Ludwig-Maximilians-Universität München
Prof. Dr. Ivo Sachs, Prof. Dr. Thomas Vogel

Lecture Winter term 2019/2020

Differentiable manifolds

Please note: These notes summarize the content of the lecture, many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. Changes to this script are made without any further notice at unpredictable times. If you find any typos or errors, please let us know.

1. Lecture on Oct. 15. – **Linear Forms**

- **Definition:** If V is a vector space over some field $\mathbb{K}$ then a *linear form* is a linear function $\phi : V \to \mathbb{K}$. The set of linear forms on V form a vector space denoted by V^* with $\dim(V^*) = \dim(V)$.
- If $\{e_i\}$, $i = 1, \cdots n = \dim(V)$ is a basis of V then $\{e^{*i}\}$, $i = 1, \cdots n$ is the *dual basis* if $e^{*i}(e_j) = \delta^i_j$.
- **Example 1:** For p, a point in $\mathbb{R}^3$ the set of vectors $q - p$ for $q \in \mathbb{R}^3$ forms a vector space denoted (somewhat redundantly) by $T_p(\mathbb{R}^3)$ over $\mathbb{R}$ with canonical basis, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_1 = (0, 0, 1)$. The dual basis is given by the **coordinate differentials**, $\{e^{*i} = \mathrm{dx^i}\}$ with
$$\mathrm{dx^i(e_j)} = \frac{\partial}{\partial \mathrm{x^j}}\mathrm{x^i} = \delta^{\mathrm{i}}_{\mathrm{j}}$$
- **Example 2:** $V = \{M \in M_2(\mathbb{C}) \,|\, M^\dagger = M\,,\ \mathrm{Tr}(M) = 0\}$ is a 3-dimensional vector space over $\mathbb{R}$ with basis $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and dual basis
$$e^{*i} = \frac{1}{2}\mathrm{Tr}(e_i\ \cdot).$$
- **Example 3:** In Quantum Mechanics $V = L^2(\mathbb{R}^3)$ is an ∞-dimensional vector space over $\mathbb{C}$ with basis $\{e_n\} = \{\frac{f_n}{\sqrt{||f_n||}}\} \equiv |e_n >$ (the **ket**'s), where $\{f_n\}$ is the set of eigenfunctions of the Laplacian Δ on $\mathbb{R}^3$. The dual basis is given by the (**bra**'s)
$$< e_n| \equiv \int d^3x\,(\bar{e}_n\ \cdot).$$

2. Lecture on Oct. 15. – **Multilinear Forms** [Ca, Na]

- **Definition** A k-linear form (or simply k-form) is an alternating k-linear function
$$\begin{aligned}\phi \quad : \quad & \underbrace{V \times V \times \ldots \times V}_{k \text{ times}} \to \mathbb{K} \\ & (x_1, \cdots, x_k) \mapsto \phi(x_1, \cdots, x_k) = \mathrm{sgn}(\sigma)\phi(x_{\sigma(1)}, \cdots, x_{\sigma(k)})\end{aligned}$$
for $\sigma \in \Sigma_k$, the symmetric group. In particular, for $\phi^i \in V^*$, $i = 1, \cdots k \le n$, $\phi^1 \wedge \phi^2 \cdots \wedge \phi^k \in \wedge^k V^*$ defined through
$$\phi^1 \wedge \phi^2 \cdots \wedge \phi^k(x_1, \cdots, x_k) = \det(\phi^i(x_j))$$

$i, j = 1, \cdots k$ is a k-form.

- The above construction gives rises to a basis in $\wedge^k V^*$: **Proposition:** any k-linear exterior form $\omega \in \wedge^k V^*$ can be expanded as
$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \omega_{i_1, \cdots, i_k} e^{*i_1} \wedge e^{*i_2} \wedge \cdots \wedge e^{*i_k}$$
where $\omega_{i_1, \cdots, i_k} \in \mathbb{K}$.
- **Definition(exterior product)** For $\omega \in \wedge^k V^*$ and $\phi \in \wedge^p V^*$ the exterior product is defined through
$$\omega \wedge \phi = \sum_{\substack{1 \le j_1 < j_2 < \cdots < i_p \le n \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} \omega_{i_1, \cdots, i_k} \phi_{j_1, \cdots, j_k} e^{*i_1} \wedge e^{*i_2} \wedge \cdots \wedge e^{*i_k} \wedge e^{*i_1} \wedge e^{*j_2} \wedge \cdots \wedge e^{*j_p}$$
- **Proposition** For $\omega \in \wedge^k V^*$, $\psi \in \wedge^k V^*$, $\phi \in \wedge^p V^*$ and $\rho \wedge^q V^*$ we have
  a) $\phi \wedge (\omega \wedge \rho) = (\phi \wedge \omega) \wedge \rho$
  b) For $p = q$ it holds that $\phi \wedge (\omega + \psi) = \phi \wedge \omega + \phi \wedge \psi$
  c) $\phi \wedge \omega = (-1)^{pk} \omega \wedge \phi$
- **Definition** A *exterior forms* or *an exterior form of degree k* on $U \subset \mathbb{R}^n$, U open, $1 \le k \le n$, is a map ω that associates to each point $p \in U$ an element $\omega(p) \in \wedge^k V^*$. Furthermore, $\omega(p)$ can be expanded as
$$\omega(p) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1, \cdots, i_k}(p) \, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$
If, the real functions $a_{i_1, \cdots, i_k}(p)$ are differentiable, then ω is called a *differential k- form.* The set of differential k-forms forms a vector space, denoted by $\Omega^k(U)$.
- For $k = 0$ we set $\Omega^1(U) = C^1(U)$.

3. Lecture on Oct. 22. – **Exterior Derivative** [Ca, Na]

- **Definition** a *vector field* on v on $\mathbb{R}^n$ assigns to each point p in $\mathbb{R}^n$ a vector $v(p) \in T_p(\mathbb{R}^n)$ smoothly. In coordinates we have
$$v(p) = v^i(p) \frac{\partial}{\partial x^i}|_p$$
with smooth functions $v^i(p)$. We denote by $\mathfrak{X}(\mathbb{R}^n)$ the set of vector fields on $\mathbb{R}^n$. Equivalently, for $f, g \in C^1(\mathbb{R}^n)$,
$$\mathfrak{X}(\mathbb{R}^n) = \mathrm{Der}\ \mathbb{F}(\mathbb{R}^n) = \{v : \mathbb{F}(\mathbb{R}^n) \to \mathbb{F}(\mathbb{R}^n), \mathbb{R}\text{-linear}, v(fg) = gv(f) + fv(g)\}$$
- **Definition** For $f \in C^1(\mathbb{R}^n)$ we denote by $\mathrm{d}f$ its differential through $\mathrm{d}f(v) = v(g)$. In coordinates $df = \frac{\partial f}{\partial x^i} \mathrm{d}x^i$. Then the map
$$\begin{aligned} \mathrm{d} \quad : \quad & \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n) \\ & \omega(p) \mapsto \mathrm{d}\omega(p) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \mathrm{d}a_{i_1, \cdots, i_k}(p) \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} \end{aligned}$$
is well-defined for $\{a_{i_1, \cdots, i_k}(p)\} \in C^1(\mathbb{R}^n)$. $\mathrm{d}\omega(p)$ is the *exterior derivative* of the differential form $\omega(p)$.
- **Proposition** For $\omega(p), \psi(p) \in \Omega^k(\mathbb{R}^n)$and $\phi \in \Omega^p(\mathbb{R}^n)$ we have
  a) $\mathrm{d}(\omega(p) + \psi(p)) = \mathrm{d}\omega(p) + \mathrm{d}\psi(p)$
  b) $\mathrm{d}(\omega(p) \wedge \phi(p)) = \mathrm{d}\omega(p) \wedge \phi(p) + (-1)^k \omega(p) \wedge \mathrm{d}\phi(p)$
  c) $\mathrm{dd}\omega(p) = 0$ assuming ω is twice differentiable

4. Lecture on Oct. 24. – **Interior Product, Pullback** [Ca]

- **Definition** For $\omega(p), \psi(p) \in \Omega^k(\mathbb{R}^n)$ and $z \in \mathfrak{X}(\mathbb{R}^n)$ we define the *interior derivative* or *interior product* as the map
$$\begin{aligned} \mathrm{i}_z \quad : \quad & \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n) \\ & \omega(p) \mapsto ((\mathrm{i}_z\omega)(p) : (x_1, \cdots, x_{k-1}) \mapsto \omega(z, x_1, \cdots, x_{k-1})) \end{aligned}$$
is well-defined for $(x_1, \cdots, x_{k-1}) \in \underbrace{V \times V \times \cdots \times V}_{k-1 \text{ times}}$ and $\{a_{i_1,\cdots,i_k}(p)\} \in C^1(\mathbb{R}^n)$.
- **Proposition** For $\omega(p) \in \Omega^k(\mathbb{R}^n)$, $\phi \in \Omega^p(\mathbb{R}^n)$ and $z \in \mathfrak{X}(\mathbb{R}^n)$ we have
  a) $\mathrm{i}_z(\omega \wedge \phi) = (\mathrm{i}_z\omega) \wedge \phi + (-1)^k \omega \wedge (\mathrm{i}_z\phi)$
  b) $\mathrm{i}_z(\mathrm{i}_z\omega) = 0$
- **Definition** Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $x \in T_p(\mathbb{R}^n)$ and $h \in C^1(\mathbb{R}^m)$, then the *differential* $(f_*)_p$ of f at p (or the *push forward*) is defined through
$$(f_*)_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$$
$$(f_*)_p(x)(h)(p) = x(f^*h)(p),$$
and extended to $f_* : \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^m)$ point-wise as above.
- **Definition** The *pull back* of a differential form on $\mathbb{R}^m$ to $\mathbb{R}^n$ is defined by the map
$$\begin{aligned} f^* \quad : \quad & \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n) \\ & \omega \mapsto ((f^*\omega) : (x_1, \cdots, x_k) \mapsto \omega(f(p))(f_*x_1, \cdots, f_*x_k)) \end{aligned}$$
for $(x_1, \cdots, x_k) \in \underbrace{\mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \times \cdots \times \mathfrak{X}(\mathbb{R}^n)}_{k \text{ times}}$.
- **Example 4:** Let
$$\omega = -\frac{y}{x^2+y^2}\mathrm{dx} + \frac{\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2}\mathrm{dy} \qquad \in \Omega^1(\mathrm{U})$$
with $U = \mathbb{R}^2 - \{0\}$, $V = \{r > 0, 0 < \vartheta < 2\pi\}$ and
$$\left(\begin{array}{c} f : U \to V \\ (r, \vartheta) \mapsto \begin{pmatrix} x = r\cos\vartheta \\ y = r\sin\vartheta \end{pmatrix} \end{array} \right)$$
Then $f^*\omega = \mathrm{d}\vartheta$.

5. Lecture on Oct. 29. – **Pullback** [Ca]

- **Proposition** For $g \in C^1(\mathbb{R}^p, \mathbb{R}^n)$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $h \in C^1(\mathbb{R}^m, \mathbb{R})$; $\omega, \psi \in \Omega^k(\mathbb{R}^m)$ and $\phi \in \Omega^k(\mathbb{R}^m)$ we have
  a) $f^*(\omega + \psi) = f^*\omega + f^*\psi$
  b) $f^*(h\omega) = f^*(h)g^*(\omega)$
  c) $f^*(\omega \wedge \phi) = (f^*\omega) \wedge (f^*\phi)$
  d) $(f \circ g)^*\omega = g^*(f^*\omega)$
  e) $\mathrm{d}f^*(\omega) = f^*(\mathrm{d}\omega)$
- **Remark** We choose canonical coordinates $\{y^i\}$ on $\mathbb{R}^m$ and take
$$\omega(p) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1,\cdots,i_k}(p)\, \mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k}$$

Then we have, using b) and c)

$$(f^*\omega)(p) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} (f^*a_{i_1,\cdots,i_k})(p)\, f^*\mathrm{d}y^{i_1}\wedge f^*\mathrm{d}y^{i_2}\wedge\cdots\wedge f^*\mathrm{d}y^{i_k}$$
$$= \sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1,\cdots,i_k}(f(p))\,\mathrm{d}f^{i_1}\wedge \mathrm{d}f^{i_2}\wedge\cdots\wedge \mathrm{d}f^{i_k}$$

where $f^i = y^i(f)$. This follows for the fact that if $\{y^i\}$ are canonical coordinates on $\mathbb{R}^n$ and $z\in\mathfrak{X}(\mathbb{R}^n)$, then by **Example 1** $f^*\mathrm{dy^i(z)} = \mathrm{dy^i(f_*z)}$ is given by the $i-th$ coordinate of f_*z. This is just $\mathrm{df^i(z)}$. This formula gives a simple and intuitive expression for the pull back of a generic differential form.

