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Exercises on Open Quantum Systems
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Exercise 1 *General Lindblad form of a time local master equation.*

We have seen that the reduced density operator of an open quantum system can in general be described in terms of a universal dynamical map, ϕ_t. Importantly, we have to prepare the system and the environment in an initially decorrelated state of the form, such that the density operator for the total system (including the open system and its environment) is

$$\rho(0) = \rho_s(0) \otimes \rho_B, \tag{1}$$

where $\rho_s(0) = \mathrm{Tr}_B\{\rho(0)\}$ is the reduced density operator of the system and ρ_B is the environment density operator. This operator has a spectral decomposition

$$\rho_B = \sum_q \lambda_q |E_q\rangle\langle E_q|, \tag{2}$$

in terms of its eigenvectors $|E_q\rangle$, and with $\lambda_q \geq 0$. Under these conditions, the reduced density matrix at a time t can then be written in terms of a Kraus decomposition,

$$\begin{aligned}
\rho_s(t) &= \mathrm{Tr}_B\{\mathcal{U}^{-1}(t)\rho_s(0) \otimes \rho_B(0)\mathcal{U}(t)\} = \sum_{q'}\sum_q \lambda_q \langle E_{q'}|\mathcal{U}^{-1}(t)|E_q\rangle \rho_s(0) \langle E_q|\mathcal{U}^{-1}(t)|E_{q'}\rangle \\
&= \sum_l E_l(t)\rho_s(0)E_l^\dagger(t) = \phi_t[\rho_s(0)],
\end{aligned} \tag{3}$$

where $E_l = \sqrt{\lambda_q}\langle E_{q'}|\mathcal{U}^{-1}(t)|E_q\rangle$ ($l \equiv \{q, q'\}$) are Kraus operators fulfilling the property

$$\sum_l E_l^\dagger E_l = \mathbb{1}_S. \tag{4}$$

One of the most desirable properties to describe the dynamics of an open system is that, provided that their map ϕ_t is invertible and differentiable, the reduced density matrix can be shown to evolve according to a time-local master equation of the form

$$\begin{aligned}
\frac{d\rho_s(t)}{dt} &= -i[\hat{H}_S(t), \rho_s(t)] + \sum_{k=1}^{d^2-1} \gamma_k(t) \Big(L_k(t)\rho_s(t)L_k^\dagger(t) \\
&- \frac{1}{2}\{L_k^\dagger(t)L_k(t), \rho_s(t)\} \Big).
\end{aligned} \tag{5}$$

Question: By considering the general form (3), probe that this is the case. Guide:

1. Formally write the derivative of (3) as a time local master equation of the form

$$\begin{aligned}
\frac{d\rho_s(t)}{dt} &= \dot{\phi}_t \phi_t^{-1} \rho_s(t) \\
&= \sum_k A_k(t)\rho_s(t)B_k^\dagger(t) = \Lambda_t[\rho_s(t)],
\end{aligned} \tag{6}$$

and specify the form of A_k and B_k in term of the Kraus operators E_l and considering that the inverse of the map can be written as

$$\phi_t^{-1}[\rho_s] = \sum_m F_m(t)\rho_s Q_m(t). \tag{7}$$

Note that the index k in (6) combines the indexes l and m.

2. Consider a complete set of $N = d^2$ basis operators $\{G_i\,;\, i = 1, \cdots, N-1\}$ for the open system, where $N = d^2$ and d is its dimension. These operators have the properties

$$\begin{aligned} G_0 &= \mathbb{1}_S/\sqrt{d}, \\ G_i &= G_i^\dagger, \\ \mathrm{Tr}\{G_i\} &= (d/\sqrt{d})\delta_{i0}, \\ \mathrm{Tr}\{G_iG_j\} &= \delta_{ij} \end{aligned} \tag{8}$$

But most importantly, because they are a completer basis of operators of the system, we have that

$$\sum_m G_m A G_m = \mathbb{1}\,\mathrm{Tr}_S\{A\}, \tag{9}$$

where A and $\mathbb{1}$ are, respectively, any arbitrary operator and the unit operator in the open system Hilbert space. Since the basis is complete, any operator of the system, A can be written as

$$A = \sum_j G_j \mathrm{Tr}_S\{G_j A\}. \tag{10}$$

Write the equation (6), in particular $\frac{d\rho_s(t)}{dt} = \sum_k A_k(t)\rho_s(t)B_k^\dagger(t)$ in terms of such a basis, such that

$$\frac{d\rho_s(t)}{dt} = \sum_{ij=0}^{N-1} c_{ij} G_i \rho_s(t) G_j. \tag{11}$$

To this aim, take into account that $A_k(t)$ and $B_k(t)$ are operators in the open system Hilbert space. Specify formally c_{ij}. Considering that ρ_s and therefore $\dot{\rho}_s$ are Hermitian, what property have the elements c_{ij}?

3. Separate from the equation the $i = 0$ and the $j = 0$ terms. Reformulate the equation by expressing such terms as a function of the operator

$$C = \frac{c_{00}}{d} + \sum_i \frac{c_{i0}}{\sqrt{d}} G_i. \tag{12}$$

Considering the trace preservation, i.e. $\mathrm{Tr}_s\{\dot{\rho}_s(t)\} = 0$, write $C + C^\dagger$ in terms of c_{ij}, G_i and G_j (for $i, j = 1, \cdots, N-1$).

4. Define a new Hamiltonian operator $H = \frac{i}{2}(C - C^\dagger)$, and show that one finally obtains

$$\frac{d\rho_s}{dt} = -i[H, \rho_s] + \sum_{i,j=1}^{N-1} c_{ij}\left(G_i\rho_s G_j - \frac{1}{2}\{G_jG_i, \rho_s\}\right). \tag{13}$$

5. By using the property (9), rewrite the coefficients c_{ij} as

$$c_{ij}(t) = \sum_k \mathrm{Tr}_S\{G_i A_k\}\mathrm{Tr}_S\{G_j B_k^\dagger\} = \sum_{m=0}^{N-1} \mathrm{Tr}_S\{G_m G_i \Lambda_t[G_m] G_j\}, \tag{14}$$

   where $\Lambda_t[G_m] = \sum_k A_k(t) G_m B_k^\dagger(t)$.

6. We now define the decoherence matrix $\mathbf{d}$ with elements $d_{ij}(t) = c_{ij}(t)$ for $i, j = 1, N-1$, and consider that it is Hermitian. Therefore, it can be written in a diagonal form

$$d_{ij} = \sum_k U_{ik} \gamma_k U_{jk}^*, \tag{15}$$

   where the eigenvalues γ_k are real, but not necessarily positive at all times, and the U_{ik} are elements of a unitary $(N-1)\times(N-1)$ matrices formed by the corresponding eigenvectors of $\mathbf{d}$. Apply such unitary transformation to Eq. (13), to find indeed Eq. (5). What is the form of L_k in terms of the basis operators G_i?