the gravitino field

$$G_\alpha = -i\frac{2\pi}{e}\frac{\delta S}{\delta\bar{\chi}^\alpha}\,.$$

Analogously to what was in the bosonic case the stress tensor $T_{\alpha\beta}$ will generate conformal transformations, while the new object G_α appears to be a generator of the supersymmetries. Equations of motion

$$T_{\alpha\beta} = 0 = G_\alpha$$

are constraints on the dynamics of our system. Using the action we find

$$
\begin{aligned}
T_{\alpha\beta} &= \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}\eta_{\alpha\beta}\partial_\gamma X_\mu \partial^\gamma X^\mu + \frac{i}{4}\bar{\psi}\rho_\alpha\partial_\beta\psi_\mu + \frac{i}{4}\bar{\psi}\rho_\beta\partial_\alpha\psi_\mu \\
G_\alpha &= \frac{1}{4}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu
\end{aligned}
$$

Note that G_α is ρ-traceless, i.e.

$$\rho^\alpha G_\alpha = 0\,.$$

This equation is an analog of $T^\alpha_\alpha = 0$. Finally, by using equations of motion one can show that the stress tensor and the supercurrent are conserved

$$\partial^\alpha T_{\alpha\beta} = 0\,, \qquad \partial^\alpha G_\alpha = 0\,.$$

These conservation laws lead to existence of infinite number of conserved charges. To analyze the algebra of contraints in more detail it is convenient to use the world-sheet light-cone coordinates $\sigma^\pm$. In the light-cone coordinates the action becomes

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\sigma \Big(\partial_+ X\partial_- X + i(\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_-)\Big)\,.$$

Equations of motion are

$$\partial_+\partial_- X^\mu = 0\,, \qquad \partial_-\psi^\mu_+ = \partial_+\psi^\mu_- = 0\,.$$

Solving equations of motion for fermions we get

$$\psi^\mu_+ = \psi^\mu_+(\sigma^+)\,, \qquad \psi^\mu_- = \psi^\mu_-(\sigma^-)\,.$$

This, it appears that two components of the Majorana fermion are left- and right-moving fields on the world-sheet.

The components of the stress-tensor $T_{+-} = 0 = T_{-+}$, while the other components are

$$
\begin{aligned}
T_{++} &= \frac{1}{2}\partial_+ X\partial_+ X + \frac{i}{2}\psi_+\partial_+\psi_+\,, \\
T_{--} &= \frac{1}{2}\partial_- X\partial_- X + \frac{i}{2}\psi_-\partial_-\psi_-\,.
\end{aligned}
$$

The components of the supercurrent are

$$
\begin{aligned}
G_+ &= \frac{1}{2}\psi_+\partial_+ X\,, \\
G_- &= \frac{1}{2}\psi_-\partial_- X\,.
\end{aligned}
$$

The conservation laws look in the light-cone coordinates as

$$\partial_- G_+ = \partial_+ G_- = 0\,, \qquad \partial_- T_{++} = \partial_+ T_{--} = 0\,.$$

Now we note that in addition to the conserved charges generated by the stress tensor:

$$Q_{\xi^\pm} = \int \mathrm{d}\sigma \; \xi^\pm(\sigma^\pm) T_{\pm\pm}(\sigma, \tau)$$

we will have the conserved charges generated by the supercurrent

$$G_{\epsilon^\pm} = \int \mathrm{d}\sigma \; \epsilon^\pm(\sigma^\pm) G_\pm(\sigma, \tau) \, .$$

6.5 Boundary conditions

Varying the action to derive the equations of motion we will get the following boundary term

$$\int \mathrm{d}\sigma \partial_\sigma \Big(\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big) \, .$$

For the case of closed string, to make this term vanishing one has to impose the following condition

$$\Big(\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big)(\sigma) - \Big(\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big)(\sigma + 2\pi) = 0 \, .$$

Since the fermions ψ_+ and ψ_- are independent this equation implies that

$$\begin{aligned} \psi_+(\sigma) &= \pm \psi_+(\sigma + 2\pi) \, , \\ \psi_-(\sigma) &= \pm \psi_-(\sigma + 2\pi) \, . \end{aligned}$$

The following terminology is standard

- Periodic boundary conditions in σ are called *Ramond boundary conditions* and they are denoted by the letter "R".
- Anti-periodic boundary conditions in σ are called *Nevew-Schwarz boundary conditions* and they are denoted by the letter "NS".

Universally, all fermionic quantities on the world-sheet have the following boundary conditions

$$\psi(\sigma + 2\pi) = e^{2\pi i \theta} \psi(\sigma) \, ,$$

where $\theta = 0$ in the R-sector and $\theta = 1/2$ in the NS sector.

Boundary conditions for ψ_+ and ψ_- can be chosen independently, which gives in total four possibilities

$$(\mathrm{R}, \mathrm{R}) \, , \quad (\mathrm{NS}, \mathrm{NS}) \, , \quad (\mathrm{R}, \mathrm{NS}) \, , \quad (\mathrm{NS}, \mathrm{R})$$

The boundary conditions for the two components of the supersymmetry parameter should be chosen in such a way as to make the variation $\delta X^\mu = i\bar{\epsilon}\psi^\mu$ periodic.

As we will see the states in the R sector give the space-time fermions, while the states in the NS sector are the space-time bosons. This further gives that (R,R) and (NS,NS) sectors are space-time bosons while (R,NS) and (NS,R) are fermions.

For the open string case we have that

$$\psi_+\delta\psi_+ - \psi_-\delta\psi_-$$

must vanish at $\sigma = 0$ and $\sigma = \pi$. If we assume that $\psi_+ = \alpha\psi_-$ at the end point of string, then

$$(\alpha^2 - 1)\psi_-\delta\psi_- = 0$$

which allows for $\alpha = \pm 1$. Thus, at each end of the string we should have $\psi_+ = \pm\psi_-$. We can always agree to choose $\psi_+(0,\tau) = \psi_-(0,\tau)$ as it is a matter of convention, then on the other hand of the string we have two possibilities

$$\begin{aligned} \psi_+(\pi,\tau) &= \psi_-(\pi,\tau) \qquad\qquad (\mathrm{Ramond})\,, \\ \psi_+(\pi,\tau) &= -\psi_-(\pi,\tau) \qquad\quad (\mathrm{Neveu-Schwarz})\,. \end{aligned}$$

6.6 Superconformal algebra

In order to compute the Poisson bracket between the components of the stress tensor and the supercurrent we need the fundamental Poisson bracket for the fermions. The Dirac action leads to the following bracket

$$\begin{aligned} \{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} &= -2\pi i\delta(\sigma-\sigma')\eta^{\mu\nu} \\ \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\} &= -2\pi i\delta(\sigma-\sigma')\eta^{\mu\nu}\,. \end{aligned}$$

Using these brackets together with brackets between the bosonic fields one find the following Poisson algebra of the constraints

$$\begin{aligned} \{T_{++}(\sigma), T_{++}(\sigma')\} &= -2\pi\Big(2T_{++}(\sigma')\partial' + \partial' T_{++}(\sigma')\Big)\delta(\sigma-\sigma')\,, \\ \{T_{++}(\sigma), G_+(\sigma')\} &= -2\pi\Big(\frac{3}{2}G_+(\sigma')\partial' + \partial' G_+(\sigma')\Big)\delta(\sigma-\sigma')\,, \\ \{G_+(\sigma), G_+(\sigma')\} &= -i\pi T_{++}(\sigma)\delta(\sigma-\sigma')\,. \end{aligned}$$

This is the so-called $\mathcal{N}=1$ *superconfomal algebra in 2dim.* Here $\mathcal{N}=1$ refers to the fact that supersymmetry transformations are performed with the help of one Majorana spinor.

The action of the supercurrent on the bosonic and fermionic fields generate supersymmetry transformations (bosonic field transforms into fermionic one and vice-versa):

$$\begin{aligned} \{G_+(\sigma), X^\mu(\sigma')\} &= -\pi\ \psi_+^\mu(\sigma)\delta(\sigma-\sigma')\,, \\ \{G_+(\sigma), \psi^\mu(\sigma')\} &= -i\pi\ \partial_+ X_+^\mu(\sigma)\delta(\sigma-\sigma') \end{aligned}$$

One can also check that the world-sheet fermion transforms under conformal transformations as the conformal field with weigh 1/2.

Consider closed strings. Using the mode expansion

$$\begin{aligned}\psi_+(\sigma,\tau) &= \sum_{r\in\mathbb{Z}+\theta} \bar{b}^\mu_r e^{-ir(\tau+\sigma)}\,,\\ \psi_-(\sigma,\tau) &= \sum_{r\in\mathbb{Z}+\theta} b^\mu_r e^{-ir(\tau-\sigma)}\,,\end{aligned}$$

where $\theta = 0$ in the R-sector and $\theta = \frac{1}{2}$ in the NS-sector, we obtain the Poisson algebra of oscillators

$$\begin{aligned}\{b^\mu_r, b^\nu_s\} &= -i\eta^{\mu\nu}\delta_{r+s}\,,\\ \{\bar{b}^\mu_r, \bar{b}^\nu_s\} &= -i\eta^{\mu\nu}\delta_{r+s}\,,\\ \{b^\mu_r, \bar{b}^\nu_s\} &= 0\,.\end{aligned}$$

The reality of the Majorana spinor implies that

$$(b^\mu_r)^\dagger = b^\mu_{-r}\,, \qquad (\bar{b}^\mu_r)^\dagger = \bar{b}^\mu_{-r}\,.$$

Introducing the modes of the stress tensor and the supercurrent

$$L_m = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma e^{-im\sigma} T_{--}\,, \qquad G_m = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\sigma e^{-ir\sigma} G_-\,.$$

Notice that the supercurrent G_- satisfies the same boundary condition as the fermion ψ_-. Substituting the mode expansion we get

$$\begin{aligned}L_m &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\alpha_{m+n} + \frac{1}{2}\sum_r \Big(r + \frac{m}{2}\Big) b_{-r}b_{m+r}\,,\\ G_r &= \sum_{n\in\mathbb{Z}} \alpha_{-n} b_{r+n}\,.\end{aligned}$$

These generators generate the classical super-Virasoro algebra

$$\begin{aligned}\{L_m, L_n\} &= -i(m-n)L_{m+n}\,,\\ \{L_m, G_r\} &= -i\Big(\frac{1}{2}m - n\Big)G_{m+r}\,,\\ \{G_r, G_s\} &= -2iL_{r+s}\,.\end{aligned}$$

7. Quantum fermionic string

Canonical quantization is again performed by substituting the Poisson bracket for the (anti)-commutator: $\{\ ,\ \}_{\rm pb} \to \frac{1}{i}[\ ,\ ]$. Therefore, the anti-commutators for the quantum fermionic fields are

$$\begin{aligned}\{\psi^\mu_+(\sigma), \psi^\nu_+(\sigma')\} &= 2\pi\delta(\sigma-\sigma')\eta^{\mu\nu}\\ \{\psi^\mu_-(\sigma), \psi^\nu_-(\sigma')\} &= 2\pi\delta(\sigma-\sigma')\eta^{\mu\nu}\,.\end{aligned}$$

Anti-commutators of the modes are

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}\,.$$

We again see that we can split oscillators into creation and annihilation operators according to the sign of their index, namely

- Oscillators with $r > 0$ are annihilation operators ,
- Oscillators with $r < 0$ are creation operators .