6. Lecture on Oct. 31. – **Integration, vector fields revisited**

[Ca, Ar, Va, Na]

- An n-form ω on $\mathbb{R}^n$ expressed in cartesian coordinates is necessarily of the form
$$\omega = a\,\mathrm{dx}^1\wedge\cdots\wedge\mathrm{dx^n}$$
with $a\in\mathfrak{F}(\mathbb{R}^n)$. The integral of ω over a Polygon in D in $\mathbb{R}^n$ is then defined as
$$\int_D \omega = \int_D a\,dx^1\cdots dx^n$$
where the r.h.s. is the usual integral of a function on $\mathbb{R}^n$ defined in terms of a Riemann sum (e.g. [Ar])
- For $f:\mathbb{R}^n\to\mathbb{R}^m$, $m\ge n$, $\sigma\subset\mathbb{R}^m$ with $\sigma = f(D)$ and $\omega\in\Omega^n(\mathbb{R}^n)$ we define
$$\int_\sigma \omega = \int_D f^*\omega$$
- **Definition :** As a set the *tangent bundle* over an open subset $U\subset\mathbb{R}^n$ is defined as
$$TU \equiv \sqcup_{p\in U} T_pU \xrightarrow{pr} U$$
- **Def :** (Alternative but equivalent definition of a vector field): A smooth vector field v on $U\subset\mathbb{R}^n$, U open, is a smooth map
$$v: U\to TU$$
s.t. $\mathrm{pr}(v_p) = p$, $\forall p\in U$.
- **Notation:** $v\in\mathfrak{X}(U)\equiv\Gamma(TU) = \{\mathbb{C}^\infty(U,TU)|pr(v_p) = p\}$. $\Gamma(TU)$ is called the space of sections in TU.
- By duality
$$T^*U \equiv \sqcup_{p\in U} T^*_pU \xrightarrow{pr} U$$
is the *cotangent bundle.*
- **Definition :** A *Lie algebra* is a vector space V over $\mathbb{R}$ endowed with a bi-linear paring
$$[\,,\,]: V\times V\to V$$
which is anti0symmetric, $[u,v] = -[v,u]$, $u,v\in V$, and satisfies the *Jacobi identity*
$$[[u,v],z]+[[v,z],u]+[[z,u],v] = 0$$

- **Proposition:** $\mathfrak{X}(\mathbb{R}^n)$ is an (infinite dimensional) Lie algebra with $[u,v](f) = u(v(f)) - v(u(f))$, for $f \in C^\infty(\mathbb{R}^n)$.
- **Definition (Flow)** An *integral curve* for a vector field v is a smooth curve $\phi : (a,b) \to \mathbb{R}^n$ satisfying $\dot{\phi}(t) = v_{\phi(t)}$ for all $t \in (a,b)$. Let $\phi_t(p)$ be such a curve that passes through p at $t = 0 \in (a,b)$. The map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is then the *flow* of the vector field v. For existence and uniqueness see [Va].
- **Definition (Lie Derivative)** Let $z \in \mathfrak{X}(\mathbb{R}^n)$ be a differentiable vector field, ϕ_t its flow and $\omega \in \Omega^k(\mathbb{R}^n)$, then the Lie derivative of ω is defined as

$$L_z\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^*\omega)\bigg|_{t=0}$$

In components we have

$$\begin{aligned}(L_z\omega)(p) \quad &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} (i_z da_{i_1,\cdots,i_k}) \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &\quad + \sum_{s=1}^{k} \sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1,\cdots,i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}(i_z \mathrm{d}x^{i_s}) \wedge \cdots \wedge \mathrm{d}x^{i_k}\end{aligned}$$

Useful formula: $L_z\omega = (\mathrm{d}\, i_z + i_z\, \mathrm{d})\omega$.

7. Lecture on Nov. 5. – **Lie derivative of a vector field** []

- **Definition:** Let φ be a diffeomorphism of $\mathbb{R}^n$ and $v \in \mathfrak{X}(\mathbb{R}^n)$. Then

$$\varphi^* v := \varphi_*^{-1} v \text{ is the pull back of } v \text{ with } \varphi$$

In particular, a flow ϕ_t, t fixed is a diffeo on $\mathbb{R}^n$ with $\varphi^{-1} = \phi_{-t}$.
- **Definition:** Let $u, v \in \mathfrak{X}(\mathbb{R}^n)$. The *Lie-derivative* of v in direction u is

(1) $$L_u v := \frac{d}{dt}\bigg|_{t=0} \phi_t^* v$$

where ϕ_t is the flow of u.
- $(\phi_t^* v)(p)$ is a smooth family of vectors in $T_p\mathbb{R}^n$.
- **Lemma:** Let u, v be smooth vector fields on $\mathbb{R}^n$ and $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$. We write ϕ_t respectively ψ_s for the flow of u respectively v. Then
  a) $\varphi^* v = \frac{d}{ds}\big|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi$.
  b) $\varphi^* v = v \Leftrightarrow \varphi \circ \psi_s = \psi_s \circ \varphi$ for all s.
  c) $L_u v = 0 \Leftrightarrow \phi_t \circ \psi_s = \psi_s \circ \phi_t$ for all s, t.
- **Lemma:** Let u, v be smooth vector fields on $\mathbb{R}^n$ and let ϕ_t respectively ψ_s be the flow of u respectively v. Then
  a) $L_u v = \frac{\partial^2}{\partial s \partial t}\Big|_{(0,0)} \phi_{-t} \circ \psi_s \circ \phi_t$.
  b) $(L_u v)(f) = u(v(f)) - v(u(f))$ for all smooth functions f on $\mathbb{R}^n$.
- **Lemma:** Let u, v be smooth vector fields on $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a diffeomorphism. Then
  a) $[u, v]$ is $\mathbb{R}$- bilinear in u and v.
  b) $[u, v] = -[v, u]$.
  c) The Jacobi identity holds.
  d) $[u, fv] = f[u, v] + u(f)v$.
  a) $\varphi_*[u, v] = [\varphi_* v, \varphi_* u]$.

8. Lecture on Nov. 7. – **Stokes' Theorem on $\mathbb{R}^n$** [Ca]

- For f a function (0-form) on $[a,b] \subset \mathbb{R}$ we have $\int_a^b \mathrm{df} = \int_\mathrm{a}^\mathrm{b} \partial_\mathrm{x}\mathrm{f}\ \mathrm{dx} = \mathrm{f(b)} - \mathrm{f(a)}$.
- For $\omega = a_i \mathrm{dx}^\mathrm{i}$ a 1-form on $U = [0,1] \times [0,1] \subset \mathbb{R}^2$ we have

$$\int_S \mathrm{d}\omega = \int_\mathrm{S} (\partial_\mathrm{x}^1 \mathrm{a}_2)\mathrm{dx}^1 \wedge \mathrm{dx}^2 + (\partial_\mathrm{x}^2 \mathrm{a}_1)\mathrm{dx}^2 \wedge \mathrm{dx}^2 = \int_{\partial \mathrm{S}} \omega$$

- More generally if S is a compact subset of $\mathbb{R}^2$ with piece-wise regular boundary ∂S (piecewise homeomorphic to intervals in $\mathbb{R}$) then we obtain by decomposing S in terms of little squares and interpreting $\int_U \mathrm{d}\omega$ as a Riemann sum over the squares the result

$$\int_U \mathrm{d}\omega = \int_{\partial \mathrm{U}} \omega$$

- This result generalises immediately to compact co-dimension zero subsets of $\mathbb{R}^n$ with piece-wise regular boundary.
- Finally, if $\mathcal{M}$ is a compact subset of dimension $m \leq n$ in $\mathbb{R}^n$ (with piecewise regular boundary, $\partial\mathcal{M}$), diffeomorphic to a compact subset of $U \subset \mathbb{R}^m$ (i.e. $\mathcal{M} = f(U)$) and $\omega \in \Omega^{m-1}(\mathcal{M})$, then

$$\int_\mathcal{M} d\omega = \int_U f^* \mathrm{d}\omega = \int_\mathrm{U} \mathrm{df}^*\omega = \int_{\partial \mathrm{U}} \mathrm{f}^*\omega = \int_{\partial \mathcal{M}} \omega$$

- More generally, the parametrisation of $\partial\mathcal{M}$ may be different from that induced by $\mathcal{M}$. Then we have

$$\int_\mathcal{M} d\omega = \int_{\partial\mathcal{M}} i^*\omega$$

where $i : \partial\mathcal{M} \to \mathcal{M}$ is the inclusion map of $\partial\mathcal{M}$ into $\mathcal{M}$.

9. Lecture on Nov. 12. – **Poincaré Thm for 1-forms** [Ca]

- **Definition:**
  - a) If $\omega \in \Omega^k(U)$ s.t. $\mathrm{d}\omega = 0$, then ω is *closed*.
  - b) If there exsits $\alpha \in \Omega^{k-1}(V)$, $V \subset U$, s.t. $\omega = \mathrm{d}\alpha$ in V then ω is *exact*.
- **Proposition A:** The following three assertions are equivalent
  - 1) $\omega \in \Omega^1(U)$ is exact in a connected open subset $V \subset U$.
  - 2) For any curve $\gamma : (a,b) \to U$, $\int_\gamma \omega$ depends only on the end points $\gamma(a), \gamma(b)$.
  - 3) $\int_\gamma \omega$ for any closed curve γ in V.
- **Rem:** Not every closed form in $\Omega^1(U)$, U open subset of $\mathbb{R}^n$ is exact as can be see for example in **Example 4**. Indeed for a closed curve encircling the origin of $\mathbb{R}^2$ we have

$$\int_\gamma \omega = \int_\gamma \mathrm{d}\vartheta = 2\pi.$$

and therefore ω cannot be exact by the above proposition. However, ω is locally exact. More gnerally,
- **Thm:** (Poincaré Thm for 1-forms on $\mathbb{R}^n$): Let $\omega \in \Omega^1(U)$, $U \subset \mathbb{R}^n$, U open. Then $\mathrm{d}\omega = 0$ iff for each $p \in U$ there is a neighbourhood $V \subset U$ of p and a differentiable function $f : V \to \mathbb{R}$, s.t. $\omega = \mathrm{df}$.

10. Lecture on Nov. 14. – **Poincaré Lemma**

- Let $U \subset \mathbb{R}^n$, open. Then, $\omega \in \Omega^k(U)$ is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\gamma$ globally for some $\gamma \in \Omega^{k-1}(U)$. Since $\mathrm{d}^2 = 0$ every exact form is closed.The converse is not true but we want to show that every closed form is nevertheless exact in the nbhd of some point.
- **Defiition** An open subset U of $\mathbb{R}^n$ is *contractible* to some point $p_0 \in U$ if there exists a differentiable map
$$\begin{aligned} H \quad &: \quad U \times \mathbb{R} \to U \\ &(p,t) \mapsto H(p,t) \in U \end{aligned}$$
such that $H(p,1) = p$ and $H(p,0) = p_0 \ \forall p \in U$.
- To every $\omega \in \Omega^k(U)$ we can the associate a k-form $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ as
$$\bar{\omega} = H^*\omega$$
On the other hand, any $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ has a unique decomposition of the form
$$\bar{\omega} = \omega_1 + \mathrm{d}t \wedge \eta$$
with $i_{\partial_t}\omega_1 = 0$ and $i_{\partial_t}\eta_1 = 0$
- Conversely we can associate a k-form $\omega \in \Omega^k(U)$ to each $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ with the help of the inclusion map
$$\begin{aligned} i_t \quad &: \quad U \to U \times \mathbb{R} \\ &i_t(p) = (p,t) \in U \times \mathbb{R} \end{aligned}$$
Then $i_t^*\bar{\omega} \in \Omega^k(U)$ provided $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$.
- Let us furthermore define the map
$$\begin{aligned} I \quad &: \quad \Omega^k(U \times \mathbb{R}) \to \Omega^{k-1}(U) \\ (I\eta)(z_1, \cdots, z_{k-1}) &= \int_0^1 \eta(p,t)(\partial_t, i_{t*}z_1, \cdots, i_{t*}z_{k-1})dt. \end{aligned}$$
The key result which then establishes local exactness is the
- **Lemma**
$$i_1^*\bar{\omega} - i_0^*\bar{\omega} = d(I\bar{\omega}) + I(d\bar{\omega}) \tag{2}$$
Indeed, since $H \circ i_1 = \mathrm{id}$ and $H \circ i_1 = p_0, \forall p \in U$ we have
$$\omega = (H \circ i_1)^*\omega = i_1^*\bar{\omega}$$
and
$$0 = (H \circ i_0)^*\omega = i_0^*\bar{\omega}$$
From this the desired result he follows:
- **Theorem** Let U be a contractible, open subset of $\mathbb{R}^n$ and $\omega \in \Omega^k(U)$ with $\mathrm{d}\omega = 0$. Then there exists a $k-1$ form $\alpha \in \Omega^{k-1}(U)$ such that $\omega = \mathrm{d}\alpha$.

