

LMU, Winter Term 2019/20

Exercises on Open Quantum Systems

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Exercise 1 *Light-Matter interaction: Back to Numerics*

Consider a onedimensional array of M coupled electromagnetic cavities for which the dispersion relation reads $\omega(k) = A + B \cos(kh)$. In addition, assume two atoms located in different cavities which are coupled via the electromagnetic field modes such that the Hamiltonian of the full system is given by

$$\hat{H}/\hbar = \sum_{q=1}^M \omega_q \hat{a}_q^\dagger \hat{a}_q + \sum_{j=1}^2 \omega_j \hat{\sigma}_j^+ \hat{\sigma}_j^- + g \sum_{q=1}^M \sum_{j=1}^2 (\hat{a}_q \hat{\sigma}_j^+ e^{ik(q)r_j} + \hat{a}_q^\dagger \hat{\sigma}_j^- e^{-ik(q)r_j}) \quad (1)$$

with $r_1 = hn_1$ and $r_2 = hn_2$ are the two atomic locations, with n_1 and n_2 integer numbers that we can change, and the discretization:

$$k = k(q) = -\frac{\pi}{h} + (q-1)dk, \quad dk = \frac{2\pi}{hM}, \quad \text{and } 1 \leq q \leq M.$$

Assume that the atoms (the system (S)) can be treated as two-level systems with internal states given by $\{|0\rangle_j, |1\rangle_j\}$.

1. Treating the electromagnetic field as the environment, obtain the shape of the corresponding environment correlation function $C_E(t)$, and express it in terms of the spectral function, for an environment initial state given by the vacuum state: $\rho_E = |\text{vac}\rangle\langle\text{vac}|$, and for the thermal state $\rho_E^{\text{th}} = \exp(-\beta H_E)/Z$; Z being the partition function and $\beta = 1/\kappa T$ the inverse of the temperature. Plot $\text{Re}\{C_E(t)\}$ for $M = \{100, 500\}$, $h = 1$, $\beta = 1$, $A = 100$, $B = 50$, $g = \{0.1, 0.5\}$.
2. Let us consider the environment in a vacuum state, $\rho_E = |\text{vac}\rangle\langle\text{vac}|$, and the two atoms at different positions, with initial states such that one of them is in the excited state while the second is in its ground one. Compute the atom-atom correlation function $C_{12}(t) = \text{tr}\{\sigma_1^x \otimes \sigma_2^x \rho_s(t)\}$. Discuss the dependency of the atom-atom correlation function on the separation $|n_1 - n_2| = 0, 1, 2, 3$ between the atoms. (Fix $M = 100$, $\omega_1 = \omega_2 = 55$ and $\omega_1 = \omega_2 = 49$ and the rest of the parameters as in the previous item).
3. Given that the entire setup is a three-partite system, i.e., Atom+Atom+Environment, compute, and plot as a function of time, the entanglement entropy (von Neumann entropy) between the atoms for the same initial state as in item 2 and $\omega_1 = \omega_2 = 55$. **Compare and discuss the results for entanglement entropy with the one for the correlation function in item 2 when $|n_1 - n_2| = 1$.** What happens if we consider an entangled initial state between the atoms, i.e., an initial state $|\psi_{\text{at}}\rangle = (|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2)/\sqrt{2}$?
4. Discuss item 3. in the case that the energies of the atoms are in taken within the gap.

... One-excitation sector: a follow up:

5. For the same situation as in items 3, and 4, compute the quantum mutual information for atoms 1 and 2

$$I(1 : 2) := S(\rho^1) + S(\rho^2) - S(\rho^{12}), \quad (2)$$

where ρ^j ($j = 1, 2$) is the reduced density matrix of each atom, and ρ^{12} is the reduced density matrix of both of them, while $S(\rho) = -\text{Tr}\{\rho \log \rho\}$ is the von Neumann entropy. Analyze how the mutual information is affected by the environment in the two cases under analysis, when $\omega_1 = \omega_2 = 55$ and $\omega_1 = \omega_2 = 49$, and for $|n_1 - n_2| = 0, 1, 2, 3$. When is the mutual information more affected by the environment, when atoms are at closer or farther away?

