LMU, Winter Term 2019/20

Exercises on Open Quantum Systems
Inés de Vega (Teacher)
Carlos Parra (Tutor)

Exercise 1 *Density Operator: Properties.*

Suppose that we have a system described by the density operator $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $\sum_i p_i = 1$. Show that

a) $\hat{\rho}^2 \neq \hat{\rho}$. What is the condition for $\hat{\rho}^2 = \rho$?

b) $\hat{\rho}^\dagger = \hat{\rho}$

c) $\mathrm{tr}\{\hat{\rho}\} = 1$

d) $\mathrm{tr}\{\hat{\rho}^2\} < 1$

e) $\mathrm{tr}\{(\hat{\rho}(t))^2\} = \mathrm{tr}\{(\hat{\rho}(0))^2\}$

Define the density matrix as

$$\hat{\rho} = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a) Is $\hat{\rho}$ a permissible density matrix? Give your reasoning.

b) Assume that it is permissible. Does it describe a pure or mixed state? Give your reasoning.

c) Compute expectation value of the following operator

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Exercise 2 *Pauli Matrices: Properties.*

The following matrices are called Pauli Matrices[1]:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If we use the eigenvectors of Z, $|0\rangle = (1,0)^T$ and $|1\rangle = (0,1)^T$, these can be also written using the brak-ket notation as follows (verify):

$$\begin{aligned} X &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ Y &= -i\left(|0\rangle\langle 1| - |1\rangle\langle 0|\right) \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned}$$

[1] For these matrices the most common notation is: $\sigma_1 = \sigma_x = X$, $\sigma_2 = \sigma_y = Y$, and $\sigma_3 = \sigma_z = Z$.

1) Show that the Pauli matrices are all Hermitian, unitary, traceless and they square to the identity

2) Compute the commutator $[A, B] = AB - BA$ and the anticommutator $\{A, B\} = AB + BA$ of the Pauli matrices, and verify the these can be casted in the compact way:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \text{and} \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\,,$$

where ϵ_{ijk} is the *Levi-Chivita* symbol, while δ_{ij} represents the Kronecker delta function, and we use the label $\sigma_0 = \mathbf{1}$, $\sigma_1 = X$, $\sigma_2 = Y$ and $\sigma_3 = Z$.

3) Find the eigenstates, eigenvalues and diagonal representation of X and Y.

4) For a state $|\psi\rangle$ write the possible states it can collapse to after the measurement of Y observable, and finde the corresponding probabilities when it can happen.

5) Write all tensor products of Pauli matrices as 4×4 matrices.

6) Show the identity

$$\exp(i\alpha\hat{n} \cdot \vec{\sigma}) = \mathbf{1}\cos(\alpha) + i(\hat{n} \cdot \vec{\sigma})\sin(\alpha)$$

where $\hat{n}$ is an unitary vector, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices and α is a scalar.

Exercise 3 *Density Operator: Reconstruction.*

Let assume an experiment from which the expectation values of the spin 1/2 can be measured, that is, we have the information:

$$\langle S_i \rangle = \mathrm{tr}\{\hat{S}_i \hat{\rho}\}\,, \quad \text{with} \quad \hat{S}_i = \frac{\hbar}{2}\sigma_i \quad (i = 0, 1, 2, 3)$$

1) Verify that the knowledge of $\langle S_i \rangle$ is enough for the reconstruction of the density operator $\hat{\rho}$

2) Use the reconstructed density operator $\hat{\rho}$ and compute the *von-Neumann* entropy $S[\rho]$ and its purity $\gamma[\rho]$ which are defined as

$$S[\hat{\rho}] = -\mathrm{tr}\{\hat{\rho}\ln\hat{\rho}\} \quad \text{and} \quad \gamma[\hat{\rho}] = \mathrm{tr}\{\hat{\rho}^2\}\,.$$

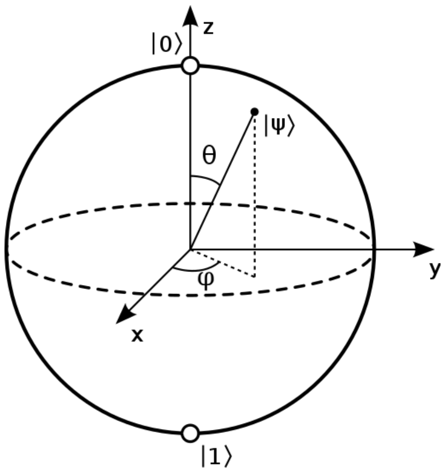
Discuss the nature of the state represented by $\hat{\rho}$.

Exercise 4 *Bloch State*

Consider a two dimensional Hilbert space spanned by the states $|0\rangle$ and $|1\rangle$.

1. Show that any pure state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ can be express as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)\,, \quad 0 \le \theta \le \pi\,, \ \text{and} \ 0 \le \phi \le 2\pi\,.$$



This state is called Bloch state and it can be graphically represented by a point on the surface of a sphere of unit radius.

2. For a pure state $|\psi\rangle$ the respective density operator representation can be computed as $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$. Show that this density operator can be rewritten in terms of the Pauli matrices as

$$\hat{\rho}_\psi = \frac{1}{2}(\mathbf{1} + \mathbf{a} \cdot \sigma) = \frac{1}{2}(\mathbf{1} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)\,.$$

   What are the components of the Bloch vector $\mathbf{a} = (a_1, a_2, a_3)$, and its norm $|\mathbf{a}| =?$

3. Show that in general, in a two dimensional Hilbert space, for any density operator the condition $|\mathbf{a}| \leq 1$ has to be fulfilled. * $|\mathbf{a}| < 1$ corresponds to states that "lives" inside the sphere and cannot, in general, be represented by a coherent linear combination of $|0\rangle$ and $|1\rangle$. Those states are reffered to as "mixed states".

4. Show that any two dimensional density operator can be written in terms of the components of the Bloch vector as

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} 1 + a_3 & a_1 + ia_2 \\ a_1 - ia_2 & 1 - a_3 \end{pmatrix}\,.$$

Why can the coefficient a_3 be though as a measure for quantum "population inversion"? Prove that if $a_1 = a_2 = 0$, then, the state is mixed.