11. Lecture on Nov. 19. – **Homotopic Curves** [Ca]

- **Def:** Two continuous open curves $\gamma_1, \gamma_2 : [a,b] \to U$
  a) are *freely homotopic* if there exists a continuous map
$$\begin{aligned} H \quad : \quad &[a,b] \times [0,1] \to U \qquad \text{s.t.} \\ &H(s,0) = \gamma_1(s) \\ &H(s,1) = \gamma_2(s) \end{aligned} \tag{3}$$

b) are homotopic *relative to the set* $\{\gamma_1(a), \gamma_1(b)\}$ if $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$ and $\forall t$,

$$\begin{aligned} H(a,t) &= \gamma_1(a) \\ H(b,t) &= \gamma_1(b) \end{aligned} \tag{4}$$

c) wo continuous *closed* curves are freely homotopic $\{\gamma_1(a) = \gamma_1(b)\}$, $\gamma_2(a) = \gamma_2(b)$ and $\forall t$,

$$H(a,t) = H(b,t) \tag{5}$$

- **Rem:** In c), instead of $H(a,t) = H(b,t)$ we can replace $[a,b]$ by S^1 in which case closedness of the interpolating curves follows from continuity. Then there is a topological statment,
- **Def:** A connected open set U in $\mathbb{R}^n$ is *simply connected* if every continuous closed curve is freely homotopic to a point.
- **Example:** $\mathbb{R}^n$ a ball in $\mathbb{R}^n$ and it homeomorphic images are simply connected.
- contractible $\implies$ simply connected but the converse is not true (e.g. 2-sphere).
- As a consequence of **Proposition A**, every closed form on a simply connected subset U is exact.
- **Thm:** Let $\omega \in \Omega^1(U)$, with $\mathrm{d}\omega = 0$ and let $\gamma_1, \gamma_2 : [a,b] \to U$ with $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$, be two continuous, homotopic curves, then
$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$
- **Proposition:** Let $\omega \in \Omega^1(U)$, with $\mathrm{d}\omega = 0$ and let γ_1, γ_2 be continuous, closed, freely homotopic curves, then
$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$
In particular, if γ_1 is freely homotopic to a point, then $\int_{\gamma_1} \omega = 0$. In important step in the proof of the Thm. in the last lecture is to replace homotopies (which are continuous) by smooth homotopies. For this we need the following two results (e.g. [L]) which we cite without proof.
- **Thm 1:** (Witney approximation on $\mathbb{R}^n$) If γ is a continuous map between subsets U and V in $\mathbb{R}^n$, then γ is homotopic to a smooth map $\tilde{\gamma}$. If γ is smooth on a closed subset D of U then the homotopy can be taken relative to D.
- **Thm 2:** If γ_1 and γ_2 are homotopic maps from U to V in $\mathbb{R}^n$, then they are smoothly homotopic. If they are homotopic, relative to D, the they are smoothly homotopic relative to D.

12. Lecture on Nov. 21. – **deRham Cohomology** [Na]

- $\Omega^k(M)$ is a vector space over $\mathbb{R}$. As such it elements form a group with respect to addition. It turns out, however that there are invariant sub group which we will now review.
- **Definition** Let M^m be a subset of $\mathbb{R}^n$ of dimension $m \leq n$. The the set of
  a) closed k-form is the k-th *cocycle group*, with real coefficients $Z^k(M, \mathbb{R})$
  b) exact k-form is the k-th *coboundary group*, with real coefficients $B^k(M, \mathbb{R})$

c) $H^k(M,\mathbb{R}) = Z^k(M,\mathbb{R})/B^k(M,\mathbb{R})$ is the k-th *deRham cohomology group* with real coefficients.

In c) taking the quotient $Z^k(M,\mathbb{R})/B^k(M,\mathbb{R})$ means that we consider *equivalence classes* by identifying two closed forms that differ by an exact form. That is, for $\alpha, \beta \in Z^k(M,\mathbb{R})$ we say that $\alpha \simeq \beta$ id $\alpha = \beta + \gamma$ where $\gamma \in B^k(M,\mathbb{R})$.

- **Examples:** for any subset U of $\mathbb{R}^n$, $H^0(U,\mathbb{R}) = \mathbb{R}$. For $U = \mathbb{R}^2 - \{0\}$ we have $H^1(U,\mathbb{R}) = \mathbb{R}$ and $H^2(U,\mathbb{R}) = 0$.

13. Lecture on Nov. 26. – **deRham Complex** [Na]

- Let $M \subset \mathbb{R}^n$ of dimension m and $\Omega^*(M,\mathbb{R}) = \oplus_{k=0}^m \Omega^k(M,\mathbb{R})$ with $\omega = (\omega_1, \cdots, \omega_m)$ for $\omega \in \Omega^*(M,\mathbb{R})$. The exterior product $\wedge$ endows $\Omega^*(M,\mathbb{R})$ with the structure of a ring. Let furthermore $\mathrm{d} : \Omega^{\mathrm{k}}(\mathrm{M},\mathbb{R}) \to \Omega^{\mathrm{k+1}}(\mathrm{M},\mathbb{R})$ be the exterior derivative then
- **Definition** The *deRham complex*, $\Omega^*(M,\mathrm{d})$ is defined by $\Omega^*(M,\mathbb{R})$ as above, together with the sequence

$$\cdots \overset{\mathrm{d_{k-2}}}{\to} \Omega^{k-1} \overset{\mathrm{d_{k-1}}}{\to} \Omega^k \overset{\mathrm{d_k}}{\to} \Omega^{k+1} \overset{\mathrm{d_{k+1}}}{\to} \Omega^{k+2} \cdots$$

with $\mathrm{Im}(\mathrm{d_{i-1}}) \subset \mathrm{Ker}(\mathrm{d_i})$.

- **Rem:** $H^i(M,\mathbb{R}) = \mathrm{Ker}(\mathrm{d_{i+1}})/\ \mathrm{Im}(\mathrm{d_i})$.
- **Definition** If sequence $H^i(M,\mathbb{R}) = 0$, $k = 1, \cdots m$ then the sequence is *exact.*
- **Definition** The *deRham cohomology ring* $H^*(M,\mathbb{R})$ is defined as

$$H^*(M,\mathbb{R}) = \oplus_{k=0}^m H^k(M,\mathbb{R})$$

- Let U, V be subsets of $\mathbb{R}^n$ and $\phi : U \to V$ a continuous map. If ϕ is a diffeomorphism, then ϕ^* is an isomorphism and thus diffeomorphic subsets have the same cohomology groups. However we can start with a weaker assumption
- **Definition** Two subsets U, V of $\mathbb{R}^n$ are *homotopy equivalent* if there exist *continuous maps*, $\phi : U \to V$ and $\psi : V \to U$ such that $\psi \circ \phi$ is homotopic to $\mathrm{id}|_U$ and $\phi \circ \psi$ is homotopic to $\mathrm{id}|_V$. Each of the two maps are called *homotopy equivalences.*
- **Examples:**
  - 1) Any homeomorphism $\phi : U \to V$ is a homotopy equivalence but the converse is not always true: a disk is not homeomorphic to a point but it is homotopy equivalent to a point.
  - 2) S^{n-1} is homotopy equivalent to $\mathbb{R}^n - \{0\}$.
- **Lemma** Let ϕ and $\mathrm{id}|_V : V \to V$ be homotopy equivalent maps, $\phi \simeq \mathrm{id}|_V$. Then ϕ^* and $\mathrm{id}|_V^*$ induce the same map on $H^*(V,\mathbb{R})$.
- **Thm** Let $\phi : U \to V$ be a homotopy equivalence with homotopy inverse ψ then ϕ^* induces an isomorphism between cohomology groups, ie. $H^k(U) \cong H^k(V)$.

14. Lecture on Nov. 28. – **Zigzac lemma** [Ha]

- **Definition** Let A^* and B^* be two (co)chain complexes. A (co)chain map $\phi^* : A^* \to B^*$ is a collection of maps $\phi^* : A^n \to B^n$ (same symbol) s.t. $\mathrm{d} \circ \phi^* = \phi^* \circ \mathrm{d} : \mathrm{A^n} \to \mathrm{B^{n+1}}$.
- **Definition** A *short exact sequence* (SES) is a collection of 3 (co)chain complexes A^*, B^*, C^* and (co)chain maps $\phi^* : A^* \to B^*$, $\psi^* : B^* \to C^*$, s.t. for each n

$$0 \to A^* \overset{\phi^*}{\to} B^* \overset{\psi^*}{\to} C^* \to 0$$

is an exact sequence.

- **Lemma** (Zigzac lemma) For A^*, B^*, C^* and ϕ^*, ψ^* as above, then for all n there exists a linear map δ, s.t.

$$\cdots \to H^{n-1}(C^*) \xrightarrow{\delta} H^n(A^*) \xrightarrow{\phi^*} H^n(B^*) \xrightarrow{\psi^*} H^n(C^*) \xrightarrow{\delta} H^{n+1}(A^*) \to \cdots$$

is an exact sequence.

- In order to apply this lemme to compute cohomology groups we need to identify A^*, B^*, C^* and ϕ^*, ψ^* in the deRham complexes of subsets of $\mathbb{R}^n$. For that let M be a subset of dimension m in $\mathbb{R}^n$, $m \leq n$, such that $M = f(U) \cup g(V)$, f and g continuous maps from $U, V \subset \mathbb{R}^m \to \mathbb{R}^n$. Furthermore, let $i : U \cap V \to U$ and $j : U \cap V \to V$ be inclusion maps and

$$\begin{aligned} (f^* \oplus g^*) \quad &: \quad \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V) \\ & \qquad \omega \mapsto (f^*(\omega), g^*(\omega)) \\ (i^* - j^*) \quad &: \quad \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \\ & \qquad (\omega, \eta) \mapsto (i^*(\omega) - j^*(\eta)) \end{aligned}$$

- **Thm** For each n there exists a linear map δ, s.t.

$$\cdots \to H^n(M) \overset{(f^* \oplus g^*)}{\to} H^n(U) \oplus H^n(V) \overset{i^* - j^*}{\to} H^n(U \cap V) \xrightarrow{\delta} H^{n+1}(M) \to \cdots$$

is an exact sequence.

15. Lecture on Dec. 3. – **Submanifolds of $\mathbb{R}^n$** [C, Jä-V, L, W]

- We give four equivalent definitions of the notion of submanifold of dimension $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. In all four of them, $M \subset \mathbb{R}^n$.
- **Condition (a) Local parametrizations:** For all $p \in M$ there is an open set $U \subset \mathbb{R}^m$, a neighbourhood $V \subset \mathbb{R}^n$ of p and a smooth map $\phi : U \longrightarrow \mathbb{R}^n$ such that
  1. φ is a homeomorphism onto $V \cap M$, and
  2. for all $x \in U$ the differential $\phi_((x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is injective.
- **Condition (b) Locally flat:** For all $p \in M$ there are an open neigbourhood $V \subset \mathbb{R}^n$ of p and $W \subset \mathbb{R}^n$ of 0 and a diffeomorphism $\Phi : V \longrightarrow W$ such that $\Phi(p) = 0$ and $\Phi(V \cap M) = (\mathbb{R}^m \times \{0 \in \mathbb{R}^{n-m}\}) \cap W$.
- **Condition (c) Locally a zero set:** For all $p \in M$ there is an open neighbourhood U and a smooth function $F : V \longrightarrow \mathbb{R}^{n-m}$ such that
  1. $F^{-1}(0) = (V \cap M)$, and
  2. for all $q \in M \cap V$ the differential $F_*(q) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$ is surjective.
- **Condition (d) Locally a graph:** For all $p \in M$ there is an open neighbourhood $V \subset \mathbb{R}^n$ and a smooth function $g : U \to \mathbb{R}^{n-m}$ defined on an open subset of $U \subset \mathbb{R}^m$ together with a permutation $\sigma \in S_n$ such that

$$V \cap M = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \,|\, (x, g(x)) \text{ with } x \in U\}.$$

- **Theorem:** For a given subset $M \subset \mathbb{R}^n$ the conditions (a)–(d) are equivalent.
- *Proof:* The proof of the implications $(b) \Rightarrow (c), (d) \Rightarrow (a)$ are trivial. For the proof of $(c) \Rightarrow (d)$ one uses the implicit function theorem, for the proof of $(a) \Rightarrow (b)$ one applies the inverse function theorem to a function Φ extending ϕ (the local inverse of Φ satisfies (b)).
- **Definition:** A subset $M \subset \mathbb{R}^n$ is a *submanifold* of dimension k if any of the conditions (a)–(d) is satisfied.