However, the modes with $r = 0$ which occur in the Ramond sector only require special care. Indeed, in the bosonic modes α_0^μ and $\bar{\alpha}_0^\mu$ correspond to the center of mass momentum of the string. Analogously, b_0^μ and $\bar{b}_0^\mu$ are distinguished from all the other modes, in particular, they form the Clifford algebra

$$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$$

and analogously for $\bar{b}_0^\mu$.

The super-Virasoro generators are again defined as normal ordered expressions

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\alpha_{m+n}: + \frac{1}{2}\sum_r \Big(r + \frac{m}{2}\Big) :b_{-r}b_{m+r}:\,, \\ G_r &= \sum_{n\in\mathbb{Z}} \alpha_{-n}b_{r+n}\,. \end{aligned}$$

Only the generator L_0 is ambiguous doe to the undetermined normal ordering constant. Ignoring this constant for the moment we obtain the following answer for the quantum super-Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{d}{8}m(m^2 - 2\omega)\delta_{m+n}\,, \\ [L_m, G_r] &= \Big(\frac{1}{2}m - n\Big)G_{m+r}\,, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{d}{2}\Big(r^2 - \frac{\omega}{2}\Big)\delta_{r+s}\,. \end{aligned}$$

Here $\omega = 0$ for the R-sector and $\omega = \frac{1}{2}$ for the NS-sector. Both the R- and NS-algebras formally agree except the linear terms in anomalies. The linear term can be changed by shifting the L_0 generator. Indeed, one can see that if one shifts $L_0^{\rm R} \to L_0^{\rm R} + \frac{d}{16}$ then both algebras have formally the same structure with $\omega = \frac{1}{2}$. Still the R- and NS-algebras are very different. For instance, in the NS-sector the five generators $L_1, L_0, L_{-1}, G_{1/2}, G_{-1/2}$ form a closed superalgebra known as OSp(1|2). In the R-sector just adding to the generators L_1, L_0, L_{-1} the generator G_0 one generates the whole infinite-dimensional algebra.

The oscillator ground state is defined in both sectors as

$$\alpha_m^\mu|0\rangle = b_r^\mu|0\rangle = 0\,, \qquad m, r > 0\,.$$

Here the dependence on the center of mass momentum is suppressed. In the Ramond sector one also has the zero mode b_0^μ. The level number operator is

$$N = N^{(\alpha)} + N^{(b)}\,,$$

where

$$N^{(\alpha)} = \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m\,,$$
$$N^{(b)} = \sum_{r\in\mathbb{Z}+\theta>0}^{\infty} r b_{-r}b_r\,.$$

Note that the zero mode in the Ramond sector does not contribute to the number operator! This leads to the fact the mass operator commutes with b_0^μ: $[b_0^\mu, M^2] = 0$, i.e. the states $|0\rangle$ and $b_0^\mu|0\rangle$ have the same mass. These states are degenerate. On the other hand, all other oscillators α_n^μ, b_r^μ with $n, r < 0$ increase $\alpha' M^2$ by $2n$ and $2r$ units respectively. This means that in the NS-sector the ground state is unique and it has Lorentz spin zero. In the R-sector the ground state is degenerate and since b_0^μ form the Clifford algebra the ground state is a spinor of the Lorentz group $\mathrm{SO}(d-1,1)$. This explains why in the NS-sector all the states are space-time bosons, while in the R-sector they are all fermions. Indeed, all creation operators have vector Lorentz index and by this reason they cannot convert a space-time boson into a space-time fermion or vice versa. If we will write the Ramond ground state as $|a\rangle$, where a is a $\mathrm{SO}(d-1,d)$ spinor index, the b_0^μ act on it as the usual Γ-matrices

$$b_0^\mu|a\rangle = \frac{1}{\sqrt{2}}(\Gamma^\mu)^a{}_b|b\rangle\,.$$

Here Γ^μ are the usual Γ-matrices of the d-dimensional Minkowski space and they satisfy the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$.

We will not go into discussion of the covariant quantization but will just state that consistency of the quantum theory will impose the following restrictions on the constant a of the normal ordering ambiguity (for the Ramond and Neveu-Schwarz sectors) and the dimension d of the target space-time:

$$a_{\mathrm{NS}} = \frac{1}{2}\,, \qquad a_{\mathrm{R}} = 0\,, \qquad d = 10\,.$$

The same result follows from the condition of non-anomalous Lorentz algebra in the light-cone gauge. Instead of $d = 26$ found for bosonic string, quantum fermionic string chooses to live in a ten-dimensional world.

7.1 Light-cone quantization and superstring spectrum

As we have established imposition of the superconformal gauge does not completely remove the gauge (unphysical) degrees of freedom. The superconformal transformations which do not destroy the superconformal gauge choice are left. In order to remove the remaining unphysical degrees of freedom one can try to fix the light-cone gauge, similar to as was done for bosonic string. We can also fix

$$X^+ = \alpha' p^+ \tau$$

in our fermionic theory and this choice will completely remove the reparametrization invariance. However, the local supersymmetry transformations obeying the equations

$$\partial_+ \epsilon^- = \partial_- \epsilon^+ = 0$$

are still left over. These transformations can be used in order to completely eliminate the fermionic field ψ^+, where this time $\psi^\pm$ refer to the target-space light-cone components

$$\psi^\pm = \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^{d-1}) .$$

This is equivalent to putting to zero the modes b_r^+ for all r. After this gauge choice is done we can solve the super-Virasoro constraints and find the longitudinal modes (remember that $T = \frac{1}{2\pi\alpha'}$)

$$\begin{aligned} \partial_\pm X^- &= \frac{1}{\alpha' p^+}\Big((\partial_\pm X^i)^2 + i\psi_\pm^i \partial_\pm \psi_\pm^i\Big) \\ \psi_\pm^- &= \frac{2}{\alpha' p^+}\psi_\pm^i \partial_\pm X^i . \end{aligned}$$

This shows that only the transversal components X^i and ψ^i are physical degrees of freedom. In terms of oscillators the previous equations read

$$\begin{aligned} \alpha_m^- &= \frac{1}{\sqrt{2\alpha'}p^+}\Big(: \alpha_n^i \alpha_{m-n}^i : + \sum_r \big(\frac{m}{2} - r\big) : b_r^i b_{m-r}^i : - 2a\delta_m\Big) \\ b_r^- &= \frac{2}{\alpha' p^+}\sum_q \alpha_{r-q}^i b_q^i . \end{aligned}$$

For the case of closed strings these expressions must be supplemented by the analogous ones for the left-moving modes. Here we also include a normal ordering constant a which is $a = \frac{1}{2}$ in the NS sector and $a = 0$ in the R sector.

The mass operator is

$$M^2 = M_R^2 + M_L^2 ,$$

where

$$\alpha' M_R^2 = 2\Big(\sum_{n>0}\alpha_{-n}^i \alpha_n^i + \sum_{r>0} r b_{-r}^i b_r^i - a\Big)$$

and similar for M_L^2. For the case of open string we have

$$\alpha' M_R^2 = \Big(\sum_{n>0} \alpha^i_{-n} \alpha^i_n + \sum_{r>0} r b^i_{-r} b^i_r - a \Big) .$$

α'mass^2	rep of SO(8)	little group	$(-1)^F$	rep of little group
		NS − sector		
$-\frac{1}{2}$	$\lvert 0\rangle$	SO(9)	-1	$\mathbf{1}$
0	$\underbrace{b^i_{-1/2}\lvert 0\rangle}_{8_v}$	SO(8)	$+1$	$\mathbf{8_v}$
$+\frac{1}{2}$	$\underbrace{\alpha^i_{-1}\lvert 0\rangle}_{8_v}, \quad \underbrace{b^i_{-1/2} b^j_{-1/2}\lvert 0\rangle}_{28}$	SO(9)	-1	$\mathbf{36}$
$+1$	$\underbrace{b^i_{-1/2} b^j_{-1/2} b^k_{-1/2}\lvert 0\rangle}_{56_v}$ $\underbrace{\alpha^i_{-1} b^j_{-1/2}\lvert 0\rangle}_{1+28+35_v}, \quad \underbrace{b^i_{-3/2}\lvert 0\rangle}_{8_v}$	SO(9)	$+1$ $+1$	$\mathbf{84+44}$
		R − sector		
0	$\underbrace{\lvert a\rangle}_{8_s}$ $\underbrace{\lvert \dot{a}\rangle}_{8_c}$	SO(8)	$+1$ -1	$\mathbf{8_s}$ $\mathbf{8_c}$
$+1$	$\underbrace{\alpha^i_{-1}\lvert a\rangle}_{8_c+56_c}, \quad \underbrace{b^i_{-1}\lvert \dot{a}\rangle}_{8_s+56_s}$ $\underbrace{\alpha^i_{-1}\lvert \dot{a}\rangle}_{8_s+56_s}, \quad \underbrace{b^i_{-1}\lvert a\rangle}_{8_c+56_c}$	SO(9)	$+1$ -1	$\mathbf{128}$ $\mathbf{128}$

Tab. 5. The lowest levels of the open fermionic string spectrum.

In the closed string case we have in addition the condition of level-matching which requires that for physical states

$$M_L^2 = M_R^2 \,.$$

Finally, we give expressions for the light-cone action

$$S_{l.c} = \frac{1}{8\pi} \int \mathrm{d}^2\sigma \Big((\dot{X}^i)^2 - (X'^i)^2 - 2i\bar{\psi}^i \rho^a \partial_\alpha \psi^i \Big) \tag{7.1}$$

and the Hamiltonian

$$H = (p^i)^2 + \sum_{n>0} (\alpha^i_{-n}\alpha^i_n + \bar{\alpha}^i_{-n}\bar{\alpha}^i_n) + \sum_{r>0} r(b^i_{-r}b^i_r + \bar{b}^i_{-r}\bar{b}^i_r) - 2a \,.$$

Note that every sector, R and NS, has its own Hamiltonian.

Let us now analyze the closed string spectrum. We first discuss the right-moving part which (up to a mass rescaling by 2) is equivalent to the spectrum of open fermionic string.

- *NS-sector*. The ground state is the oscillator vacuum $|0\rangle$ with $\alpha' M^2 = -a$. The first excited state is $b^i_{-1/2}|0\rangle$ with $\alpha' M^2 = \frac{1}{2} - a$. This is a vector of $\mathrm{SO}(d-2)$, where the critical dimension $d = 10$. Since the little Lorentz group for massless states in d-dimensions is $\mathrm{SO}(d-2)$ this state must be massless which gives the normal ordering constant to be $a = \frac{1}{2}$. At the next level one has the states $\alpha^i_{-1}|0\rangle$ and $b^i_{-1/2}b^j_{-1/2}|0\rangle$ with $\alpha' M^2 = \frac{1}{2}$. The number of these bosonic states is $8 + 28 = 36 = \frac{9\times 8}{2}$, they are comprise an antisymmetric representation of $\mathrm{SO}(9)$, the little Lorentz group for massive states.

