LMU, Winter Term 2019/20

Exercises on Open Quantum Systems
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Exercise 1 *Dephasing Discrete Map*

Consider a qubit system A described in the computational basis $\{|0\rangle_A, |1\rangle_A\}$ and an environment represented by the set of states: $\{|0\rangle_E, |1\rangle_E, |2\rangle_E\}$. Let $\mathbf{T}$ be a discrete quantum map defined as:

$$\begin{aligned} |0\rangle_A \otimes |0\rangle_E &\mapsto \sqrt{1-p}|0\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |1\rangle_E \\ |1\rangle_A \otimes |0\rangle_E &\mapsto \sqrt{1-p}|1\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |2\rangle_E\,, \end{aligned}$$

where no qubit flipping occurs after a elapsed time Δt.

a) Show that the respective *Kraus* operators take the form

$$E_0 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and that the reduced density operator can be written as

$$\rho_s = \phi[\rho_s(0)] = \left(1 - \frac{1}{2}p\right)\rho_s(0) + \frac{1}{2}p\,\sigma_3\rho_s(0)\sigma_3 \tag{1}$$

where σ_i's are the Pauli matrices. Check that an alternative way of representing the previous equation is:

$$\phi\left[\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\right] = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}$$

where ρ_{ij} are the matrix elements of the initial density operator. Why is this Map referred to as *Dephasing Map*?

b) Consider now that we apply the map n times until a time $t = n\Delta t$ (Δt time interval between "kicks"). Define Γ as the scattering event per unit time and show that after n kicks, with $n \to \infty$. Show that for a system initial state $\alpha|0\rangle_A + \beta|1\rangle_A$, the density operator evolves to

$$\rho_s(t \to \infty) = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

c) Find the *Bloch* vector representation of the density operator. Discuss geometrically the effect of continous dephasing and why this map is not completely positive.

d) **Extra Map**: Consider the following Map:

$$\begin{aligned} |0\rangle_A \otimes |0\rangle_E &\mapsto \sqrt{1-p}|0\rangle_A \otimes |0\rangle_E \\ |1\rangle_A \otimes |0\rangle_E &\mapsto \sqrt{1-p}|1\rangle_A \otimes |0\rangle_E + \sqrt{p}|0\rangle_A \otimes |1\rangle_E\,, \end{aligned}$$

and show that

$$\rho_s(t) = \begin{pmatrix} \rho_{00} + (1-e^{-\Gamma t})\rho_{11} & e^{-\frac{1}{2}\Gamma t}\rho_{01} \\ e^{-\frac{1}{2}\Gamma t}\rho_{10} & e^{-\Gamma t}\rho_{11}\,. \end{pmatrix}$$

Discuss the difference of this map and the dephasing map.

Exercise 2 *Non-Markovianity*

In a previous exercise we derive the map

$$\phi_t[\rho_0] = \begin{pmatrix} 1 & 0 & 0 & 1-|G(t)|^2 \\ 0 & G(t) & 0 & 0 \\ 0 & 0 & G^*(t) & 0 \\ 0 & 0 & 0 & |G(t)|^2 \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix}$$

keeping in mind that $G(t)$ is a solution of

$$\frac{d}{dt}G(t) = -\int_0^t C(t-\tau)G(\tau)d\tau\,.$$

Let us assume that the qubit is coupled to a single mode such that $C(t) = g_k^2 e^{-i(\omega_k-\omega_s)t}$

a) Prove that ϕ_t is invertible

b) The exact master equation for the reduced density operator can be obtain using the map ϕ_t as

$$\frac{d}{dt}\rho_s(t) = \dot{\phi}_t[\phi_t^{-1}[\rho_s(t)]]\,.$$

Show that

$$\frac{d}{dt}\rho_s(t) = -\frac{i}{2}S(t)[\sigma^+\sigma^-, \rho_s(t)] + \gamma(t)\left(\sigma^-\rho_s(t)\sigma^+ - \frac{1}{2}\{\sigma^+\sigma^-, \rho_s(t)\}\right)$$

where

$$\gamma(t) = -2\mathrm{Re}\left\{\frac{\dot{G(t)}}{G(t)}\right\} \qquad S(t) = -2\mathrm{Im}\left\{\frac{\dot{G(t)}}{G(t)}\right\}.$$

c) Integrate the master equation of the previous item and plot the expectation values $\mathrm{Re}(\langle\sigma^\pm\rangle)$ as a function of the time t. Consider the qubit initial state to be $\rho_s(0) = |1\rangle\langle 1|$.

d) Find the solution $G(t)$ for $g_k = 1$, $\omega_s = 49$, $\omega_k = 49$ and $\omega_k = 0.1$ (Use the previous results from Exercise sheet 3). Compute the non-Markovianity measure BLP (Breuer, Laine and Piilo) given the system initial states

$$\rho_s^1(0) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad \rho_s^2(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$