- **Remark:** When M is a non-empty submanifold the number m is then the same in all the conditions, in particular the dimension of a non-empty submanifold is well-defined. By convention, the empty subset is a submanifold of any dimension (including negative integers).
- **Corollary:** Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension m, $U, U' \subset \mathbb{R}^m$ open and

$$\phi : U \longrightarrow M \qquad\qquad \phi' : U' \longrightarrow M$$

local parametrizations of M. Then

$$\phi^{-1} \circ \phi' : \phi'^{-1}(\phi(U) \cap \phi'(U')) \longrightarrow \phi^{-1}(\phi(U) \cap \phi'(U'))$$

is a diffeomorphism.

16. Lecture on Dec. 5. – **Examples of Submanifolds of $\mathbb{R}^n$, Abstract manifolds 1**[C, Jä-V, L, W]

- **Examples :** $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n, k \leq n$, $S^k = \{(x_1, \ldots, x_{k+1}) \,|\, x_1^2 + \ldots + x_n^2 = 1\} \subset \mathbb{R}^{k+1}$, and more interestingly

$$\mathrm{O}(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \,|\, AA^T = E\}$$

are submanifolds. To prove this for $\mathrm{O}(n)$ verify condition (c) for

$$\begin{aligned} F : \mathrm{Mat}(n \times n, \mathbb{R}) &\longrightarrow \mathrm{Sym}(n, \mathbb{R}) = \{B \in \mathrm{Mat}(n \times n, \mathbb{R}) \,|\, B = B^T\} \\ A &\longmapsto AA^T - E. \end{aligned}$$

The dimension of $O(n)$ is $\frac{n(n-1)}{2}$.
- We now start discussing manifolds without reference to an ambient space. The first attempt is preliminary.
- **Definition:** Let M be a set. A *smooth k-dim. atlas* $\mathcal{A}$ on M is a collection of maps $\phi_i : U_i \longrightarrow M, i \in I$ (called *charts*) such that
  1. $U_i \subset \mathbb{R}^k$ is open and $\phi_i : U_i \longrightarrow \phi_i(U_i)$ is bijective,
  2. $\cup_i \phi_i(U_i) = M$,
  3. for all $i, j \in I$ such that $\phi_i(U_i) \cap \phi_j(U_j) \neq \emptyset$ the preimage under ϕ_i, ϕ_j are open in $\mathbb{R}^k$ and

$$\phi_i^{-1} \circ \phi_j : \phi_j^{-1}(\phi_i(U_i) \cap \phi_j(U_j)) \longrightarrow \phi_i^{-1}(\phi_i(U_i) \cap \phi_j(U_j))$$

  is a diffeomorphism.
- **Definition:** Two such atlases $\mathcal{A}, \mathcal{A}'$ for M are *equivalent* if their union is still a smooth atlas.
- **Preliminary definition:** A manifold of dimension k is a set with an equivalence class of smooth k-dim atlases.
- **Example:** Submanifolds of $\mathbb{R}^n$ and products of such have natural smooth atlases where ϕ_i are smooth homeomorphisms.
- **Example:** For $k \geq 0$ let real projective space $\mathbb{RP}^k$ be the set of lines through the origin in $\mathbb{R}^{k+1}$, i.e.

$$\mathbb{RP}^k = (\mathbb{R}^{k+1} \setminus \{0\}) / \sim$$

where $(x_0, x_1, \ldots, x_n) \sim (x_0', \ldots, x_n')$ if and only if there is $\lambda \in \mathbb{R}$ such that $\lambda \cdot (x_0, \ldots, x_n) = (x_0', \ldots, x_n')$. Elements of this set are denoted by *homogeneous* *coordinates* $[x_0 : \ldots : x_n]$. There is an atlas for $\mathbb{RP}^k$ with $k+1$ charts: For $i \in \{0, \ldots, k\}$ let

$$\begin{aligned} \phi_i : \mathbb{R}^k &\longrightarrow \mathbb{RP}^k \\ (x_1, \ldots, x_k) &\longmapsto [x_1 : \ldots : 1 : \ldots : x_k] \end{aligned}$$

Here the 1 occupies the ith slot of the homogenous coordinate. One obtains the complex projective space $\mathbb{CP}^n$ of dimension $2n$ when one replace $\mathbb{R}$ by $\mathbb{C}$.

17. Lecture on Dec. 10. – **Functions, vector fields and forms on abstract manifolds** [Ca, C, Jä-V, L, W]

- **Definition:** $f : M \longrightarrow \mathbb{R}$ is *smooth* near $p \in M$ if there is a local parametrization $\phi : U \subset \mathbb{R}^k \longrightarrow M$ such that $p \in \phi(U)$ and $f \circ \phi$ is smooth.
- **Remark:** This is independent from the choice of ϕ by the corollary in lecture 15.
- The next goal is the construction of sufficiently many smooth functions on smooth manifolds with positive dimension.
- **Reminder:** The function

$$\begin{aligned} \lambda : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} 0 & t \leq 0 \\ \exp(-t^{-1}) & t > 0 \end{cases} \end{aligned}$$

  is smooth. The same holds for $\psi_\varepsilon(x) = \frac{\lambda(x)}{\lambda(x)+\lambda(\varepsilon - x)}$ for $\varepsilon > 0$. This function is nowhere negative and $\equiv 1$ on $\{x > \varepsilon\}$ while it is $\equiv 0$ on $\{x < 0\}$. Finally, the function f_ε on $\mathbb{R}^n$ defined by

$$f_\varepsilon(x) = 1 - \psi_\varepsilon(\|x\| - \varepsilon)$$

  is smooth, it vanishes outside of a 2ε-ball around the origin and is $\equiv 1$ on the ε-ball around the origin.
- Let $p \in M$, $\phi : U \longrightarrow M$ a chart mapping the origin to p. Then for small enough $\varepsilon > 0$, the function

$$\begin{aligned} g : M &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} 0 & x \notin \phi(U) \\ f_\varepsilon(\varphi^{-1}(x)) & x \in \phi(U) \end{cases} \end{aligned}$$

  is well-defined and smooth. Using this construction it easy to show that the vector space of smooth functions $\mathfrak{F}(M)$ on M has infinite dimension (provided that the dimension of M is positive (and M nonempty)).
- **Definition:** Let $(U_i)_{i \in I}$ be an open covering of M, M compact. A partition of unity subordinate to the covering is a collection of smooth functions $(\varphi_j)_{j \in J}$ such that
  1. for all $p \in M$ there is a neighbourhood V_p such that all but finitely many φ_j vanish on V_p,
  2. for all $j \in J$ there is $i(j) \in I$ such that support$(\varphi_j) \subset U_{i(j)}$, and
  3. $\varphi_j \geq 0$ and $\sum_{j \in J} \varphi_j = 1$.

  Because of the first condition one does not have to worry about convergence of the series in the third condition.
- **Proposition:** Let $(U_i)_{i \in I}$ be an open covering of M. Then there exists a partition of unity subordinate to $(U_i)_i$.

- Vector fields on M are defined as before as derivations on $\mathfrak{F}(M)$,

$$\mathfrak{X}(M) = \mathrm{Der}(\mathfrak{F}(M)) = \{v : \mathfrak{F}(M) \to \mathfrak{F}(M)\,, \mathbb{R}\text{-linear, Leibnitz}\}$$

Note that everything happens on the charts. In particular on U_α:

$$v_{(\alpha)} = v^i_{(\alpha)} \partial_{x^i}$$

with $v^i_{(\alpha)}$ functions on U_α.

- Differential form on M are again defined as before

$$\Omega^k(M) = \{\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{F}(M)\,, \text{mult-linear, alternating, differentiable}\}$$

In particular, in the standard basis on U_α:

$$\omega(p) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (a_{i_1, \cdots, i_k})_{(\alpha)}(p)\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

and

$$\omega_{(\alpha)}(v_{1(\alpha)}, \cdots, v_{k(\alpha)}) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \sum_{\sigma(i_1, \cdots, i_k)} \mathrm{sign}(\sigma) (a_{i_1, \cdots, i_k})_{(\alpha)} v^{\sigma(i_1)}_{1\,(\alpha)} \cdots v^{\sigma(i_k)}_{k\,(\alpha)}$$

18. Lecture on Dec. 12. – **Differential forms on smooth manifolds**

- **Definition:** A smooth manifold M is *orientable* if there exists an atlas $\mathcal{A}$ such that for each pair $\alpha, \beta \in I$ with $\phi(U_\alpha) \cap \phi(U_\beta) \neq \emptyset$ the differential

$$(\phi_\beta^{-1} \circ \phi_\beta)_* : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta$$

has positive determinant. Otherwise M in *non-orientable.*

- If $\omega \in \Omega^n(M)$, M oriented, $n = \dim(M)$ has compact support, $K \subset M$, then this form can be integrated over M as follows: Suppose first that $K \subset \phi_\alpha(U_\alpha)$ for some $\alpha \in I$ and $\{x^i\}$, $i = 1, \cdots, n$, cartesian coordinates on $\mathbb{R}^n$, then we define

$$\int_M \omega = \int_{U_\alpha} \omega_\alpha = \int_{U_\alpha} a_\alpha \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \int_{U_\alpha} a_\alpha \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n$$

where the last expression is the Lebesgue integral defined for continuous functions on U_α. The last step in the above definition proceeds through evaluation of ω_α on an infinitesimal hypercube in $\mathbb{R}^n$ spanned by the vectors $\mathrm{dx}^{\mathrm{i}}\partial_{x^i}$ where the dx^{i} are the coordinate differentials.

- **Remark** Under a change of coordinates, $\phi = (\phi_\alpha^{-1} \circ \phi_\beta) : U_\beta \to U_\alpha$ such that ϕ_* has positive determinant with $\{x^i = f^i(y)\}$ coordinates on U_α and $\{y^i\}$

coordinates on U_β we have

$$\begin{aligned}\int_{U_\beta} a_\beta \mathrm{d}y^1\mathrm{d}y^2\cdots\mathrm{d}y^n &= \int_{U_\beta}\omega_\beta = \int_{U_\beta} f^*\omega_\alpha \\ &= \int_{U_\beta} a_\alpha(f(y))f^*\mathrm{d}x^1\wedge\cdots f^*\mathrm{d}x^n \\ &= \int_{U_\alpha} a_\alpha(f)\mathrm{d}f^1\wedge\cdots\mathrm{d}f^n \\ &= \int_{U_\alpha} a_\alpha(x)\mathrm{d}x^1\wedge\cdots\mathrm{d}x^n \\ &= \int_{U_\alpha} a_\alpha(x)\mathrm{d}x^1\cdots\mathrm{d}x^n.\end{aligned}$$

- If the support K is not contained in any coordinate nbhd $\phi_\alpha(U_\alpha)$ we construct a *partition of unity* $\{\varphi_i\}$ *subordinate to* the covering $\{U_\alpha\}$ and define

$$\int_M \omega := \sum_{i=1}^m \int_M \varphi_i\omega$$

.