- *R-sector*. The Ramond ground state is a spinor of $\mathrm{SO}(9,1)$. The dimension of the Dirac spinor in $d = 10$ is $2^{\frac{d}{2}} = 2^5 = 32$, i.e. it has 32 complex or 64 real components. In ten dimensions it is possible to impose both Majorana and Weyl conditions[20] which reduce the number of independent components to $\frac{64}{2\times 2} = 16$. On shell the number of components is further reduced by two because the Dirac equation $\Gamma^\mu \partial_\mu \psi$ relates half of the components to the other half (which satisfies the Klein-Gordon equation). The 8 remaining components can be viewed as the components of the Majorana-Weyl spinor of $\mathrm{SO}(8)$, the latter being the little Lorentz group for massless states in $d = 10$. Indeed, the spinor of $\mathrm{SO}(8)$ should have

$$(2^{\frac{8}{2}} \text{ complex components})/(\text{Majorana} \times \text{Weyl}) = 32/4 = 8$$

[20]In general, for the groups $\mathrm{SO}(p,q)$ the Majorana and Weyl conditions can be simultaneously imposed if and only if $p - q = 0 \mod 8$. For Minkowski space, $p = d-1$, $q = -1$, this gives $d = 2 + 2n$ and for Euclidean space, $p = d$, $q = 0$, this gives $d = 2n$.

real components. Thus, the Ramond ground state is massless.

It turns out that the group SO(8) has three inequivalent representations of dimension 8: two of them are spinor representations and another is the vector one. Spinor representations are commonly denoted by $\mathbf{8}_s$ and $\mathbf{8}_c$ and the corresponding representation bases are depicted as

$$|a\rangle \qquad \text{and} \qquad |\dot{a}\rangle\,.$$

The vector representation is $\mathbf{8}_v$ and the basis is $|i\rangle$.

The first excited level consist of states $\alpha^i_{-1}|a\rangle$ and $b^i_{-1}|a\rangle$ and their chiral partners with $\alpha' M^2 = 1$. Once again, for $d = 10$, all the massive light-cone states can be uniquely assembled into representations of SO(9), the little Lorentz group for massive states.

GSO projection

It turns out that fermionic string with all the states we found in the R and NS sectors is inconsistent. This can seen, for instance, from the fact that the 1-loop amplitude is not modular invariant. In order to construct a consistent modular invariant theory one should truncate the string spectrum in a specific way. This truncation is known as the GSO (Gliozzi-Scherk-Olive) projection. It restores the modular invariance, removes from the theory the tachyon and, in addition, provides the *space-time supersymmetry* of the resulting string spectrum. Below we will use an inverse argument to motivate the GSO projection – we will show that it allows to achive a spectrum which exhibits space-time supersymmetry.

Looking at the massless states in the Ramond sector we see that one has two SO(8) spinors 8_s and 8_c. On the other hand, the massless states of the NS sector comprise a vector $\mathbf{8}_v$. If we project one of the two spinors out then there will be match of NS bosonic (8) and R fermionic (also 8) degrees of freedom. These massless vector and the massless spinor is indeed a content of the $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions.

One has also to get rid of tachyon which is in the NS sector. This all can be achieved if one first defines an operator

$$G = (-1)^F\,, \qquad F = \sum_{r=\frac{1}{2}}^{\infty} b^i_{-r} b^i_r - 1$$

and then requires that all allowed states should have $G = 1$:

$$G|\Phi\rangle = |\Phi\rangle.$$

$\alpha'\text{mass}^2$	rep of SO(8)	little group	$(-1)^F$	$(-1)^{\bar{F}}$	rep of little group
	$(\mathbf{NS},\mathbf{NS})-\textbf{sector}$				
-2	$\vert 0\rangle_L \times \vert 0\rangle_R$	SO(9)	-1	-1	$\mathbf{1}$
0	$\underbrace{b^i_{-1/2}\vert 0\rangle_L}_{8_v} \times \underbrace{b^j_{-1/2}\vert 0\rangle_R}_{8_v}$	SO(8)	$+1$	$+1$	$\mathbf{1}+\mathbf{28}+\mathbf{35_v}$
	$(\mathbf{R},\mathbf{R})-\textbf{sector}$				
0	$\underbrace{\vert a\rangle_L}_{8_v} \times \underbrace{\vert b\rangle_R}_{8_v}$	SO(8)	$+1$	$+1$	$\mathbf{1}+\mathbf{28}+\mathbf{35_s}$
	$\underbrace{\vert \dot{a}\rangle_L}_{8_c} \times \underbrace{\vert \dot{b}\rangle_R}_{8_c}$		-1	-1	$\mathbf{1}+\mathbf{28}+\mathbf{35_c}$
	$\underbrace{\vert \dot{a}\rangle_L}_{8_c} \times \underbrace{\vert b\rangle_R}_{8_s}$		-1	$+1$	$\mathbf{8_v}+\mathbf{56_v}$
	$\underbrace{\vert a\rangle_L}_{8_s} \times \underbrace{\vert \dot{b}\rangle_R}_{8_c}$		$+1$	-1	$\mathbf{8_v}+\mathbf{56_v}$
	$(\mathbf{R},\mathbf{NS})-\textbf{sector}$				
0	$\underbrace{\vert a\rangle_L}_{8_s} \times \underbrace{b^i_{-1/2}\vert 0\rangle_R}_{8_v}$	SO(8)	$+1$	$+1$	$\mathbf{8}_c+\mathbf{56}_c$
	$\underbrace{\vert \dot{a}\rangle_L}_{8_c} \times \underbrace{b^i_{-1/2}\vert 0\rangle_R}_{8_v}$		-1	$+1$	$\mathbf{8}_s+\mathbf{56}_s$
	$(\mathbf{NS},\mathbf{R})-\textbf{sector}$				
0	$\underbrace{\bar{b}^i_{-1/2}\vert a\rangle_L}_{8_v} \times \underbrace{\vert a\rangle_R}_{8_s}$	SO(8)	$+1$	$+1$	$\mathbf{8}_c+\mathbf{56}_c$
	$\underbrace{\bar{b}^i_{-1/2}\vert a\rangle_L}_{8_v} \times \underbrace{\vert \dot{a}\rangle_R}_{8_c}$		$+1$	-1	$\mathbf{8}_s+\mathbf{56}_s$

Tab. 6. The lowest levels of the closed fermionic string spectrum.

Since a general state in the NS sector has the form

$$|\Phi\rangle = \alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} b^{j_1}_{-r_1} \cdots b^{j_M}_{-r_M} |0\rangle$$

we get

$$G|\Phi\rangle = (-1)^{M-1}|\Phi\rangle .$$

Thus, all states with M even are projected out, in particular, the tachyon. This removes the tachyon and all the states with half-integer $\alpha' M^2$. Indeed,

$$\alpha' M^2 = \sum_{i=1}^{r_M} r_i - \underbrace{\frac{1}{2}}_{a}$$

and for M even the sum of half-integers is always an integer $\alpha' M^2$ is half-integer number.

In the Ramond sector the operator G is defined as follows

$$G = (-1)^F = b_0^1 \cdots b_0^8 (-1)^{\sum_{n=1}^{\infty} b^i_{-n} b^i_n} .$$

The transversal zero modes b_0^i form the Clifford algebra $\{b_0^i, b_0^j\} = \delta^{ij}$. Thus, these operators can be represented by SO(8) γ-matrices Γ^i which have size 16 by 16. These matrices act on the 16-dimensional Majorana (i.e. real) spinor whose components can be thought to combine two Weyl projections, which are precisely $|a\rangle$ and $|\dot{a}\rangle$:

$$|\psi\rangle = \begin{pmatrix} |a\rangle_{\mathbf{8_s}} \\ |\dot{a}\rangle_{\mathbf{8_c}} \end{pmatrix} .$$

In the Majorana representation these matrices Γ^i can be taken in the block-diagonal form as

$$\Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ (\gamma^i)^t & 0 \end{pmatrix} .$$

The fact that Γ^i obey the standard Glifford algebra $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$ implies that 8×8 real matrices γ^i satisfy the following algebra

$$\gamma^i(\gamma^j)^t + \gamma^j(\gamma^i)^t = 2\delta^{ij} .$$

If we introduce the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then the matrices γ^i can be defined as

$$\begin{aligned} \gamma^1 &= -i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 , & \gamma^2 &= i\mathbb{I} \otimes \sigma_1 \otimes \sigma_2 , \\ \gamma^3 &= i\mathbb{I} \otimes \sigma_3 \otimes \sigma_2 , & \gamma^4 &= i\sigma_1 \otimes \sigma_2 \otimes \mathbb{I} , \\ \gamma^5 &= i\sigma_3 \otimes \sigma_2 \otimes \mathbb{I} , & \gamma^6 &= i\sigma_2 \otimes \mathbb{I} \otimes \sigma_1 , \\ \gamma^7 &= i\sigma_2 \otimes \mathbb{I} \otimes \sigma_3 , & \gamma^8 &= \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} . \end{aligned}$$

The matrix γ^i cab be understood as carrying the matrix indices a and $\dot{a}$: $\gamma^i_{a\dot{a}}$. The chiral and anti-chiral representations of SO(8) are constructed then with the help of matrices

$$\begin{aligned}\gamma_s^{ij} &= \frac{1}{2}\left[\gamma^i(\gamma^j)^t - \gamma^j(\gamma^i)^t\right],\\ \gamma_c^{ij} &= \frac{1}{2}\left[(\gamma^i)^t\gamma^j - (\gamma^j)^t\gamma^i\right],\end{aligned}$$

The operator $\Gamma^9 = b_0^1 \cdots b_0^8$ is the chirality operator (the analog of the γ^5-matrix in 4dim.) and it projects out one of the two Weyl components of the Ramond ground state $|\psi\rangle$. We see that G anti-commutes with any mode b_{-n}: $\{G, b^i_{-n}\} = 0$ and, therefore, the eigenvalues of G in the Ramond sector are ± 1, depending on their chirality, if we define $G|a\rangle = |a\rangle$ and $G|\dot{a}\rangle = -|\dot{a}\rangle$. Further, a general state in the R sector is

$$|\Phi\rangle_a = \alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} b^{j_1}_{-m_1} \cdots b^{j_M}_{-m_M} |a\rangle$$

or

$$|\Phi\rangle_{\dot{a}} = \alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} b^{j_1}_{-m_1} \cdots b^{j_M}_{-m_M} |\dot{a}\rangle$$

We therefore find that

$$\begin{aligned}G|\Phi\rangle_a &= (-1)^M (-1)^{\sum_i \delta_{m_i,0}} |\Phi\rangle_a\,,\\ G|\Phi\rangle_{\dot{a}} &= -(-1)^M (-1)^{\sum_i \delta_{m_i,0}} |\Phi\rangle_{\dot{a}}\,.\end{aligned}$$

The GSO projection consists in leaving the states which have either $G = 1$ or $G = -1$.