6. In the same context, consider now imperfections in the setup of our system in such a way that atoms are located at positions $r_1 = hn_1 + \delta h_1$ and $r_2 = hn_2 + \delta h_2$, where δh_j is a location variation due to fabrication defects, $\delta h_j/h \ll 1$. Assume that $err = \{\delta h_1, \delta h_2\}$ is a set of random numbers and compute the atom-atom correlation function defined in the item 2 for $|n_1 - n_2| = 1, 2, 3$. (Keep in mind that an average over many realisations is needed, therefore use up to 100 sets err). Compare your results with the one in item 2, i.e. without fabrication errors.
7. *Fidelity measure:* Consider an initial state for the two atoms given by $|\psi_{\text{at}}\rangle = (|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2)/\sqrt{2}$, and the environment initial state defined as the vacuum. Compute the fidelity of the system initial state defined as

$$\mathcal{F}(\rho_s(0), \rho_s(t)) = (\text{tr} \left\{ \sqrt{\sqrt{\rho_s(0)}\rho_s(t)\sqrt{\rho_s(0)}} \right\})^2$$

for the same parameters as in the previous items. Can we, from these analysis, infer the loss of the entanglement of the initial state?

... Now consider only one atom:

8. In this exercise, we compare the results obtained with the Markov and secular approximations, leading to the Lindblad equation, with the exact ones that we have obtained by solving the problem in the one excitation sector. We consider as usual $M = \{100, 500\}$, $A = 100$, $B = 50$. In detail:
- a.) Derive a Lindblad master equation using the perturbative expansion, as well as the Markov and secular approximations as explained in class. Please specify and justify the steps and approaches used.
- b.) Compare the result from the master equation and the exact one (diagonalize the full Hamiltonian and compute the same as in the previous item) for $g = 0.05, 0.1, 0.5$.
9. Consider now the exact master equation of the problem, which we obtained in the Exercise Sheet 4, i.e.

$$\frac{d}{dt}\rho_s(t) = -\frac{i}{2}S(t)[\sigma^+\sigma^-, \rho_s(t)] + \gamma(t) \left(\sigma^-\rho_s(t)\sigma^+ - \frac{1}{2}\{\sigma^+\sigma^-, \rho_s(t)\} \right)$$

where

$$\gamma(t) = -2\text{Re} \left\{ \frac{\dot{G}(t)}{G(t)} \right\} \quad S(t) = -2\text{Im} \left\{ \frac{\dot{G}(t)}{G(t)} \right\}.$$

Use the function $\gamma(t)$ to study the non-markovianity of the dynamics. Represent the non-Markovianity with respect to values of $\omega_1 \in [45, 55]$, so that we can analyze this quantity in the band-gap edge that is located at $\omega_1 = 5$. Where is the non-Markovianity higher? Discuss the results!

... Beyond the one-excitation sector:

10. Consider again only one atom, but this time assume that the environment is initialized in a thermal state. This implies that we will no longer be able to describe the problem within the one excitation sector, since the initial thermal state will contain more than one excitation. We consider as usual $M = \{100, 500\}$, $A = 100$, $B = 50$.
 - Similar to Item 1, compute the environment correlation function and represent its real part for different values of the inverse temperature, $\beta = 0, 1, 0.5, 1, 10$. How does the decay time of the correlation function, also known as correlation time, change with the temperature?
 - Obtain the Lindblad master equation considering such an initial thermal state for the environment, and integrate it numerically.
 - What is the expression of the decay rates of the Lindblad master equation, as given by the Fermi Golden rule?
 - The decay rates depend on the spectral density of the environment. Compute this function for the present system. Where is the spectral density larger, nearby the band-gap edge or in the middle of the band?
 - The steady state of the Lindblad master equation is known to be the thermal state $\rho_S^{\text{th}} = e^{-\beta H_S} / Z_S$, i.e. the system is expected to thermalize under such equation. Compute and represent the trace distance between the time evolving $\rho_S(t)$ and $\rho_S^{\text{th}} = e^{-\beta H_S} / Z_S$ for different values of $\omega_1 = 45, 49, 50, 51, 55$. Chose a temperature $\beta = 1$. Does the system thermalize in all cases? Assume that the initial state of the system is the excited state $\rho_S(0) = |1\rangle\langle 1|$. What happens when the system is initialized in a thermal state?
 - For $\rho_S(0) = |1\rangle\langle 1|$, represent the thermalization time (defined at the time in which the trace distance between $\rho_S(t)$ and ρ_S^{th} is smaller than 10^{-3}) for: (a) $\omega_1 = 55$ and $\beta \in [0.1, 1]$ with a discretization $\Delta\beta = 0.125$., and (b) $\beta = 1$ and $\omega_1 \in [50, 55]$ with a discretization $\Delta\omega = 0.25$. How does the thermalization time behave with the temperature? what is the reason? When is the thermalization time faster, near the band-gap edge or farther away from it, and why?