- **Remark** The convergence of the above sum is guaranteed by the assumption of compactness.
- In order to parametrise manifolds with boundary we consider maps from open sets in $H^n = \{x^1,\ldots,x^n \in \mathbb{R}^n | x_1 \geq 0\}$.
- **Definition** An open set, V, in H^n is the intersection of an open set $U \subset \mathbb{R}^n$ with H^n. A function $f : V \to \mathbb{R}$ is differentiable if there exists an open set $U \subset \mathbb{R}^n$ such that $V \subset U$ together with a differentiable function $\bar{f} : U \to \mathbb{R}$ such that $\bar{f}|_V = f|_V$.
- A smooth manifold with boundary is then defined in complete analogy with a smooth manifold without boundary by replacing $\mathbb{R}^n$ by H^n everywhere.
- A point $P \in M$ is on the boundary ∂M if for some parametrisation $\phi : V \subset H^n \to M$ we have $\phi(0, x^2, \cdots, x^n) = P$.
- **Lemma** This definition of a point on ∂M is independent of the choice of parametrisation.
- **Proposition** The boundary ∂M of an n-dimensional smooth manifold with boundary is an $(n-1)$-dimensional smooth manifold. Furthermore,the orientation on M induces an orientation on ∂M.
- Let ω be an $(n-1)$-form on a smooth manifold M of dimension n with boundary. Then $\mathrm{d}\omega$ can be integrated on M.
- **Theorem (Stokes)** Let M be a smooth, compact, oriented manifold of dim n with boundary and $i : \partial M \to M$ be the inclusion map of the boundary into M. Then for $\omega \in \Omega^{(n-1)}$ we have

$$\int_{\partial M} i^*\omega = \int_M \mathrm{d}\omega.$$

19. Lecture on Dec. 17 – **Abstract manifolds, topology** [Jä-T, Q, Jä-V]

- So far, we considered submanifolds of $\mathbb{R}^n$ and spaces with a smooth atlas as a generalization. The submanifold definition is alright, but many spaces we want to consider appear naturally without an embedding into $\mathbb{R}^n$ and it can be a non-trivial task to find such an embedding. An example where this happens are Grassmannians like $\mathbb{CP}^n$, the space of complex lines in $\mathbb{C}^{n+1}$. We have seen that this space can be covered by a smooth atlas. The correct definition of an abstract manifold should have the following properties:
    1. There should be a natural way in which submanifolds appear as abstract manifolds.
    2. Abstract manifolds should be spaces which enjoy all the properties that submanifolds of $\mathbb{R}^n$ have. So far, we have only discussed a semilocal property (locally Euclidean and smooth transition functions). There are additional requirements of a more global nature which pertain to the *topology* of the space. For example, they should admit a metric (i.e. the topology should be induced by a metric, see below).
- **Non-example:** Consider the set $M = (-\infty, 0) \cup (0, \infty) \cup \{a, b\}$. There are obvious maps φ_a, φ_b identifying the subsets $U_a = M \setminus \{b\}$ and $U_b = M \setminus \{a\}$ with $\mathbb{R}$. The transition function
$$\varphi_b \circ \varphi_a^{-1} : \varphi_a(U_a \cap U_b) = \mathbb{R} \setminus \{0\} \longrightarrow \varphi_b(U_b \cap U_a) = \mathbb{R} \setminus \{0\}$$
is the identity (and smooth). Still, M does not feel like a manifold: A smooth function on M has to attain equal values on a and b.
- **Definition:** A *topology* on a set M is a subset $\mathcal{O} \subset \mathcal{P}(M)$ of the power set such that
    1. $\emptyset, M \in \mathcal{O}$
    2. $\mathcal{O}$ closed under finite intersections: $U_1, \ldots, U_n \in \mathcal{O} \Longrightarrow U_1 \cap \ldots U_n \in \mathcal{O}$.
    3. $\mathcal{O}$ closed under unions: $U_i \in \mathcal{O}, i \in I$, then $\cup_{i \in I} U_i \in \mathcal{O}$

  Sets in $\mathcal{O}$ are called *open*, $A \subset M$ is closed if $M \setminus A$ is open. A subset $V \subset M$ containing p is a *neighborhood* of p if there is an open set $U \subset V$ containing p.
- **Remark:** There are other versions of the same notion, for example in terms of collections of closed (as opposed to open) sets.
- **Definition:** Let $V \subset M, \mathcal{O}$ be arbitrary. The closure $\overline{V}$ of V is the smallest closed subset of M containing V. The interior $\mathring{V}$ is the largest open subset contained in V.
- **Definition:** A map $f : X \longrightarrow Y$ between topological spaces is *continuous* if $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}_Y$. f is a *homeomorphism* if it is bijective, continuous and f^{-1} is continuous.
- **Examples:** The powerset and $\{\emptyset, M\}$ are topologies.
- **Example:** If M has a metric d, then $B_\varepsilon(x)$ denotes the ε-ball around x. Then
$$\mathcal{O}_d = \{U \subset M \,|\, \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subset U\}$$
defines a topology. If X, Y are metric spaces, then the old definition of continuity coincides with the new one.
- **Remark:** If $\mathcal{O}_1, \mathcal{O}_2$ are two topologies on M, then the intersection of the two topologies is again a topology. Therefore, given a collection of subsets of M one can ask for the smallest topology on M which contains the subset. In particular, if a set, like M in the non-example, is covered by charts, then we can define a topology which makes the parametrizations into homeomorphisms.

- **Definition:** A topological space $(M, \mathcal{O})$ is *Hausdorff* if for all $p, q \in M$ there is a pair of open sets $p \in U_p, q \in U_q$ so that $U_p \cap U_q = \emptyset$.
- **Example:** Let M be a topological space and $\sim$ an equivalence relation on M. Then the quotient topology on $M/\sim$ is the largest topology so that $M \longrightarrow M/\sim$ is continuous. Very often, quotient topologies are not Hausdorff.
- **Example:** Let M be a set. Let U be open if and only if the complement is finite. This defines a topology which is not Hausdorff as soon as M is infinite.
- **Lemma:** If $N \subset M$ is a subspace of a Hausdorff space, then N is Hausdorff.
- **Example:** Let $(M, \mathcal{O}_M)$ be a topological space and $N \subset M$ a subset. Then N carries a topology: The subspace topology is the smallest topology so that the inclusion $N \longrightarrow M$ is continuous, i.e. $V \subset N$ is open if and only if there is an open $U \subset M$ so that $U \cap N = V$.
- **Remark:** $\mathbb{R}^n$ with the metric topology has another property It has a countable basis.
- **Definition:** Let $M, \mathcal{O}$ be a topological space and $\mathcal{B} \subset \mathcal{O}$. Then $\mathcal{B}$ is a *basis* for $\mathcal{O}$ if every open set is a union of sets in $\mathcal{B}$.
- **Example:** Let $\mathcal{B} = \{B_\varepsilon(x) \,|\, \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}^n\}$. This is a basis of the metric topology of $\mathbb{R}^n$.
- **Lemma:** If $A \subset M$ and M has a countable basis, then the subspace topology on A has a countable basis (second countable).
- **Definition:** A *manifold* M of dimension n is a topological space which is Hausdorff and has a countable basis together with an atlas $(\varphi_i : U_i \longrightarrow V_i)$ where V_i is an open subset of $\mathbb{R}^n$ and $U_i \subset M$ is open so that the transition functions
$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$
are smooth.

  Two atlases are *equivalent* if their union is still a smooth atlas. More precisely, a manifold is then a topological space with an equivalence class of smooth atlases. Alternatively, one can use one maximal atlas.
- **History:** This definition, in its version for Riemann surfaces, was given by Hermann Weyl and later generalized/popularized by Hassler Whitney.
- The definition above defines C^∞-manifolds. Similarly, one can define C^k-manifolds for $k \geq 0$. The case $k = 0$ is fundamentally different from the case $k \geq 1$. (A theorem of Whitney asserts that a C^k atlas is equivalent to a smooth atlas for $k \geq 1$. This is wrong for $k = 0$.)
- **Fact:** A theorem due to Hassler Whitney states that every manifold of dimension n can be realized as a submanifold of $\mathbb{R}^{2n+1}$ which is closed as a subset. This seems to say that the notion of manifold is not new. This, however, is misleading. Keeping track of embeddings can be intricate. More importantly, the above definition is versatile: Complex manifolds have a similar definition, one only needs to replace smooth transition function by holomorphic transition function. It is a consequence of the maximum principle for holomorphic functions that no compact complex manifold is a submanifold of $\mathbb{C}^N$.

20. Lecture on Dec. 19 – **Compactness, Partition of unity, paracompactness**

- It is not clear why second countability matters in the definition of an abstract manifold. On a technical level, many things that we like need this axiom for the validity of

  - Sard's theorem: Regular values of smooth maps are dense. Most theorem which are called *transversality theorems* (many of them due to R. Thom) rely on this.
  - Manifolds are metrizable, i.e. there is a metric on M so that the topology on M is induced from the metric.
- **Definition:** A topological space $(M, \mathcal{O}_M)$ is *compact* if for every collection $U_i, i \in I$, of open sets covering $M = \cup_i U_i$ there is a finite subcollection $U_{i_1}, \ldots, U_{i_k}$ covering M.
- **Warning:** Several sources require in addition, that M is Hausdorff. We do not.
- **Warning:** Compact spaces are not sequentially compact (every sequence contains a convergent subsequence), in general. This is the case when the space is first countable. This means that for each point there is a countable collection of neighborhoods $N_i(x)$ so that every neighborhood contains one of the N_i.
- **Lemma:** Let $f : M \longrightarrow N$ be a continuous map between topological spaces and assume that $X \subset M$ is compact. Then $f(X)$ is compact.
- **Proof:** Let $U_i = V_i \cap f(X)$ be an open cover of $f(X)$ in the subspace topology. Then $f^{-1}(V_i), i \in I$, is an open (f continuous) cover of X. Since X is compact, so $f^{-1}(V_i), i \in I$, admits a finite subcover. From this subcover we obtain a subcover of $U_i, i \in I$, covering $f(X)$.
- **Theorem:** A subset of $\mathbb{R}^n$ with the metric topology is compact if and only if it is closed and bounded.
- **Corollary:** A continuous $f : X \longrightarrow \mathbb{R}$ on a compact X attains its maximum/minimum.
- **Lemma:** Let $A \subset M$ be a closed subset of a compact M. Then A is compact. Conversely, assume that M is Hausdorff and compact, and that A is compact. Then A is closed.
- **Proof:** Let $x \in X \setminus A$. We will show that $X \setminus A$ contains an open set $Xx)$ containing x which is disjoint from A. By Hausdorff, for each $a \in A$ there is a pair of disjoint open sets $U(a) \ni a$ and $V_x(a) \ni x$ and $U(a), a \in A$, induces an open covering of A. By compactness, there is a finite subcover $U(a_1), \ldots, U(a_k)$ whose union contains A. Then $V(x) = V_x(a_1) \cap \ldots V_x(a_k)$ has the desired properties.
- **Lemma:** Assume that X is compact and $A \subset X$ is closed. Then A (with the subspace topology) is compact.
- **Proof:** Let $U_i = V_i \cap A, i \in I$, be an open cover of A (with $V_i \subset X$ open). Then $V_i, i \in I$, together with $X \setminus A$ (A is closed) is an open cover of X, so admits a finite subcover of X. From this one extracts a finite subcover of U_i of A.
- **Theorem:** Let $f : X \longrightarrow Y$ be continuous, bijective and assume that X is compact and that Y is Hausdorff. Then f is a homeomorphism.
- **Proof:** It remains to show that f^{-1} is continuous. One first show that a map between topological spaces is continuous if and only if preimages of *closed* sets are closed. Then one combines the previous three lemmas with this fact to show that $f^{-1} : Y \longrightarrow X$ is continuous.
- **Remark:** Second countablity allows us to probe the manifold effectively with compact sets: Even if M is not compact, there is a countable collection of compact subspaces $K_1 \subset K_2 \subset K_3, \ldots$ so that K_i is a compact manifold with boundary, K_i lies in the interior of K_{i+1}, and $\cup_i K_i = M$. In other words:
- **Theorem:** Every manifold admits a compact exhaustion.
- **Definition:** A topological space $M, \mathcal{O}$ is *paracompact*, if for every open cover $U_i, i \in I$ there is another open cover $V_j, j \in J$ so that V_j is contained in one

of the U_i (refinement) and for every point p in M there is an open set $W_p \ni p$ which intersects only finitely many V_j (locally finite).

- **Theorem:** Manifolds are paracompact.
- **Theorem (Stone):** A metric space is paracompact.
- **Definition:** Let M be a manifold and $U_i, i \in I$, an open covering. A *partition of unity* subordinate to U_i is a locally finite subcover V_j and a collection of smooth functions $f_k : M \longrightarrow [0,1]$ such that the closure of $f_k^{-1}((0,1])$ is contained in one of the V_j, the supports are locally finite, and
$$\sum_k f_k \equiv 1.$$
- **Theorem:** Every open covering of a manifold admits a smooth partition of unity subordinate to the covering.