To construct the spectrum of the closed superstring we have to tensor left and right-moving states (such that the level matching constraint $M_L^2 = M_R^2$ is satisfied) and then impose the GSO projection. Here we have to distinguish four different sectors

$$\underbrace{(\mathrm{R},\mathrm{R})\,, \quad (\mathrm{NS},\mathrm{NS})}_{\text{space}-\text{time bosons}}\,, \quad \underbrace{(\mathrm{R},\mathrm{NS})\,, \quad (\mathrm{NS},\mathrm{R})}_{\text{space}-\text{time fermions}}$$

The GSO projection is imposed separately for the left- and right-moving modes. In the NS sector one keeps the states with

$$G = (-1)^F = +1\,, \qquad \bar{G} = (-1)^{\bar{F}} = +1\,.$$

In the Ramond sector there are essentially two possibilities which lead to supersymmetric and tachyonic-free spectrum. One of them is to take $G = \bar{G} = 1$. The massless spectrum is

$$\begin{aligned}&\text{Bosons}: &&\Big[(\mathbf{1}) + (\mathbf{28}) + (\mathbf{35})_{\mathbf{v}}\Big] + \Big[(\mathbf{1}) + (\mathbf{28}) + (\mathbf{35})_{\mathbf{s}}\Big]\\ &\text{Fermions}: &&\Big[(\mathbf{8})_{\mathbf{c}} + (\mathbf{56})_{\mathbf{c}}\Big] + \Big[(\mathbf{8})_{\mathbf{c}} + (\mathbf{56})_{\mathbf{c}}\Big]\,.\end{aligned}$$

In total there are 128 bosonic and 128 fermionic states. The GSO projection imposed in this way defines the so-called *Type IIB superstring* and its massless spectrum is that of Type IIB supergravity in ten dimensions. In particular, $\mathbf{35_v}$ are on-shell degrees of freedom of the graviton, two $\mathbf{28}$ are two anisymmetric tensor fields and $\mathbf{35}_s$ is a rank four antisymmetric self-dual tensor. In addition one has two real scalars. The fermionic degrees of freedom are two spin-3/2 gravitinos, $\mathbf{56}_c$, and two spin-1/2 fermions. The presence of two gravitinos indicates that the corresponding theory has $\mathcal{N} = 2$ supersymmetry. Since the gravitions and the fermions are of the same chirality this theory is *chiral*.

One can make another choice of the GSO projection by requiring that $G = -\bar{G} = 1$. This gives the following massless spectrum

$$\begin{aligned} &\text{Bosons :} \qquad \Big[(\mathbf{1}) + (\mathbf{28}) + (\mathbf{35})_\mathbf{v}\Big] + \Big[(\mathbf{8})_\mathbf{v} + (\mathbf{56})_\mathbf{v}\Big] \\ &\text{Fermions :} \quad \Big[(\mathbf{8})_\mathbf{c} + (\mathbf{56})_\mathbf{c}\Big] + \Big[(\mathbf{8})_\mathbf{s} + (\mathbf{56})_\mathbf{s}\Big] \,. \end{aligned}$$

There are on-shell degrees of freedom of the graviton $(\mathbf{35_v})$, antisymmetric rank three tensor $(\mathbf{56_v})$, an antisymmetric rank two tensor $(\mathbf{28})$, one vector $\mathbf{8_v}$) and one real scalar, which is called dilaton. The fermionic degrees of freedom comprise two spin-3/2 gravitinos and two spin-1/2 fermions. Gravitions and fermions are of opposite chirality. Thus, this theory has $\mathcal{N} = 2$ supersymmetry also but it is non-chiral. This string theory is called *Type IIA*, and the corresponding supergravity is Type IIA supergravity.

Appendices

A. Dynamical systems of classical mechanics

To motivate the basic notions of the theory of Hamiltonian dynamical systems consider a simple example.

Let a point particle with mass m move in a potential $U(q)$, where $q = (q^1, \dots q^n)$ is a vector of n-dimensional space. The motion of the particle is described by the Newton equations

$$m\ddot{q}^i = -\frac{\partial U}{\partial q^i}$$

Introduce the momentum $p = (p_1, \dots, p_n)$, where $p_i = m\dot{q}^i$ and introduce the energy which is also know as the *Hamiltonian* of the system

$$H = \frac{1}{2m}p^2 + U(q)\,.$$

Energy is a conserved quantity, i.e. it does not depend on time,

$$\frac{dH}{dt} = \frac{1}{m}p_i\dot{p}_i + \dot{q}^i\frac{\partial U}{\partial q^i} = \frac{1}{m}m^2\dot{q}_i\ddot{q}_i + \dot{q}^i\frac{\partial U}{\partial q^i} = 0$$

due to the Newton equations of motion.

Having the Hamiltonian the Newton equations can be rewritten in the form

$$\dot{q}^j = \frac{\partial H}{\partial p_j}\,, \qquad \dot{p}_j = -\frac{\partial H}{\partial q^j}\,.$$

These are the fundamental Hamiltonian equations of motion. Their importance lies in the fact that they are valid for arbitrary dependence of $H \equiv H(p,q)$ on the dynamical variables p and q.

The last two equations can be rewritten in terms of the single equation. Introduce two $2n$-dimensional vectors

$$x = \begin{pmatrix} p \\ q \end{pmatrix}\,, \qquad \nabla H = \begin{pmatrix} \frac{\partial H}{\partial p_j} \\ \frac{\partial H}{\partial q^j} \end{pmatrix}$$

and $2n \times 2n$ matrix J:

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

Then the Hamiltonian equations can be written in the form

$$\dot{x} = J \cdot \nabla H\,, \qquad \text{or} \qquad J \cdot \dot{x} = -\nabla H\,.$$

In this form the Hamiltonian equations were written for the first time by Lagrange in 1808.

Vector $x = (x^1, \dots, x^{2n})$ defines a state of a system in classical mechanics. The set of all these vectors form a *phase space* $M = \{x\}$ of the system which in the present case is just the $2n$-dimensional Euclidean space with the metric $(x, y) = \sum_{i=1}^{2n} x^i y^i$.

The matrix J serves to define the so-called *Poisson brackets* on the space $\mathcal{F}(M)$ of differentiable functions on M:

$$\{F, G\}(x) = (\nabla F, J\nabla G) = J^{ij}\partial_i F \partial_j G = \sum_{j=1}^{n} \Big(\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q^j} - \frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p_j} \Big) .$$

Problem. *Check that the Poisson bracket satisfies the following conditions*

$$\{F, G\} = -\{G, F\} ,$$
$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$$

for arbitrary functions F, G, H.

Thus, the Poisson bracket introduces on $\mathcal{F}(M)$ the structure of an infinite-dimensional Lie algebra. The bracket also satisfies the Leibnitz rule

$$\{F, GH\} = \{F, G\}H + G\{F, H\}$$

and, therefore, it is completely determined by its values on the basis elements x^i:

$$\{x^j, x^k\} = J^{jk}$$

which can be written as follows

$$\{q_i, q_j\} = 0 , \qquad \{p^i, p^j\} = 0 , \quad \{p^i, q_j\} = \delta^i_j .$$

The Hamiltonian equations can be now rephrased in the form

$$\dot{x}^j = \{H, x^j\} \qquad \Leftrightarrow \qquad \dot{x} = \{H, x\} = X_H .$$

A Hamiltonian system is characterized by a triple $(M, \{,\}, H)$: a phase space M, a Poisson structure $\{,\}$ and by a Hamiltonian function H. The vector field X_H is called the *Hamiltonian vector field* corresponding to the Hamiltonian H. For any function $F = F(p, q)$ on phase space, the evolution equations take the form

$$\frac{dF}{dt} = \{H, F\}$$

Again we conclude from here that the Hamiltonian H is a time-conserved quantity

$$\frac{dH}{dt} = \{H, H\} = 0\,.$$

Thus, the motion of the system takes place on the subvariety of phase space defined by $H = E$ constant.

In the case under consideration the matrix J is non-degenerate so that there exist the inverse

$$J^{-1} = -J$$

which defines a skew-symmetric bilinear form ω on phase space

$$\omega(x, y) = (x, J^{-1}y)\,.$$

In the coordinates we consider it can be written in the form

$$\omega = \sum_j dp_j \wedge dq^j\,.$$

This form is closed, i.e. $d\omega = 0$.

A non-degenerate closed two-form is called symplectic and a manifold endowed with such a form is called a symplectic manifold. Thus, the phase space we consider is the symplectic manifold.

Imagine we make a change of variables $y^j = f^j(x^k)$. Then

$$\dot{y}^j = \underbrace{\frac{\partial y^j}{\partial x^k}}_{A^j_k}\dot{x}^k = A^j_k J^{km}\nabla^x_m H = A^j_k J^{km}\frac{\partial y^p}{\partial x^m}\nabla^y_p H$$

or in the matrix form

$$\dot{y} = AJA^t \cdot \nabla_y H\,.$$

The new equations for y are Hamiltonian if and only if

$$AJA^t = J$$

and the new Hamiltonian is $\tilde{H}(y) = H(x(y))$.

Transformation of the phase space which satisfies the condition

$$AJA^t = J$$

is called canonical. In case A does not depend on x the set of all such matrices form a Lie group known as the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The term "symplectic group" was introduced by Herman Weyl. The geometry of the phase space which is invariant under the action of the symplectic group is called *symplectic geometry.* Symplectic (or canonical) transformations do not change the symplectic form ω:

$$\omega(Ax, Ay) = -(Ax, JAy) = -(x, A^t JAy) = -(x, Jy) = \omega(x, y)\,.$$

In the case we considered the phase space was Euclidean: $M = \mathbb{R}^{2n}$. This is not always so. The generic situation is that the phase space is a manifold. Consideration of systems with general phase spaces is very important for understanding the structure of the Hamiltonian dynamics.

Dynamical systems with symmetries

Let $g(t)$ will be a one-parametric group of transformations of the phase space: $x \to g(t)x$. This group does not need to coincide with the one generated by the Hamiltonian H. The action of this group is called Hamiltonian if there exists a function C such that

$$\frac{d}{dt} g(t)x|_{t=0} = J \cdot \nabla C\,.$$

The flow of any function F under the one-parameter group generated by C is then

$$\delta F \equiv \frac{d}{dt} F(g(t)x)|_{t=0} = \nabla F \cdot \frac{d}{dt} g(t)x|_{t=0} = (\nabla F, J \cdot \nabla C) = \{F, C\}$$

If we take $F = H$, we get

$$\delta H \equiv \{H, C\}\,.$$

Thus, if $\dot{C} = \{H, C\} = 0$, i.e. if C is an integral of motion, then it generates the symmetry transformations which leave the Hamiltonian invariant. Infinitezimally, the symmetry transformations are realized as

$$\delta F = \{F, C\}\,.$$

There could be several one-parametric groups which are one-parametric subgroups of a non-abelian Lie G, the latter being the symmetry of the Hamiltonian. Accordingly, there are the integrals of motion C_i, $i = 1, \ldots, \dim G$. Since C_i are integrals of motion, from the Jacobi identity

$$\{\{C_i, C_j\}, H\} + \{\{H, C_i\}, C_j\} + \{\{C_j, H\}, C_i\} = 0$$

we conclude that $\{\{C_i, C_j\}, H\} = 0$, i.e. $\{C_i, C_j\}$ is an integral of motion. If one can chose the functions C_i is such a way that they form the Lie algebra of G under the Poisson bracket:

$$\{C_i, C_j\} = f_{ij}^k C_k\,,$$

then the corresponding action of G on the phase space is called *Poisson.* Here f_{ij}^k are the structure constants of the Lie algebra of G.

B. OPE and conformal blocks

Every conformal (primary) operator enters in the Operator Product Expansion with its conformal family (descendants). Contribution of the entire conformal family associated to a primary operator O into the OPE is called the *conformal block*. Conformal blocks are completely fixed by conformal symmetry. As an example, let us show how to find the conformal block associated to the scalar primary operator O of conformal dimension Δ_O, which arises in the OPE of two scalar operators A and B of conformal dimensions Δ_A and Δ_B, respectively.