21. Lecture on Jan. 7 - Smooth functions, smooth maps, tangent vectors, differentials

- **Reminder:** A manifold of dimension n is a second countable Hausdorff space together with an atlas, i.e. a collection $(U_i, \varphi_i)_{i\in I}$ of homeomorphisms $\varphi_i : U_i \subset M \longrightarrow \varphi_i(U_i) \subset \mathbb{R}^n$ with $U_i \subset M$ open such that
  - $\cup_i U_i = M$ and $U_i \subset M$, $\varphi_i(U_i) \subset \mathbb{R}^n$ are open, and
  - $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ is smooth for all $i, j \in I$.
- **Definition:** Let M be a smooth manifold. A function $M \longrightarrow \mathbb{R}$ is *smooth around* x if for a local charts (U, φ), with $x \in U \subset M$ open, the function $f \circ \varphi^{-1}$ is smooth on a neighborhood of $\varphi(x)$. f is smooth if it is smooth everywhere.
- **Fact:** This is independent of the choice of chart around x.
- **Definition:** Let M, N be smooth manifolds. A continuous map $f : M \longrightarrow N$ is *smooth around* x if there are charts (U, φ) around x and (V, ψ) around $f(x)$ so that $\psi \circ f \circ \varphi^{-1}$ is smooth on a neighborhood of $\varphi(x)$. Moreover, f is smooth if it is smooth around every point.
- **Remark:** Again, this is independent of the choice of the choice of charts. The continuity of f ensures that there is a neighborhood of $\varphi(x)$ on which $\psi \circ f \circ \varphi^{-1}$ is defined.
- **Reminder:** Let $M \subset \mathbb{R}^n$ be a submanifold and $x \in M$. The tangent space $T_xM \subset \mathbb{R}^n$ is defined using a local parametrisation $\alpha : U \longrightarrow M \subset \mathbb{R}^N$ as $d\alpha(\mathbb{R}^N) \subset \mathbb{R}^N$. Tangent vectors are then elements of tangent space. Smoothness of vector fields can then be defined in terms of smoothness of maps $\mathbb{R}^m \supset U \longrightarrow \mathbb{R}^N$. The definition of tangent vectors of abstract manifolds requires slightly more care.

  We will give three different definitions of a tangent vector of a smooth manifold M. These definitions are equivalent and should be viewed as a package.
- A smooth curve at p is a smooth map $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = p$. Two curves γ_0, γ_1 at p are equivalent ($\gamma_0 \sim \gamma_1$) if for a local chart $\varphi : U \subset M \longrightarrow \mathbb{R}^n$ around p we have
$$\frac{d}{dt}\bigg|_{t=0} (\varphi \circ \gamma_0)(t) = \frac{d}{dt}\bigg|_{t=0} (\varphi \circ \gamma_1)(t).$$
- This is independent of φ and $\sim$ is an equivalence relation.

- **Definition (geometric):** The *(geometric) tangent space* at p is
$$T_pM := \{\text{smooth curves at } p\}/\sim .$$
Elements of T_pM are tangent vectors.
- **Remark:** It is not immediately clear that the tangent space is a vector space, instead of explaining this directly we will use the equivalence of the geometric definition with the following ones.
- Let $C^\infty(M)$ be the ring (with pointwise addition and multiplication) of smooth real valued functions on M.
- **Definition (algebraic):** A *derivation at p* is a linear map
$$v : C^\infty(M) \longrightarrow \mathbb{R}$$
which satisfies the Leibniz rule, i.e.
$$v(fg) = v(f)g(p) + f(p)v(g).$$
The vector space T_pM of all derivations at p is the (algebraic) *tangent space* of M at p.
- **Remark:** Every derivation at p vanishes on constant functions. Moreover, if $f = g$ on a neighbourhood V of p then $v(f) = v(g)$. This is shown using a smooth function h with support in V which is $\equiv 1$ on a neighborhood of p in V. (Then $0 = v(h(f-g)) = v(f) - v(g)$.) In particular, instead of $C^\infty(M)$ we could have used $\mathcal{E}_p^\infty(M) = C^\infty(M)/\sim$ with $f \sim g$ if and only if f, g coincide on a neighborhood of p. Elements of $\mathcal{E}_p^\infty(M)$ are *germs of functions at p*. From now on we will frequently consider smooth functions defined on neighborhoods of p.

    In particular, we are allowed to restrict a function to coordinate charts when we want to determine how a derivation reacts to the function.
- **Lemma:** Let $U \subset \mathbb{R}^n$ be a ball around 0 and $f : U \longrightarrow \mathbb{R}$ smooth. Then there are smooth functions $f_i : U \longrightarrow \mathbb{R}$ such that
$$f(x) = f(0) + \sum_i x^i f_i(x)$$
and $f_i(0) = \frac{\partial f}{\partial x^i}(0)$. Here x^i is the i-th coordinate, not a power of something.
- **Proof:**
$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx^1, \dots, tx^n)\, dt \\ &= \int_0^1 \sum_i x^j \frac{\partial f}{\partial x^j}(tx^1, \dots, tx^n)\, dt \\ &= \sum_j x^j \underbrace{\int_0^1 \frac{\partial f}{\partial x^j}(tx^1, \dots, tx^n)\, dt}_{=: f_j(x)} . \end{aligned}$$
- Using this Lemma one shows that a derivation v at p is determined by $v(x^1), \dots, v(x^n)$ where x^i are local coordinates from a local parametrization (φ, U) near p such that $x^i(p) = 0$ for all i. Then $\dim(T_pM) = n$. For $i = 1, \dots, n$ the derivation v with $v(x^i) = \delta_{ij}$ is denoted by $\frac{\partial}{\partial x^i}$.
- **Definition:** A (physicists) tangent vector of M at p is a map
$$v : \mathcal{D}_p(M) = \{\text{local charts around } p\} \longrightarrow \mathbb{R}^n$$

such that

$$v((V,\psi)) = D_p(\psi \circ \varphi^{-1})v((U,\varphi)). \tag{6}$$

- **Remark:** v is determined on its value on one chart. One could phrase this as *a tangent vector is something that transforms like it should under coordinate transformations.*

  The vector space of such maps obviously has dimension $\leq n$ and $= n$ because of the chain rule.
- To obtain an algebraic tangent vector from a geometric one:

$$\begin{aligned}\{\text{curves at } p\}/\sim &\longrightarrow \{\text{derivation at } p\}\\ [\gamma] &\longmapsto \left(f \longmapsto \frac{d}{dt}\bigg|_{t=0} (f\circ\gamma)(t)\right).\end{aligned}$$

- To obtain a physicists tangent vector from an algebraic one:

$$\begin{aligned}\{\text{derivation at } p\} &\longrightarrow \{\text{physicists tangent vectors}\}\\ v &\longmapsto \big((U,\varphi) \longmapsto (v(x^i))_i\big)\end{aligned}$$

  where x^i are the coord. around p from φ.
- To obtain a geometric tangent vector from an physicists tangent vector:

$$\begin{aligned}\{\text{physicists tangent vectors}\} &\longrightarrow \{\text{curves at } p\}/\sim\\ v &\longmapsto \big[t \longmapsto \varphi(\varphi^{-1}(p) + tv((U,\varphi)))\big]\end{aligned}$$

  with $|t| < \varepsilon$ so that this is well-defined.
- **Fact:** All these maps are well-defined, bijective and the second map is a linear isomorphism. Moreover, passing from geometric to algebraic, then to the physicist version, then back to the geometric, a geometric tangent vector gets mapped to itself, etc.
- As we have three definitions of tangent vectors, there are three definitions of the differential of a smooth map $F : M \longrightarrow N$ between smooth manifolds at $p \in M$.
- **Definition (geometric, curves):** The differential D_pF is

$$\begin{aligned}D_pF : T_pM = \{\text{smooth curves at } p\}/\sim &\longrightarrow T_pN\\ [\gamma] &\longmapsto [F\circ\gamma].\end{aligned}$$

- **Definition (algebraic, derivations):** The differential D_pF is

$$\begin{aligned}D_pF : T_pM = \{\text{derivations at } p\} &\longrightarrow T_pN\\ v &\longmapsto \big(g \longmapsto v(g\circ F) =: v(F^*g)\big).\end{aligned}$$

- **Definition (physicists, transformation rule):** Let M be a manifold of dimension n and (V,φ) a local parametrization of M around $p \in M$. The differential D_pF is

$$\begin{aligned}D_pF : T_pM = \{v : \mathcal{D}_p(M) \longrightarrow \mathbb{R}^n \text{ and transformation rule}\} &\longrightarrow T_pN\\ v &\longmapsto \left((U,\varphi) \longmapsto D_{\varphi(p)}\left(\psi\circ F\circ\varphi^{-1}\right)(v((U,\varphi)))\right).\end{aligned}$$

- All these versions are well-defined and compatible with the identifications of the various definitions of tangent spaces discussed above.

- **Lemma (chain rule):** Let $F : M \longrightarrow M'$ and $G : M' \longrightarrow M''$ be smooth maps between smooth manifolds. Then
$$D_p(G \circ F) = \big(D_{F(p)}G\big) \circ (D_pF)\,.$$

Lecture on Jan. 9 – **Tangent bundle, Lie groups**

- **Definition:** Let M be a smooth manifold of dimension n. The set
$$TM := \bigcup_{p\in M} T_pM$$
is the tangent bundle of M. There is an obvious map pr from TM to M taking a tangent vector in T_pM to $p \in M$.
- Our goal now is to give TM the structure of a $2n$-manifold where $n = \dim(M)$. such that pr is smooth.
- Let $(U_i, \varphi_i)_{i\in I}$ be an atlas for M consisting of local charts $\varphi_i : U_i \longrightarrow \mathbb{R}^n$ of M. Then
$$\bigcup_{i\in I} \mathrm{pr}^{-1}(U_i) = TM.$$
We define local coordinates on $\mathrm{pr}^{-1}(U_i)$ by
$$\begin{aligned}\widehat{\varphi}_i : \mathrm{pr}^{-1}(U_i) \subset TM &\longrightarrow \varphi_i(U_i) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \text{ open}\\ v &\longmapsto (\varphi_i(\mathrm{pr}(v)), v((U_i, \varphi_i))).\end{aligned}$$
This is a bijective map. We choose the topology on TM so that $\widehat{\varphi}_i$ is a homeomorphism. Since M admits a countable atlas and open sets in $\mathbb{R}^{2n}$ have a countable basis it follows that TM is second countable with this topology. (The assumption that I is countable is only convenient, not necessary). It is easy to prove that it is Hausdorff.

  We used the physicist definition of tangent vectors because the transformation behavior inherent in this definition will facilitate understanding the coordinate transformation for the atlas $((U_i \times \mathbb{R}^n), \widehat{\varphi}_i)_{i\in I}$. First note, that
$$\widehat{\varphi}_i\left(\mathrm{pr}^{-1}(U_i) \cap \mathrm{pr}^{-1}(U_j)\right) = \varphi_i\left(U_i \cap U_j\right) \times \mathbb{R}^n.$$
The transition function $\widehat{\varphi}_j \circ \widehat{\varphi}_i^{-1}$ is (by (6))

(7)
$$\begin{aligned}\varphi_i\left(U_i \cap U_j\right) \times \mathbb{R}^n &\longrightarrow \varphi_j\left(U_i \cap U_j\right) \times \mathbb{R}^n\\ (x, w) &\longmapsto \left(\varphi_j \circ \varphi_i^{-1}(x), D_x(\varphi_j \circ \varphi_i^{-1})(w)\right).\end{aligned}$$
We have expressed the new transition function in terms of the transition function of the atlas we started with. Hence, $\widehat{\varphi}_j^{-1} \circ \widehat{\varphi}_i$ is smooth.