Assume the Operator Product Expansion

$$A(x)B(0) = \frac{1}{(x^2)^{\frac{1}{2}(\Delta_A+\Delta_B-\Delta_O)}} \sum_{k=0}^{\infty} \frac{1}{k!} \Lambda^k(x,\partial)O(0) + \dots \tag{B.1}$$

Here dots indicate the conformal families of other primary operators. We assume that all primary operators are orthogonal w.r.t. to the two-point functions

$$\langle O(0)O(y)\rangle = \frac{C_O}{(y^2)^{\Delta_O}}\,.$$

The three-point functions are fixed by conformal symmetry. In particular,

$$\langle A(x)B(0)O(y)\rangle = \frac{C_{ABO}}{(x^2)^{\frac{1}{2}(\Delta_A+\Delta_B-\Delta_O)}(y^2)^{\frac{1}{2}(\Delta_B+\Delta_O-\Delta_A)}((x-y)^2)^{\frac{1}{2}(\Delta_A+\Delta_O-\Delta_B)}}.$$

Plugging the OPE into the tree-point function we get

$$\langle A(x)B(0)O(y)\rangle = \frac{1}{(x^2)^{\frac{1}{2}(\Delta_A+\Delta_B-\Delta_O)}} \sum_{r=0}^{\infty} \frac{1}{r!} \Lambda^r(x,-\partial_y)\langle O(0)O(y)\rangle.$$

Thus, compatibility of the 3-point function with the OPE results into

$$\frac{C_{ABO}}{C_0}\frac{1}{(y^2)^{\frac{1}{2}(\Delta_B+\Delta_O-\Delta_A)}((y-x)^2)^{\frac{1}{2}(\Delta_A+\Delta_O-\Delta_B)}} = \sum_{k=0}^{\infty} \frac{1}{k!} \Lambda^k(x,-\partial_y)\frac{1}{(y^2)^{\Delta_O}}.$$

Taking into account that $e^{-x\partial_y}$ is the shift operator acting as

$$e^{-x\partial_y} f(y) = f(y-x)\,,$$

the last relation may be written as

$$\frac{C_{ABO}}{C_0}\sum_{k=0} \frac{1}{k!}\frac{1}{(y^2)^{\frac{1}{2}(\Delta_B+\Delta_O-\Delta_A)}}(-x\cdot\partial_y)^k\frac{1}{(y^2)^{\frac{1}{2}(\Delta_A+\Delta_O-\Delta_B)}} = \sum_{k=0}^{\infty} \frac{1}{k!}\Lambda^k(x,-\partial_y)\frac{1}{(y^2)^{\Delta_O}}.$$

It suggests to define

$$\Lambda^k(x,\partial_y) = \frac{C_{ABO}}{C_0} Q_k^{a,b}(x,\partial_y),$$

where the operator $Q_k(x,\partial_y)$ is defined by the relation

$$(y^2)^a(x\cdot\partial_y)^k(y^2)^b = Q_k(x,\partial_y)(y^2)^{a+b} \tag{B.2}$$

and a,b are given by

$$a=-\frac{1}{2}(\Delta_B+\Delta_O-\Delta_A), \qquad b=-\frac{1}{2}(\Delta_A+\Delta_O-\Delta_B).$$

The explicit form of the operator $Q_k(x,\partial_y)$ is found by the Fourier transform. Indeed, we have

$$(y^2)^a = 2^{2a+d}\pi^{d/2}\frac{\Gamma(a+\frac{d}{2})}{\Gamma(-a)}\frac{1}{(2\pi)^d}\int\frac{e^{-ip\cdot y}}{(p^2)^{a+d/2}} \tag{B.3}$$

and similar for the others. Substituting the Fourier transform of every function of y^2 one gets

$$\begin{aligned}&\frac{1}{\pi^{d/2}}\frac{\Gamma(a+\frac{d}{2})\Gamma(b+\frac{d}{2})}{\Gamma(a+b+\frac{d}{2})}\frac{\Gamma(-a-b)}{\Gamma(-a)\Gamma(-b)}\int dpdq\frac{e^{-ipy}}{(p^2)^{a+d/2}(q^2)^{b+d/2}}(-ixq)^k \\ &=\int dp\frac{e^{-ipy}}{(p^2)^{a+b+d/2}}Q_k^{a,b}(x,-ip),\end{aligned} \tag{B.4}$$

where $Q_k^{a,b}(x,-ip)$ is defined by

$$Q_k^{a,b}(x,\partial_y)e^{ipy} = Q_k^{a,b}(x,-ip)e^{ipy},$$

and we have used the change of variables $p\to p-q$.

From here one gets that $Q_k^{a,b}(x,-ip)$ is given by the integral

$$Q_k^{a,b}(x,-ip)=\frac{1}{\pi^{d/2}}\frac{\Gamma(a+\frac{d}{2})\Gamma(b+\frac{d}{2})}{\Gamma(a+b+\frac{d}{2})}\frac{\Gamma(-a-b)}{\Gamma(-a)\Gamma(-b)}(p^2)^{a+b+d/2}\int dq\frac{(-ixq)^k}{((p-q)^2)^{a+d/2}(q^2)^{b+d/2}}.$$

Thus, the problem is reduced to evaluation of the integral

$$I(\alpha_1,\alpha_2)=\int dq\frac{(-ixq)^k}{((p-q)^2)^{\alpha_1}(q^2)^{\alpha_2}}. \tag{B.5}$$

One has

$$\begin{aligned}I(\alpha_1,\alpha_2)&=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 dt t^{\alpha_1-1}(1-t)^{\alpha_2-1}(-i)^k\int dq\frac{(xq)^k}{[(q-tp)^2+t(1-t)p^2]^{\alpha_1+\alpha_2}}\\ &=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 dt t^{\alpha_1-1}(1-t)^{\alpha_2-1}(-i)^k\int dq\frac{(xq+txp)^k}{[q^2+t(1-t)p^2]^{\alpha_1+\alpha_2}}\\ &=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 dt t^{\alpha_1-1}(1-t)^{\alpha_2-1}(-i)^k\sum_{m=0}^{[k/2]}C_k^{2m}(txp)^{k-2m}\int dq\frac{(xq)^{2m}}{[q^2+t(1-t)p^2]^{\alpha_1+\alpha_2}},\end{aligned}$$

where $C_k^{2m} = \frac{k!}{(2m)!(k-2m)!}$. There appear only even powers $2m$ since for odd powers the internal integral is zero. Thus, we now need to evaluate the integral

$$L = \int dq \frac{(xq)^{2m}}{[q^2+h]^\gamma}.$$

Under the integral the symmetric product $q_{i_1}...q_{i_{2m}}$ may be substituted for

$$q_{i_1}...q_{i_{2m}} = \frac{(q^2)^m}{d(d+2)...(d+2(m-1))}(\delta_{i_1 i_2}...\delta_{i_{2m-1} i_{2m}} + \text{permutations}), \qquad \text{(B.6)}$$

where on the r.h.s. all non-trivial permutations are present. The total number of this permutations is $\frac{(2m)!}{2^m m!}$. Therefore, under the integral one may substitute

$$(xq)^{2m} = x^{i_1}...x^{i_{2m}} q_{i_1}...q_{i_{2m}} = \frac{(x^2)^m (q^2)^m}{d(d+2)...(d+2(m-1))} \frac{(2m)!}{2^m m!}.$$

Now we have

$$\int dq \frac{(q^2)^m}{[q^2+h]^\gamma} = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dq^2 \frac{(q^2)^{m+d/2-1}}{[q^2+h]^\gamma},$$

Performing the change of variables $t = \frac{h}{q^2+h}$, we arrive at

$$\begin{aligned}
\int dq \frac{(q^2)^m}{[q^2+h]^\gamma} &= \frac{\pi^{d/2}}{\Gamma(d/2)} h^{m+d/2-\gamma} \int_0^1 dt t^{\gamma-m-d/2-1}(1-t)^{m+d/2-1} \\
&= \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(\gamma-m-d/2)\Gamma(m+d/2)}{\Gamma(\gamma)} h^{m+d/2-\gamma}.
\end{aligned}$$

Thus, we evaluate the integral L

$$\begin{aligned}
L &= \int dq \frac{(xq)^{2m}}{[q^2+h]^\gamma} = \\
&= \frac{(x^2)^m}{d(d+2)...(d+2(m-1))} \frac{(2m)!}{2^m m!} \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(\gamma-m-d/2)\Gamma(m+d/2)}{\Gamma(\gamma)} h^{m+d/2-\gamma} \\
&= \pi^{d/2} \frac{(2m)!}{4^m m!} \frac{\Gamma(\gamma-m-d/2)}{\Gamma(\gamma)} (x^2)^m h^{m+d/2-\gamma}.
\end{aligned}$$

Finally, one gets

$$\begin{aligned}
I(\alpha_1,\alpha_2) &= \pi^{d/2} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \sum_{m=0}^{[k/2]} C_k^{2m} (-ixp)^{k-2m} (-1)^m \\
&\times \int_0^1 dt t^{\alpha_1+k-2m-1} (1-t)^{\alpha_2-1} \frac{(2m)!}{4^m m!} \frac{\Gamma(\alpha_1+\alpha_2-m-d/2)}{\Gamma(\alpha_1+a_2)} (x^2)^m (t(1-t)p^2)^{m+d/2-\alpha_1-\alpha_2},
\end{aligned}$$

where we have substituted $\gamma = \alpha_1 + \alpha_2$ and $h = t(1-t)p^2$. This is simplified to

$$I(\alpha_1, \alpha_2) = \frac{\pi^{d/2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \sum_{m=0}^{[k/2]} C_k^{2m} \frac{(2m)!}{4^m m!} (-ixp)^{k-2m} (-x^2)^m (p^2)^{m+d/2-\alpha_1-\alpha_2}$$
$$\times \int_0^1 dt t^{k-m+d/2-\alpha_2-1} (1-t)^{m+d/2-\alpha_1-1} \Gamma(\alpha_1+\alpha_2-m-d/2).$$

Thus, the final formula reads as

$$I(\alpha_1, \alpha_2) = \frac{\pi^{d/2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \sum_{m=0}^{[k/2]} C_k^{2m} \frac{(2m)!}{m!} (-ixp)^{k-2m} \left(-\frac{1}{4}x^2p^2\right)^m (p^2)^{d/2-\alpha_1-\alpha_2}$$
$$\times \frac{\Gamma(k-m+d/2-\alpha_2)\Gamma(m+d/2-\alpha_1)}{\Gamma(k+d-\alpha_1-\alpha_2)} \Gamma(\alpha_1+\alpha_2-m-d/2).$$

For $Q_k^{a,b}(x, -ip)$ one therefore finds

$$Q_k^{a,b}(x, -ip) = \frac{1}{\Gamma(a+b+d/2)} \frac{\Gamma(-a-b)}{\Gamma(-a)\Gamma(-b)} \sum_{m=0}^{[k/2]} C_k^{2m} \frac{(2m)!}{m!} (-ixp)^{k-2m} \left(-\frac{1}{4}x^2p^2\right)^m$$
$$\times \frac{\Gamma(k-m-a)\Gamma(m-b)}{\Gamma(k-a-b)} \Gamma(a+b+d/2-m).$$

By using the relation $(a)_{-m} = \frac{(-1)^m}{(1-a)_m}$ it can be further simplified to give

$$Q_k^{a,b}(x, \partial_y) = \frac{1}{(-a-b)_k} \sum_{m=0}^{[k/2]} \frac{k!(-a)_m(-b)_{k-m}}{m!(k-2m)!(-d/2-a-b+1)_m} (x \cdot \partial_y)^{k-2m} \left(-\frac{1}{4}x^2\Delta_y\right)^m.$$

Further summation gives the conformal block of the scalar field and it is performed by changing the order of the summation and the shift of the summation variable:

$$\sum_{k=0}^{\infty} \frac{1}{k!} Q_k^{a,b}(x, \partial_y) =$$
$$\sum_{m=0}^{\infty} \sum_{k=2m}^{\infty} \frac{1}{(-a-b)_k} \frac{(-a)_m(-b)_{k-m}}{m!(k-2m)!(-d/2-a-b+1)_m} (x \cdot \partial_y)^{k-2m} \left(-\frac{1}{4}x^2\Delta_y\right)^m$$
$$= \sum_{m=0}^{\infty} \frac{(-a)_m}{m!(-d/2-a-b+1)_m} \left(-\frac{1}{4}x^2\Delta_y\right)^m \sum_{k=0}^{\infty} \frac{(-b)_{k+m}}{(-a-b)_{k+2m}} \frac{(x\partial_y)^k}{k!}.$$

Since

$$\frac{(-b)_{k+m}}{(-a-b)_{k+2m}} = \frac{(-b)_m}{(-a-b)_{2m}} \frac{(-b+m)_k}{(-a-b+2m)_k}$$

we finally get

$$\sum_{k=0}^{\infty} \frac{1}{k!} Q_k^{a,b}(x, \partial_y) =$$
$$= \sum_{m=0}^{\infty} \frac{(-a)_m(-b)_m}{(-\mu-a-b+1)_m m!} \frac{1}{(-a-b)_{2m}} \, {}_1F_1(-b+m; -a-b+2m; x\partial_y) \left(-\frac{1}{4}x^2\Delta_y\right)^m,$$

where ${}_1F_1$ is a degenerate hypergeometric function:

$$ {}_1F_1(\alpha,\beta;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{x^k}{k!}. \tag{B.7}$$

C. Useful formulae

In discussion of the central charge of the Virasoro algebra we encounter the sum

$$S_n=\sum_{q=1}^{n}q^2\,.$$

Here we present a general method of computing this and similar sums. The method is based on considering the generating function

$$\wp(x)=\sum_{n=1}^{\infty}S_nx^n\,,$$

so that

$$S_n=\frac{1}{n!}\Big(\frac{\partial^n\wp}{\partial x^n}\Big)|_{x=0}\,.$$

For $x<1$ we have

$$\wp(x)=\sum_{n=1}^{\infty}S_nx^n=\sum_{n=1}^{\infty}\sum_{q=1}^{n}q^2x^n=\sum_{q=1}^{\infty}q^2\sum_{n=q}^{\infty}x^n=\sum_{q=1}^{\infty}\frac{q^2x^q}{1-x}\,.$$

We further notice that

$$\frac{\partial^2}{\partial x^2}\sum_{q=1}^{\infty}x^q=\sum_{q=2}^{\infty}q(q-1)x^{q-2}=\frac{1}{x^2}\sum_{q=2}^{\infty}q^2x^q-\frac{1}{x^2}\sum_{q=2}^{\infty}qx^q=\frac{1}{x^2}\sum_{q=1}^{\infty}q^2x^q-\frac{1}{x^2}\sum_{q=1}^{\infty}qx^q\,.$$

Thus,

$$\sum_{q=1}^{\infty}q^2x^q=x^2\Big(\frac{\partial^2}{\partial x^2}\frac{x}{1-x}+\frac{1}{x^2}\frac{\partial}{\partial x}\frac{x}{1-x}\Big)=\frac{x^2+x}{(1-x)^3}$$

and, therefore, we obtain the generating function

$$\wp(x)=\frac{x^2+x}{(1-x)^4}=\frac{1}{(1-x)^2}-\frac{3}{(1-x)^3}+\frac{2}{(1-x)^4}\,.$$

Finally we compute

$$\frac{1}{n!}\Big(\frac{\partial^n\wp}{\partial x^n}\Big)=\frac{1}{n!}\left(\frac{2\cdot3\cdots(n+1)}{(1-x)^{n+2}}-3\frac{3\cdot4\cdots(n+2)}{(1-x)^{n+3}}+2\frac{4\cdot5\cdots(n+3)}{(1-x)^{n+4}}\right)\,.$$

The last formula results into

$$S_n=(n+1)\Big(1-\frac{3}{2}(n+2)+\frac{1}{3}(n+2)(n+3)\Big)=\frac{1}{6}n(n+1)(2n+1)\,.$$

D. Riemann normal coordinates

Consider a Riemannian manifold $\mathcal{M}$ of dimension n with coordinates x^i, $1 = 1, \dots, m$. The geodesic equation is

$$\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0\,.$$

Let us consider two points p and q with coordinates x^i and $x^i + \delta x^i$ respectively. We assume that these points are closed so there is a unique geodesic connecting them.

A parameter t on the geodesic can be chosen proportional to the length of the arc connecting these two points (the natural parameter). The solution $x^i(t)$ can be chosen so that $x^i(0) \equiv x^i$ and $x^i(1) = x^i + \delta x^i$. The tangent vector to the geodesic at $t = 0$ is defined by $\xi^i = \dot{x}^i(0)$. Then equation for the geodesic can be solved perturbatively by assuming the following expansion

$$x^i(t) = x^i + c^i_1 t + c^i_2 t^2 + \cdots\,.$$

Plugging this into the geodesic equation one finds

$$x^i(t) = x^i + \xi^i t - \frac{1}{2}\Gamma^i_{j_1 j_2}(x)\xi^{j_1}\xi^{j_2}t^2 - \frac{1}{3!}\Gamma^i_{j_1 j_2 j_3}(x)\xi^{j_1}\xi^{j_2}\xi^{j_3}t^3 - \cdots\,,$$

where

$$\Gamma^i_{j_1 j_2 j_3} = \partial_{j_1}\Gamma^i_{j_2 j_3} - \Gamma^l_{j_1 j_2}\Gamma^i_{l j_3} - \Gamma^l_{j_1 j_2}\Gamma^i_{j_3 l}\,.$$

Here all the quantities are evaluated at x^i. At $t = 1$ we have $x^i(1) = x^i + \delta x^i$ so that

$$x^i + \delta x^i = x^i + \xi^i - \frac{1}{2}\Gamma^i_{j_1 j_2}(x)\xi^{j_1}\xi^{j_2} - \frac{1}{3!}\Gamma^i_{j_1 j_2 j_3}(x)\xi^{j_1}\xi^{j_2}\xi^{j_3} - \cdots$$

Hence, the point $x^i + \delta x^i$ is parameterized by the tangent vector ξ^i. We can therefore take ξ^i as the new coordinate system on our manifold. The coordinates ξ^i are called Riemann normal coordinates. They are used to define a map of an open neighbourhood U of the zero-vector $0 \in T_x\mathcal{M}$ to the manifold $\mathcal{M}$:

$$exp_x: \quad U \in T_x\mathcal{M} \to \mathcal{M}$$

which is well-defined diffeomorphism of U on its image. Now we are parameterizing the points of the curved manifold with tangent vectors. The image of the geodesic through a point x in the Riemann normal coordinates is just a straight line.

How metric will look in the new coordinate system? Upon changing the coordinates form x^i to ξ^i the metric transforms in the standard way

$$g'_{ij}(\xi) = \frac{\partial(x^k + \delta x^k)}{\partial \xi^i}\frac{\partial(x^l + \delta x^l)}{\partial \xi^j}g_{kl}(x + \delta x)$$

We have

$$\frac{\partial(x^k + \delta x^k)}{\partial \xi^i} = \delta^k_i - \Gamma^k_{ij}\xi^j + \cdots$$

and also

$$g_{kl}(x+\delta x) = g_{kl}\Big(x^i + \xi^i - \frac{1}{2}\Gamma^i_{j_1 j_2}\xi^{j_1}\xi^{j_2} + \cdots\Big) = g_{kl}(x) + \partial_k g_{ij}\xi^k + \cdots$$

Thus, at the linearized level we find that

$$\begin{aligned} g'_{ij}(\xi) &= \Big(\delta^k_i - \Gamma^k_{in}\xi^n\Big)\Big(\delta^l_j - \Gamma^l_{jm}\xi^m\Big)\Big(g_{kl}(\phi) + \partial_p g_{kl}\xi^p\Big) + \cdots \\ &= g_{ij}(x) + \underbrace{\Big(\partial_n g_{ij} - \Gamma^k_{in}g_{kj} - \Gamma^k_{jn}g_{ki}\Big)}_{D_n g_{ij}}\xi^n + \cdots \end{aligned}$$

We see that

$$g'_{ij}(\xi) = g_{ij}(x) + D_k g_{ij}(x)\xi^k + \cdots \tag{D.1}$$

It is important to realize that the coordinates ξ^k depend on x because they define the tangent vector to the manifold at the point x. Under reparametrizations of $x \to x'$ the object $\xi^k(x)$ transforms as a vector! *The expansion (D.1) is covariant under general coordinate transformations $x \to x'$ because it includes the tensorial quantities only.*

In the case of the minimal connection (connection compatible with the metric) we have $D_k g_{ij} = 0$. This, the expansion of the metric start from the quadratic order in ξ^i. Extending this calculation to higher orders in ξ one finds

$$g'_{ij}(\xi) = g_{ij}(x) - \frac{1}{3}R_{ik_1jk_2}\xi^{k_1}\xi^{k_2} - \frac{1}{3!}D_{k_1}R_{ik_2jk_3}\xi^{k_1}\xi^{k_2}\xi^{k_3} + \cdots$$

Also we recall the transformation property of the Christoffel connection

$$\Gamma^{k'}_{p'q'}(\xi) = \frac{\partial x^{k'}}{\partial x^k}\Big(\Gamma^k_{pq}\frac{\partial x^p}{\partial x^{p'}}\frac{\partial x^q}{\partial x'^{q'}} + \frac{\partial^2 x^k}{\partial x^{p'}\partial x^{q'}}\Big),$$

where $x^i \equiv \phi^i$ and $x'^i \equiv \phi^i + \pi^i$. Expanding the r.h.s. of the last equation in ξ^i it is easy to see that expansion does not contain the constant piece because the constant terms in the bracket cancel against each other, in other words, in the Riemann normal coordinates we have

$$\Gamma^{k'}_{p'q'}(\xi) = \mathcal{O}(\xi)$$

In the Riemann normal coordinate system die to the vanishing of Γ^k_{pq} at $\xi = 0$ the geodesic equation takes a form of the free motion $\ddot{\lambda} = 0$. This is coordinate system corresponds to the rest frame of a freely falling observer.

E. Exercises

Exercise 1. Show that the Hessian matrix associated to the Nambu-Goto Lagrangian has for each σ two zero eigenvalues corresponding to $\dot{X}^\mu$ and X'^μ.

Exercise 2. How reparametrization invariance can be used to bring the equation

$$\frac{\partial}{\partial\tau}\left(\frac{(\dot{X}X')X'^\mu - (X')^2\dot{X}^\mu}{\sqrt{(\dot{X}X')^2 - \dot{X}^2X'^2}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{(\dot{X}X')\dot{X}^\mu - (\dot{X})^2X'^\mu}{\sqrt{(\dot{X}X')^2 - \dot{X}^2X'^2}}\right) = 0\,.$$

to the simplest form?