  This shows that for a n-manifold M, the *tangent bundle* TM is a smooth manifold. Moreover, the projection map $\mathrm{pr} : TM \longrightarrow M$ is smooth and $DF : TM \longrightarrow TN$ is smooth for $F : M \longrightarrow N$ a smooth map between smooth manifolds.
- **Definition:** A *vector field* X on a manifold is a smooth map $X : M \longrightarrow TM$ such that $\mathrm{pr} \circ X = \mathrm{id}_M$.
- In the following Lemma/Definition, we use the algebraic view point on tangent vectors.
- **Lemma:** If X, Y are vector fields, then so is the *commutator* $[X, Y]$ with $[X, Y]\cdot f := X(Y(f)) - Y(X(f))$. Moreover, the Jacobi identity

(8)
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

- **Proof:** Direct computations show that $[X, Y]$ is a derivation at each point and the Jacobi identity.
- **Remark:** In terms of local coordinates, a vector field can be written in the form
$$X = \sum_{i=1}^{n} a^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}$$
where a^i are smooth functions and $\frac{\partial}{\partial x^i}$ is the derivation (see Remark close to previous Lemma) with $\frac{\partial}{\partial x^i} x^j = \delta_i^j$. Then, for $Y = \sum_j b^j \partial_j$
$$[X, Y] = \sum_{i,j} \left(\left(a^i \frac{\partial}{\partial x^i} b^j \right) \frac{\partial}{\partial x^j} - \left(b^j \frac{\partial}{\partial x^j} a^i \right) \frac{\partial}{\partial x^i} \right)$$
- **Reminder:** In Lecture 7, this was discussed for open sets of Euclidean space. The interpretations and Lemmas from that lecture carry over to the present setting.
- **Remark:** Examples of abstract manifolds are provided by submanifolds. It is common to use parametrisations for submanifolds (i.e. maps into the submanifold) and charts for abstract manifolds (i.e. maps defined on open subsets of the manifold).
- The most important examples of manifolds are arguably *Lie groups*.
- **Definition:** A *Lie group* G is a smooth manifold so that the multiplication $G \times G \longrightarrow G$ and the inversion $G \longrightarrow G$ is smooth.
- **Examples:** $\mathrm{Gl}(\mathbb{R}, n), \mathrm{O}(n), \mathrm{Sl}(n, \mathbb{R}), \mathrm{SO}(n), \mathrm{Gl}(n, \mathbb{C}), \mathrm{U}(n), \mathrm{Sl}(n, \mathbb{C}), \mathrm{SU}(n), \dots$
- **Extended example:**Let G be a Lie group and $g \in G$. Then
$$\begin{aligned} c_g : G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned}$$
is smooth and we have the rule $c_g \circ c_{g'} = c_{gg'}$. In particular, c_g is a diffeomorphisms with inverse $c_{g^{-1}}$. Moreover, $c_g(e) = e$. Thus, we can differentiate c_g at e and we get a linear map
$$\mathrm{Ad}_g := D_e c_g : T_e G \longrightarrow T_e G.$$
This is an isomorphism of vector spaces with inverse $Ad_{g^{-1}}$ and by the chain rule the map
$$\begin{aligned} Ad : G &\longrightarrow \mathrm{Aut}(T_e G) \\ g &\longmapsto D c_g = \mathrm{Ad}_g \end{aligned}$$
is a group homomorphism which is smooth. Note that $\mathrm{Aut}(T_e G)$ is an open subset of the vector space of all endomorphisms of $T_e G$. In particular, $\mathrm{Aut}(T_e G)$ is a Lie group. The tangent space at E (the identity automorphism) is the space of all endomorphisms of $T_e G$. Moreover, $Ad_e = \mathrm{id}_{T_e G}$. Thus, we can differentiate Ad at e and we get a linear map
$$\begin{aligned} \mathrm{ad} : T_e G &\longrightarrow T_{\mathrm{id}}(\mathrm{Aut}(T_e G)) = \mathrm{End}(T_e G) \\ X &\longmapsto \left(Y \longmapsto \left(D_e \mathrm{Ad}(X) \right)(Y) = \mathrm{ad}(X)(Y) \right). \end{aligned}$$
This map is called *adjoint representation of* $T_e G$.

LECTURE ON JAN. 14 – **More on Lie groups, manifolds with boundary, orientation, Stokes**

- **Reminder:** Let X, Y be smooth vector fields on M and $F : M \longrightarrow M$ a diffeomorphism. Then

$$DF([X,Y]) = [DF(X), DF(Y)]. \tag{9}$$

This is a consequence of the local computations in Lecture 16.

- **more on Lie groups:** A vector field X on a Lie group G is *left-invariant* if $l_{g*}X = X$ for all $g \in G$ ($l_g(h) = gh$ denotes the left multiplication with g). Let $\mathfrak{g}$ denote the vector space of left-invariant vector fields on G. Then

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_eG \\ X &\longmapsto X(e) \end{aligned}$$

is an isomorphism of vector spaces since by left-invariance $X(g) = l_{g*}(X(e))$ for a left invariant vector field X. By (9) the same is true for $[X, Y]$ when X, Y are left-invariant. Thus, the space of left-invariant vector fields forms a *Lie-algebra*, i.e. a real vector space with an antisymmetric, $\mathbb{R}$-bilinear pairing $[\cdot,\cdot]$ satisfying the Jacobi identity.

By standard theorems on solvability of ODE's, given left invariant vector field X there is a curve

$$\begin{aligned} \mathbb{R} &\longrightarrow G \\ t &\longmapsto \alpha^X(t) \end{aligned}$$

so that $\frac{d}{dt}\big|_{t=t_0} \alpha^X(t) = X(\alpha(t_0))$ and $\alpha^X(0) = e$. Left-invariance allows to describe the flow of X completely using α^X.

$$\begin{aligned} \varphi : \mathbb{R} \times G &\longrightarrow G \\ (t,h) &\longmapsto h\alpha^X(t) \end{aligned}$$

satisfies $\frac{d}{dt}\varphi(t,h) = X(h\alpha^X(t))$ since by the chain rule

$$\frac{d}{dt}\bigg|_{t=t_0} h\alpha^X(t) = l_{h*}X(\alpha^X(t_0)) = X(h\alpha^X(t_0)).$$

In particular, $\alpha^X(s+t) = \alpha^X(s)\alpha^X(t)$. One says that α^X parametrises a 1-parameter subgroup.

- **Example:** If G is a subgroup of $\mathrm{Gl}(\mathbb{R}, n)$, then $X(A) = A \cdot X$ and

$$\alpha^X(t) = \exp(tX) = \sum_{j=0}^{\infty} \frac{t^j X^j}{j!}.$$

- **continuation:** It is common to write

$$\begin{aligned} \exp : T_eG = \mathfrak{g} = LG &\longrightarrow G \\ X &\longmapsto \alpha^X(1). \end{aligned}$$

- **Fact:** This is smooth and $D\exp = \mathrm{id}_{\mathfrak{g}}$.

- **Proof:** Note that $\alpha^{tX}(1) = \alpha^X(t)$ and $T_X\mathfrak{g} = \mathfrak{g}$. Thus, if exp is differentiable, then $D\exp(Y) = Y$. Consider
$$\begin{aligned}\mathbb{R} \times G \times \mathfrak{g} &\longrightarrow G \times \mathfrak{g}\\ (t, g, X) &\longmapsto (g\alpha^X(t), X)\end{aligned}$$
This is the flow of the vector field $\underline{X}(g, X) = (X(g), 0)$ on $G \times LG$ which is smooth, the restriction to $1 \times e \times LG \longrightarrow G$, which is exp is also smooth.
- **Naturality of** exp**:** Let $f : G \longrightarrow H$ be a homomorphism of Lie group (smooth homomorphism). Then $Df : \mathfrak{g} \longrightarrow \mathfrak{h}$ satisfies
$$f(\exp_G(X)) = \exp_H(Df(X)).$$
because $f \circ \alpha^X$ parametrises the 1-parameter subgroup associated to $Df(X)$.
- **Corollary:** A smooth homomorphism between connected Lie groups is determined by its differential at the identity element.
- **Proof:** Exponential maps are local diffeomorphism at the origin, and they are natural. Thus, there is a restriction of the unit element on which the claim is true. This implies the claim since every connected Lie group is generated by every neighborhood of the identity (i.e. every element is a product of group elements in a fixed neighborhood of the identity). To see this let U be a neighborhood of id. Consider $U \cap U^{-1} = V$ and let $\cup_j V^j = H$. This is a subgroup of G, and each part of the union is open: V is open, $V^2 = \cup_{v\in V} vV$ is open, etc. Then G is the disjoint union $\cup_{[g]\in G/H} gH$, so the complement $\cup_{[e]\neq[g]\in G/H}$ is open, i.e. H is closed. Then $G = H$ since G is connected.
- **Remark:** We want to better understand $\mathrm{ad}(X)(Y)$ in the case when $G \subset \mathrm{Gl}(\mathbb{R}, n)$ is a matrix Lie group. By definition
$$\mathrm{Ad}(g)(Y) = \left.\frac{d}{dt}\right|_{t=0} g\alpha^Y(t)g^{-1}$$
and
$$\mathrm{ad}(X)(Y) = \left.\frac{d}{dt}\right|_{s=0} \left(\left.\frac{d}{dt}\right|_{t=0} \alpha^X(s)\alpha^Y(t)\alpha^X(s)^{-1} \right)$$
Then, since we know α^X, α^Y explicitly, we obtain
$$\mathrm{ad}(X)(Y) = XY - YX.$$
By part a) of the second Lemma from the 7th Lecture (p. 5), it follows that $\mathrm{ad}(X)(Y) = [X, Y]$. This is true in general, not only for matrix Lie groups (cf. [BtD], p.18f)
- **References:** Some of you have asked for references on Lie groups. I like [BtD]. There are many other books on Lie groups/Lie algebras, and I am by no means an expert. But here are a few more references:
  - Fulton-Harris, *Representation theory*, Springer GTM 128
  - Bump, *Lie groups*, Springer GTM 225
  - more of an introduction: Tapp, *Matrix groups for undergraduates*
  - more comprehensive : Knapp, *Lie groups beyond an introduction.*
- **Definition:** A manifold with boundary M of dimension n is a topological space which is Hausdorff, second countable together with a family of homeomorphisms $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0\}, i \in I$ defined on open sets of M onto open subsets of $\{x_1 \leq 0\}$, so that transition functions are smooth.

- **Definition:** A manifold (with boundary) is oriented when the atlas used in its definition has the property that all transition functions has everywhere positive Jacobi determinant.
- **Fact/Definition:** Let M be an oriented manifold with boundary of dimension $n+1$. Then ∂M is an oriented manifold of dimension n. The orientation of ∂M is determined by the outward normal first convention (explaining the choice $\{x_1 \leq 0\}$ in the definition of a manifold with boundary.
- **Remark:** We want to use differential forms to study manifolds. The extirior differential of smooth forms was already defines. We also know:
- **Stokes theorem:** Let M be a compact manifold with boundary of. Then

  $$\int_M d\omega = \int_{\partial M} \iota^* \omega$$

  for every n-form ω on M. Here $\iota : \partial M \longrightarrow M$ denotes the (smooth) inclusion map (which is often omitted).
- **Goal:** We want to understand $H^n_{dR}(M, \mathbb{R})$ for a smooth, compact, oriented manifold. We already learned about a Poincaré-Lemma, but that one is insufficient for our current purpose. We work on a smooth manifold M and compactly supported forms $\omega \in \Omega^n_c(M)$, i.e. smooth forms vanishing outside a compact set. Clearly, d maps compactly supported forms to compactly supported forms, and we still have $d^2 = 0$.

Lecture on Jan. 16 – **Compactly supported cohomology, Poincare Lemma**

- **Reference:** [BT], Section I.4
- **Definition:** The k-th compactly supported deRham cohomology of M is

  $$H^k_c(M) = \frac{\ker(d : \Omega^k_c(M) \longrightarrow \Omega^{k+1}_c(M)}{\operatorname{im}(d : \Omega^{k-1}_c(M) \longrightarrow \Omega^k_c(M)}.$$
- **Poincaré-Lemma for compactly supported forms:**

  $$H^k_c(\mathbb{R}^n) = \begin{cases} 0 & k \neq n \\ \mathbb{R} & k = n \end{cases}$$

  The isomorphism for $k+n$ is given by integration (which is well-defined since forms are compactly supported and Stokes theorem).
- **Remark:** This Lemma is obvious for $n = 0$ and $n = 1$.
- **Warning:** Let $f : M \longrightarrow N$. Then there is a well-defined map $f^* : H^*(N) \longrightarrow H^*(M)$. This is not the case for H^*_c since preimages of compact sets are not compact in general. This is the case for the projection $M \times \mathbb{R} \longrightarrow M$, for example.
- **Integration over the fiber:** In some cases, for example $\pi : M = N \times \mathbb{R} \longrightarrow N$ and a smooth manifold M, one can define a map

  $$\pi_* : \Omega^k_c(M \times \mathbb{R}) \longrightarrow \Omega^{k-1}_c(M)$$

  called *integration over the fiber*. For this note that on $M \times \mathbb{R}$, all k forms are sums (more than two summands in general) of two types of k-forms:

  $$(A)\omega = f(x,t)\pi^*\eta \text{ with } \eta \in \Omega^k(M)$$
  $$(B)\omega = f(x,t)\pi^*\eta \wedge dt \text{ with } \eta \in \Omega^{k-1}(M)$$

where f is a compactly supported function. Integration over the fiber maps forms of type (A) to zero. For forms of type (B)

$$f(x,t)\pi^*\eta \wedge dt \longmapsto \int_{-\infty}^{\infty} f(x,t)dt \cdot \eta$$

- **Lemma:** Integration over the fiber is a chain map, i.e. $\pi_* \circ d = d \circ \pi_*$.
- **Proof:** For forms of type (A) we have

$$\begin{aligned} d\left(f(x,t)\pi^*\eta\right) &= \frac{\partial f}{\partial t}(x,t)dt \wedge \pi^*\eta + \text{ Type}(A)\text{-forms} \\ &= (-1)^k \frac{\partial f}{\partial t}(x,t)\pi^*\eta \wedge dt + \text{ Type}(A)\text{-forms} \end{aligned}$$

Hence, we get

$$\pi_* \circ d\left(f(x,t)\pi^*\eta\right) = (-1)^k (f(x,\infty) - f(x,-\infty))\eta = 0.$$

This is the result of $d \circ \pi_*\left(f(x,t)\pi^*\eta\right)$. The case of forms of type (B) is similar.
- **Remark:** As a consequence, π_* defines a map

$$H_c^k(M \times \mathbb{R}) \longrightarrow H_c^{k-1}(M).$$

Now pick a function $e : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support and $\int_{\mathbb{R}} e(t)dt = 1$. Then one can define the map

$$\begin{aligned} e_* : \Omega_c^k(M) &\longrightarrow \Omega_c^{k+1}(M \times \mathbb{R}) \\ \eta &\longmapsto \pi^*\eta \wedge e(t)dt \end{aligned}$$

Obviously, $\pi_* \circ e_* = \mathrm{id}_{\Omega_c^k(M)}$ but, of course,

$$e_* \circ \pi_* \neq \mathrm{id}_{\Omega_c^{k+1}(M \times \mathbb{R})}.$$

Since e_* is also a chain map, both π_* and e_* define maps between compactly supported cohomology groups.