Exercise 3. The Polyakov string. Prove that equations of motion for the fields X^μ imply conservation of the two-dimensional stress-energy tensor

$$\nabla^\mu T_{\mu\nu} = 0$$

Exercise 4. Show that $T_{\alpha\beta} = 0$ implies that the end points of the open string move with the speed of light.

Exercise 5. Non-relativistic string.

- Consider a string in equilibrium on the x-axis between $(0,0)$ and $(L,0)$ and suppose that the infinitesimal parts of the string can move only in the y-direction. Derive the Lagrangian with μ the mass density and T the string tension.
- Derive the equation of motion from this Lagrangian and keep explicit attention to the boundary conditions.
- Analyze the boundary terms. What must you impose in order to have a stationary action?
- Construct the momentum function P.
- Calculate the time derivative of the momentum (consider the boundary conditions). What do you conclude?
- Fourier transform the x-coordinate and solve the eom. Do this for both boundary conditions.

Exercise 6. Show that the Polyakov action is invariant under reparametrizations

$$\begin{aligned}\delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu \\ \delta h_{\alpha\beta} &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \\ \delta(\sqrt{h}) &= \partial_\alpha(\xi^\alpha \sqrt{h})\end{aligned}$$

Exercise 7. Show that the Weyl invariance implies the tracelessness of the stress-energy tensor $T_{\alpha\beta}$.

Exercise 8. Show that the Gauss-Bonnet term

$$\chi = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\ \sqrt{h}R$$

is topological, i.e. it vanishes under smooth variations of the world-sheet metric $h_{\alpha\beta}$. Take into account that in 2dim the Ricci tensor is proportional to Ricci scalar and also

$$\delta(\sqrt{h}R) \sim \Big(R_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}R\Big)\delta h^{\alpha\beta}\,.$$

Exercise 9. Let $S(q,t;q_0,t_0)$ be the action of the classical path between (q_0,t_0) and (q,t). Show that

$$\frac{\partial S}{\partial q} = p(t)\,,$$

where $p(t)$ is the conjugate momentum of q at time t. Show that

$$\frac{\partial S}{\partial t} = -H(q, \frac{\partial S}{\partial q})\,,$$

where H is the Hamiltonian. Suppose that $H(q,p) = \frac{p^2}{2m} + V(q)$ and define

$$\psi(q,t) = e^{\frac{i}{\hbar}S(q,t;q_0,t_0)}\,.$$

Show that the schrödinger equation approximately holds for ψ,

$$i\hbar\frac{\partial\psi}{\partial t} = H\Big(q, -i\hbar\frac{\partial}{\partial q}\Big)\psi + O(\hbar)\,.$$

This is of course related to Dirac's idea that the phase of the wave function is proportional to the classical action.

Exercise 10.

- Show that the constraints

$$C_1 = P_\mu P^\mu + T^2 X'_\mu X'^\mu\,, \qquad C_2 = P_\mu X'^\mu$$

have the following Poisson brackets

$$\begin{aligned}
\{C_1(\sigma), C_1(\sigma')\} &= 4T^2 \partial_\sigma C_2(\sigma)\delta(\sigma-\sigma') + 8T^2 C_2(\sigma)\partial_\sigma \delta(\sigma-\sigma')\,,\\
\{C_1(\sigma), C_2(\sigma')\} &= \partial_\sigma C_1(\sigma)\delta(\sigma-\sigma') + 2C_1(\sigma)\partial_\sigma \delta(\sigma-\sigma')\,,\\
\{C_2(\sigma), C_1(\sigma')\} &= \partial_\sigma C_1(\sigma)\delta(\sigma-\sigma') + 2C_1(\sigma)\partial_\sigma \delta(\sigma-\sigma')\,,\\
\{C_2(\sigma), C_2(\sigma')\} &= \partial_\sigma C_2(\sigma)\delta(\sigma-\sigma') + 2C_2(\sigma)\partial_\sigma \delta(\sigma-\sigma')\,.
\end{aligned}$$

- Define the linear combinations

$$\begin{aligned}
T_{++} &= \frac{1}{8T^2}(C_1 + 2TC_2) = \frac{1}{8T^2}(P_\mu + TX'_\mu)^2\,,\\
T_{--} &= \frac{1}{8T^2}(C_1 - 2TC_2) = \frac{1}{8T^2}(P_\mu - TX'_\mu)^2\,.
\end{aligned}$$

and show that their Poisson algebra is

$$\begin{aligned}
\{T_{++}(\sigma), T_{++}(\sigma')\} &= \frac{1}{2T}\Big(\partial_\sigma T_{++}(\sigma)\delta(\sigma-\sigma') + 2T_{++}(\sigma)\partial_\sigma \delta(\sigma-\sigma')\Big)\,,\\
\{T_{--}(\sigma), T_{--}(\sigma')\} &= -\frac{1}{2T}\Big(\partial_\sigma T_{--}(\sigma)\delta(\sigma-\sigma') + 2T_{--}(\sigma)\partial_\sigma \delta(\sigma-\sigma')\Big)\,,\\
\{T_{++}(\sigma), T_{--}(\sigma')\} &= 0\,.
\end{aligned}$$

Exercise 11. For the closed string case define

$$\begin{aligned}
L_m &= 2T\int_0^{2\pi} \mathrm{d}\sigma\ e^{im\sigma^-} T_{--}(\sigma,\tau)\\
\bar{L}_m &= 2T\int_0^{2\pi} \mathrm{d}\sigma\ e^{im\sigma^+} T_{++}(\sigma,\tau)\,.
\end{aligned}$$

Show that for any integer m the generators L_m and $\bar{L}_m$ are time-independent.

Exercise 12. Compute the Poisson brackets of the constraints L_m, $\bar{L}_m$. What kind of constraints they are, i.e. the first or the second class?

Exercise 13. It is known in curved space-time that we can transform the metric locally in the neighborhood of a point $x^\mu = 0$ to the following form $g_{\mu\nu}(x) = \eta_{\mu\nu} -$

$\frac{1}{3}R_{\mu\sigma\nu\rho}x^\sigma x^\rho$, this are Riemann normal coordinates. So that the deviation of g from the flat metric η is only second order in x (this is a coordinate system of an observer in free fall). Suppose we have a coordinate system x with metric expanded around 0,

$$g_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(0) + \partial_\sigma g_{\mu\nu}(0)\tilde{x}^\sigma + \frac{1}{2}\partial_\sigma\partial_\rho g_{\mu\nu}(0)\tilde{x}^\sigma\tilde{x}^\rho \,.$$

We want to transform this using a coordinate transformation $\tilde{x} \to x = x(\tilde{x})$ to Riemann normal coordinates. We expand the coordinate transformation to third order (zeroth order is zero)

$$x^\mu = \frac{\partial x^\mu}{\partial\tilde{x}^\nu}\tilde{x}^\nu + \frac{1}{2}\frac{\partial x^\mu}{\partial\tilde{x}^\nu\partial\tilde{x}^\sigma}\tilde{x}^\nu\tilde{x}^\sigma + \frac{1}{6}\frac{\partial x^\mu}{\partial\tilde{x}^\nu\partial\tilde{x}^\sigma\partial\tilde{x}^\rho}\tilde{x}^\nu\tilde{x}^\sigma\tilde{x}^\rho \,.$$

We can use the first term to bring g to η. Show that after we have done this there are $\frac{1}{2}d(d-1)$ remaining degrees of freedom left for $\frac{\partial x^\mu}{\partial\tilde{x}^\nu}$, where d is the dimension. Where do these remaining degrees of freedom correspond to?
Show that we can bring $\partial_\sigma g_{\mu\nu}$ to zero using the second term of the transformation, by counting degrees of freedom.
For arbitrary dimension there are not enough degrees of freedom to put $\partial_\sigma\partial_\rho g_{\mu\nu}$ to zero. Count the number of remaining degrees of freedom and show that it equals $\frac{1}{12}d^2(d^2-1)$. This is precisely the number of independent components of the Riemann tensor.

Exercise 14. Solve equation of motion $\Box X^\mu = 0$ for the case of open string (take into account the open string boundary conditions).

Exercise 15. Consider solution of the closed string equations of motion

$$X^\mu(\sigma,\tau) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$$

where

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{p^\mu}{4\pi T}(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}$$
$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{p^\mu}{4\pi T}(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq0}\frac{1}{n}\bar{\alpha}^\mu_n e^{-in(\tau+\sigma)}$$

Derive the Poisson algebra of the variables $(x^\mu, p^\mu, \alpha^\mu_n, \bar{\alpha}^\mu_n)$ by using the fundamental Poisson brackets of $X^\mu(\sigma), P^\mu(\sigma)$ variables.

Exercise 16. Prove that that (closed) string has an infinite set of integrals of motion: for any function f the quantities

$$L_f = \int_0^{2\pi} \mathrm{d}\sigma f(\sigma^-) T_{--}\,, \qquad \bar{L}_f = \int_0^{2\pi} \mathrm{d}\sigma\; f(\sigma^+) T_{++}\,,$$

are conserved.

Exercise 17. Obtain an expression for the Virasoro generators L_m and $\bar{L}_m$ in terms of string oscillators.

Exercise 18. Show that the operators $D_n = -ie^{in\theta}\frac{d}{d\theta}$ obey the Virasoro algebra.

Exercise 19. Consider an open string solution $0 \leq \sigma \leq \pi$:

$$\begin{aligned} X^0 &= t = L\tau \\ X^1 &= L\cos\sigma\cos\tau\,, \\ X^2 &= L\cos\sigma\sin\tau\,, \\ X^i &= 0, \quad i = 3,\ldots,d-1\,. \end{aligned}$$

- Show that this solution satisfies the Virasoro constraints and open string boundary conditions.
- Compute the mass of string.
- Compute the angular momentum $J \equiv J_{12}$ of string.
- Show that $J = \alpha' M^2$, where $\alpha' = \frac{1}{2\pi T}$ is called a slope of the Regge trajectory.

Exercise 20. By using the Poisson brackets between the generators of the Poincare group

$$\begin{aligned} \{P^\mu, P^\nu\} &= 0 \\ \{P^\mu, J^{\rho\sigma}\} &= \eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}P^\sigma \\ \{J^{\mu\nu}, J^{\rho\sigma}\} &= \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} \end{aligned}$$

show that for a certain choice of a function f the following expression

$$J^2 = \frac{1}{2}\left(J_{\alpha\beta}J^{\alpha\beta} + f(P^2)P_\alpha J^{\alpha\lambda}P^\beta J_{\beta\lambda}\right)\,, \qquad P^2 \equiv P_\mu P^\mu.$$

is an invariant of the Poincare group. Find the corresponding f.

Exercise 21.* Show that for any open classical string solution the following inequality holds

$$J \equiv \sqrt{J^2} \leq \alpha' M^2 \,,$$

(Hint. Make the computation in the static gauge $X^0 = t = \tau$ and use the Schwarz inequality $|(\bar{f}, g)|^2 \leq (\bar{f}, f)(\bar{g}, g)$ which holds for any two complex functions f and g.)

Exercise 22. Show that the generators of the angular momentum commute with the Virasoso generators

$$\{L_m, J^{\mu\nu}\} = 0 \,.$$

Exercise 23. By using the first-order formalism and imposing the light-cone gauge for the open string case

- Solve the Virasoro constraints,
- Find the open string light-cone Hamiltonian.

Exercise 24. Show that Virasoro constraints $T_{\alpha\beta} = 0$ which we have found in the conformal gauge can be directly solved in the light-cone gauge without using the first-order formalism. Show how the conformal gauge Hamiltonian turns into the light-cone Hamiltonian upon substitution the light-cone gauge conditions and the Virasoro constraints.

Exercise 25. Rewrite the Poisson algebra of the Poincaré generators in terms of light-cone coordinates $P^{\pm}, P^i$ and $J^{i\pm}, J^{ij}, J^{+-}$.

Exercise 26. For the closed string case in the light-cone gauge compute the Poisson brackets between the zero mode variables p^+, p^-, p^i, x^-, x^i . The full answer is given in the lecture notes.

Exercise 27. Proof the fulfilment of the Poisson algebra relations between the generators $\{J^{i+}, J^{j+}\}$ and $\{J^{i+}, J^{j-}\}$.

Exercise 28. The Virasoro algebra relation takes the form

$$[L_m, L_n] = (m - n)L_{m+n} + A(m)\delta_{m+n}$$

where $A(m)$ is a function of m. The aim of this exercise is to find the constraints on $A(m)$ that follow from the condition that the relations above define the Lie algebra.

- That does the antisymmetry requirement on a Lie algebra tells you about $A(m)$? What is $A(0)$?
- Consider the Jacobi identity for the generators L_m, L_n and L_k with $m+n+k = 0$. Show that

$$(m-n)A(k) + (n-k)A(m) + (k-m)A(n) = 0.$$

- Use the last equation to show that $A(m) = \alpha m$ and $A(m) = \beta m^3$, for constants α and β, yield consistent central extensions.
- Consider the last equation with $k = 1$. Show that $A(1)$ and $A(2)$ determine all $A(n)$

Exercise 29.

- Use the Virasoro algebra to show that if a state is annihilated by L_1 and L_2 then it is annihilated by all L_n with $n \geq 1$.
- Consider the Virasoro generators L_0, L_1 and L_{-1}. Write out the relevant commutators. Do these operators form a subalgebra of the Virasoro algebra? Is there a central term here?

Exercise 30. Consider open string. The fundamental commutation relation is

$$[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma'), \qquad \sigma \in [0,\pi].$$

- Show that consistency with the oscillator expansion implies that

$$\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \cos n\sigma \cos n\sigma'$$

- Why the fundamental commutation relation compatible with open string boundary conditions?
- Prove this representation for the δ-function by using the fact that any function $f(\sigma)$ with $\sigma \in [0,\pi]$ and vanishing derivative at $\sigma = 0, \pi$ can be expanded as

$$f(\sigma) = \sum_{n=0}^{\infty} A_n \cos n\sigma.$$

Exercise 31. Compute in the light-cone gauge the commutator of the orbital momenta

$$[\ell^{i-},\ell^{j-}]=?$$

Exercise 32. Show that in the quantum theory the eigenvalues of the covariant number operator

$$N=\sum_{n=1}^{\infty}\alpha^{\mu}_{-n}\alpha_{n,\mu}$$

are always nonnegative.

Exercise 33. Using the previous exercise show that for any fixed state all but a finite number of positively moded Virasoro operators automatically annihilate the state without imposing any conditions. More precisely, show that any state $|\Phi\rangle$ with the number eigenvalue $N\geq 0$ automatically satisfies

$$L_n|\Phi\rangle=0 \qquad \text{for} \qquad n>N.$$

Exercise 34. Compute the open string propagator

$$\langle X(\tau,\sigma)X(\tau',\sigma')\rangle=T\big(X(\tau,\sigma)X(\tau',\sigma')\big)-:X(\tau,\sigma)X(\tau',\sigma'):\ .$$

Exercise 35. Show that the vertex operator of the open string

$$V(k,\tau)=\underbrace{e^{\frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\frac{k_\mu\alpha^{\mu}_{-n}}{n}e^{in\tau}}}_{V_-}\ \underbrace{e^{ik_\mu(x^\mu+\frac{p^\mu}{\pi T}\tau)}}_{V_0}\ \underbrace{e^{-\frac{1}{\sqrt{\pi T}}\sum_{n=1}^{\infty}\frac{k_\mu\alpha^{\mu}_{n}}{n}e^{-in\tau}}}_{V_+}$$

is the conformal operator with the conformal dimension $\Delta=\alpha' k^2$.

Exercise 36. Compute the two-point correlation function of the tachyon vertex operators

$$\langle 0|V(k_2,\tau_2)V(k_1,\tau_1)|0\rangle$$

Exercise 37. Compute the three-point correlation function of the tachyon vertex operators

$$\langle 0|V(k_3,\tau_3)V(k_2,\tau_2)V(k_1,\tau_1)|0\rangle$$

Exercise 38. Compute the three-point correlation function of the tachyon vertex operators

$$\langle 0|V(k_4,\tau_4)V(k_3,\tau_3)V(k_2,\tau_2)V(k_1,\tau_1)|0\rangle$$

Exercise 40. Verify that $\partial_-^n X^\mu$ is a conformal operator. Find the corresponding conformal dimension. Find the singular terms of the OPE

$$T_{--}(\tau,\sigma)\,\partial_-^n X^\mu(\tau',\sigma')\,.$$

Exercise 41. Find the singular terms of the OPE

$$T_{--}(\tau,\sigma)\,V(k,\tau',\sigma')\,,$$

where $V(k,\tau',\sigma')$ is the vertex operator of tachyon.

Exercise 42. Find the singular terms of the OPE

$$T_{--}(\tau,\sigma)\,T_{--}(\tau',\sigma')\,.$$

Exercise 43. Suppose that there are $i=1,\dots,n$ grassman (anticommuting) variables η_i and $\bar\eta_i$. Let us define the integration rules as

$$\int \mathrm{d}\eta = 0\,,\qquad \int \mathrm{d}\eta\eta = 1$$

for any η_i and $\bar\eta_i$. Show that for any $n\times n$ matrix M the following formula is valid

$$\mathrm{det}M = \int \mathrm{d}\eta\mathrm{d}\bar\eta\; e^{\bar\eta M\eta}\,.$$

Here $\bar\eta M\eta \equiv \bar\eta_i M_{ij}\eta_j$.

Exercise 44. Show that the stress-tensor for the ghost fields implies the following expression for the ghost Virasoro generators of the closed string

$$L_m^{\mathrm{gh}} = \sum_{n=-\infty}^{\infty}(m-n) : b_{m+n}c_{-n} :$$

$$\bar L_m^{\mathrm{gh}} = \sum_{n=-\infty}^{\infty}(m-n) : \bar b_{m+n}\bar c_{-n} :$$

Exercise 45. Consider conformal transformations $z \to z' = f(z)$. By definition, primary fields are the fields which transform as tensors under conformal transformations

$$\phi(z,\bar z) \to \phi'(z,\bar z) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar z'}{\partial \bar z}\right)^{\bar h} \phi(z'(z), \bar z'(\bar z))$$

Find how a primary field transforms under infinitezimal conformal transformations $\phi \to \phi + \delta_{\xi,\bar\xi}\phi$, where $z' = z + \xi(z)$.

Exercise 46. Show that conformal fields of weight h have the following mode expansion

$$\phi(z) = \sum_{n \in \text{integer}} z^{-n-h} \phi_n \,.$$

Exercise 47. In the radial quantization products of fields is defined by putting them in the radial order. Using the radial order prescription show that the conformal transformation

$$\delta_\xi \phi(w) = [T_\xi, \phi(w)]$$

can be written in the form

$$\delta_\xi \phi(w) = \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \xi(z) T(z) \phi(w)$$

Here C_w is a small contour in a complex plane around point z.

Exercise 48. Using the previous exercise together with the Cauchy-Riemann formula

$$\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{f(z)}{(z-w)^n} = \frac{f^{(n-1)}(w)}{(n-1)!}$$

show that any conformal field must have the following R-ordered operator product with $T(z)$:

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \text{regular terms}$$

Exercise 49. Show that the following operator product of the stress tensor

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{regular terms}$$

corresponds to the following transformation law

$$\delta_\xi T(z) = \frac{c}{12}\partial^3 \xi(z) + 2\partial\xi(z) T(z) + \xi(z)\partial T(z)$$

Exercise 50. Under finite transformations $z \to f(z)$ the stress tensor transforms as

$$T(z) \to T'(z) = (\partial f)^2 T(f(z)) + \frac{c}{12} D(f)_z$$

Here

$$D(f)_z = \frac{\partial f(z)\partial^3 f(z) - \frac{3}{2}(\partial^2 f(z))^2}{(\partial f)^2}$$

is the Schwarzian derivative. Show that if $f(z) = \frac{az+b}{cz+d}$ then $D(f)_z = 0$.

Exercise 51. Consider a parallelogram on the complex plane determined by the following identification

$$z \equiv z + n\lambda_1 + m\lambda_2\,, \qquad n, m \in Z$$

Show that a general transformation

$$\lambda_1 \to d\lambda_1 + c\lambda_2\,, \qquad \lambda_2 \to b\lambda_1 + a\lambda_2$$

with the condition $ad - bc = 1$ preserves the area of the parallelogram.

Exercise 52. Show that the modular transformations

$$T: \quad \tau \to \tau + 1\,, \qquad S: \quad \tau \to -\frac{1}{\tau}$$

applied to the fundamental region of the modular group

$$\mathcal{M}_{g=1} = \Big\{ -\frac{1}{2} \leq \mathrm{Re}\tau \leq 0, |\tau|^2 \geq 1 \cup 0 < \mathrm{Re}\tau < \frac{1}{2}, |\tau|^2 > 1 \Big\}$$

generate the whole upper-half plane.

Exercise 53. Show that $\tau = i$ and $\tau = e^{\frac{2\pi i}{3}}$ are the fixed points of S and ST transformations respectively.

Exercise 54. Verify that the action

$$S = -\frac{1}{8\pi} \int \mathrm{d}^2\sigma \Big(\partial_\alpha X^\mu \partial^\alpha X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \Big)\,.$$

is invariant under supersymmetry transformations w

$$\begin{aligned} \delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu\,, \\ \delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^\alpha \partial_\alpha X^\mu \epsilon \\ \delta_\epsilon \bar{\psi}^\mu &= -\frac{1}{2}\bar{\epsilon}\rho^\alpha \partial_\alpha X^\mu \end{aligned}$$

provided the parameter ϵ satisfies the following equation

$$\rho^\beta \rho_\alpha \partial_\beta \epsilon = 0\,.$$

Exercise 55. If in the previous exercise the parameter ϵ is constant then we deal with global supersymmetry transformations. Derive the Noether current which corresponds to this global symmetry of the action.

Exercise 56.* Express the spin connection ω via vielbein e by using the condition of vanishing of the torsion

$$T^a_{\alpha\beta} = D_\alpha e^a_\beta - D_\beta e^a_\alpha = 0\,.$$

Exercise 57. Derive the on-shell algebra of supersymmetry transformations

$$[\delta_1, \delta_2] = ?\,.$$

Exercise 58. Using the Noether method derive the currents corresponding to the Poincaré symmetry for the fermionic string. Express the corresponding Noether charges via oscillators for both Neveuw-Schwarz and Ramond sectors.