Our goal is to prove that the induced maps on cohomology are inverses of each other. For this, one defines the following homotopy operator (for the terminology see (2))

$$\begin{aligned} K : \Omega_c^k(M \times \mathbb{R}) &\longrightarrow \Omega_c^{k-1}(M) \\ \omega = f(x,t)\pi^*\eta \text{ of type } (A) &\longmapsto 0 \\ \omega = f(x,t)dt \wedge \pi^*\eta \text{ of type } (B) &\longmapsto \left(\int_{-\infty}^{t} f(x,s)ds - F(x)A(t)\right)\pi^*\eta \end{aligned}$$

where $F(x) = \int_{-\infty}^{\infty} f(x,s)ds$ and $A(t) = \int_{-\infty}^{t} e(s)ds$.
- **Proposition:** $1 - e_*\pi_* = (-1)^{k-1}(dK - Kd)$ on $\Omega_c^k(M \times \mathbb{R})$.

- **Proof:** On type (A)-forms

$$
\begin{aligned}
(1-e_*\pi_*)f(x,t)\pi^*\eta &= f(x,t)\pi^*\eta \\
(dK-Kd)f(x,t)\pi^*\eta &= 0 - K\left(\frac{\partial f}{\partial t}dt\wedge\pi^*\eta + \text{ Type (A) forms }\right) \\
&= -(-1)^k K\left(\frac{\partial f}{\partial t}\pi^*\eta\wedge dt + \text{ Type (A) forms }\right) \\
&= \left(\int_{-\infty}^{t}\frac{\partial f}{\partial s}(x,s)ds - \int_{-\infty}^{\infty}\frac{\partial f}{\partial s}(x,s)ds\cdot A(t)\right)\pi^*\eta \\
&= (-1)^{k-1}f(x,t)\pi^*\eta.
\end{aligned}
$$

Now, for forms of type (B):

$$
\begin{aligned}
(1-e_*\pi_*)f(x,t)\pi^*\eta\wedge dt &= f(x,t)\pi^*\eta\wedge dt - e_*\left(F(x)\eta\right) \\
&= f(x,t)\pi^*\eta\wedge dt - e(t)F(x)\pi^*\eta\wedge dt \\
(dK-Kd)f(x,t)\pi^*\eta\wedge dt &= d\left(\left(\int_{-\infty}^{t}f(x,s)ds - F(x)A(t)\right)\pi^*\eta\right) \\
&\quad - K\left(\frac{\partial f}{\partial x}dx\wedge\pi^*\eta\wedge dt + f(x,t)\pi^*d\eta\wedge dt\right) \\
&= f(x,t)dt\wedge\pi^*\eta + \int_{-\infty}^{t}\frac{\partial f(x,s)}{\partial x}ds\,dx\wedge\pi^*\eta - A(t)dF\wedge\pi^*\eta \\
&\quad - F(x)e(t)dt\wedge\pi^*\eta + \left(\int_{-\infty}^{t}f(x,s)ds - F(x)A(t)\right)\pi^*d\eta \\
&\quad - \int_{-\infty}^{t}\frac{\partial f(x,s)}{\partial x}ds\,dx\wedge\pi^*\eta + A(t)\underbrace{\int_{-\infty}^{\infty}\frac{\partial f(x,s)}{\partial x}ds\,dx}_{=dF}\wedge\pi^*\eta \\
&\quad - \left(\int_{-\infty}^{t}f(x,s)ds - F(x)A(t)\right)\pi^*d\eta
\end{aligned}
$$

The claim follows by inspection.
- **Conclusion:** The proof of the Poincaré-Lemma is now an easy inductive proof using the above discussion in the case when $M = \mathbb{R}^{n-1}$. The statement of the Poincaré-Lemma was formulated for $\mathbb{R}^n$, but works equally well for open subsets of manifolds which are diffeomorphic to $\mathbb{R}^n$. This includes all convex subsets U of $\mathbb{R}^n$ contained in a manifold. The form $\eta \in \Omega_c^{n-1}(U)$ with $d\eta = \omega$ produced by the Poincaré-Lemma for a compactly supported form ω extends by zero to a smooth form on the entire manifold. There will be no distinction in the notation for the extended form and the form on U.

Lecture on Jan. 21 – **Top-dimensional cohomology, mapping degree**

- **Reference:** This is an adaptation of standard material often described in almost every book on Algebraic topology, for example [Ha], to differential forms. It is implicit in [BT].

- **Theorem:** The map

$$\begin{aligned} H^n(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

is well-defined and surjective.
- **Proof:** Since $\partial M = \emptyset$, we have $\int_M \omega + d\eta = \int_M \omega$. This implies that integration is well-defined on cohomology classes. Let (U_i, φ_i) be a fixed chart from the oriented atlas. In U_i, in local coordinates let $\omega = f(x_1, \dots, x_n) dx^1 \wedge \ldots \wedge dx^n$, where $f > 0$ is smooth. Then $\int_M \omega > 0$. This implies that M is surjective by linearity.
- **Informal Definition:** A *bump form* ω on a n-manifold is a form of degree n supported in a coordinate domain $(U, \varphi : U \longrightarrow \mathbb{R}^n)$ such that $\omega = \rho(x) dx^1 \wedge \ldots \wedge dx^n$ with ρ_x a smooth function with compact support and $\left|\int_U \omega\right| = 1$.
- **Proposition:** Let M be connected of dimension n. Then all bump forms are cohomologous up to multiplication with ± 1.
- **Proof:** Let ω, ω' we bump forms in coordinate charts $(U, \varphi), (U', \varphi')$. If $(U, \varphi) = (U', \varphi')$, then the claim follows from the Poincare Lemma with compact support and the transformation formula. If $U \cap V \neq \emptyset$, then ω is cohomologous to a (U, φ)-bump-form in $U \cap V$. Then ω is cohomologous to $\pm\omega'$. Now let p, p' be points in the support of ω, ω'. Since M is connected, there is a path from p to p' in M. This path can be covered by a finite number of charts $(U_i, \varphi_i : U_i \longrightarrow \mathbb{R}^n), i = 0, \dots, k$, with $U_i \cap U_{i+1} \neq \emptyset$ and $(U_0, \varphi_0) = (U, \varphi)$, $(U_k, \varphi_k) = (U', \varphi')$.
- **Lemma:** Let ω be a n-form on a manifold of dimension n. Then ω is a locally finite linear combination of bump forms.
- **Proof:** Direct application of a partition of unity subordinate to a cover by charts $(U_i, \varphi_i : U_i \longrightarrow \mathbb{R}^n)$.
- **Theorem:** $\dim(H^n(M)) \leq 1$ for connected, smooth manifold M of dimension n. If M is closed and oriented, then $H^n(M) \simeq \mathbb{R}$ (the isomorphism is given by integration).
- **Proof:** Direct consequence of the above.
- **Terminology:** A manifold is closed if it is compact and $\partial M = \emptyset$.
- **Theorem:** Let M be a smooth connected manifold which is not closed or not orientable. Then $H^n(M) = 0$.
- **Proof:** In the various cases we will show that every bump form is cohomologous to zero.
- **Case $\partial M \neq 0$:** If one bump form is cohomologous to zero, then all of them are. Let $(U, \varphi : U \longrightarrow \varphi(U) \subset \{x_1 \leq 0\})$ be a chart of a boundary point $p \in \partial M \neq \emptyset$ and $q \in \{x_1 < 0\} \cap \varphi(U)$ so that the straight line between p, q is parallel to the x_1-axis and contained in $\varphi(U)$. This straight line has a neighborhood V which is open in $\mathbb{R}^n$, convex and $\overline{V} \cap \{x_1 \leq 0\} \subset \varphi(U)$.

  Pick a bump forms ω_+ respectively ω_- with compact support in $V \cap \{x_1 > 0\}$ respectively $V \cap \{x_1 < 0\}$ so that $\int_V (\omega_+ + \omega_-) = 0$. By the Poincaré-Lemma with compact support there is $\eta \in \Omega_c(V)$ with $d\eta = (\omega_+ + \omega_-)$. The restriction of η to $\varphi(U)$ extends to a $n-1$-form on M whose differential is ω_+.
- **Case M not orientable:** If M is not orientable, then there is a chain of charts $U_0, \dots, U_k = U_0$ with $U_i \cap U_{i+1} \neq \emptyset$ and the number of transition functions with negative Jacobi determinant is odd when one moves from U_0 to U_k. Otherwise,

one obtains an oriented atlas. This implies that a bump form ω sopported in U_0 is cohomologous to bump form $-\omega$. This implies that 2ω is null-homologous.

- **Case M not compact:** Homework.
- **Remark:** The following statement has a similar proof as the previous one.
- **Theorem:** Let M be a non-compact, connected manifold. Then

$$H^n_c(M) \simeq \begin{cases} \mathbb{R} & \text{if } M \text{ is orienteable and has empty boundary} \\ 0 & \text{otherwise} \end{cases}$$

In the first case the isomorphism is given by integration.

- **Remark:** We want to give applications of deRham cohomology. We will male use of the fact that a homotopy invariant, i.e. $f_0 \sim f_1 : M \longrightarrow N$, then f_0^* and f_0^* induce the *same* map on cohomology.
- **Lemma:** $H^{n-1}(\overline{B_1(0)}) = 0$ for the n-ball for $n \geq 2$.
- **Proof:** The identity of $\overline{B_1(0)}$ is homotopic to the constant map and the deRham cohomology in degree $k > 0$ of a point is trivial (i.e. $= \{0\}$).
- **Brouwer fixed point theorem:** Let $f : \overline{B_1(0)} \longrightarrow \overline{B_1(0)} \subset \mathbb{R}^{n+1}$ be a smooth map. Then f has a fixed point.
- **Proof:** We give the proof for $n \geq 1$. The cases $n = -1$ respectively $n = 0$ are trivial respectively simple applications of the intermediate value theorem.

  Assume not. Then the following definition is possible

$$F : \partial\overline{B_1(0)} \longrightarrow S^n$$
$$x \longmapsto \left\{ \begin{array}{l} \text{unique intersection point of the ray} \\ \text{starting at } f(x) \text{ through } x \text{ with } S^n. \end{array} \right.$$

One has to show that this is smooth, it is when the same is true for f. It is immediate that $F(x) = x$ for $x \in S^n$. Let $\iota : S^{n-1} \longrightarrow \overline{B^1(0)}$ denote the inclusion. Then $F \circ \iota = \mathrm{id}_{S^{n-1}}$. Therefore, F^* has to be injective as a map $F^* : \mathbb{R} \simeq H^n(S^n) \longrightarrow H^n(\overline{B_1(0)}) = \{0\}$. But it is not.

- **Remark:** Using the Stone-Weierstraß theorem (polynomials are dense in the space of continuous functions) one can generalize the statement to *continuous* maps.
- **Assumption:** In the following, we assume that N, M are closed, oriented, connected manifolds of the same dimension n.
- **Definition:** Let $f : M \longrightarrow N$ be smooth. Fix $\omega \in \Omega^n(M)$ so that $\int_N \omega \neq 0$. Then the *degree* of f is

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}$$

- **Remark:** This is independent of the choice of ω. It is also homotopy invariant, i.e. if f_0 and f_1 are homotopic through smooth maps, then they have the same degree.
- **Example:** Let $M = N = S^1 \subset \mathbb{C}$ with $\omega = d\alpha$ (not exact in spite of the notation). Then $f_k(z) = z^k$ is a map of degree k for $k \in \mathbb{Z}$.
- **Theorem:** For all $f : M \longrightarrow N$, the degree is an integer. It is the signed count of preimages of a regular value.
- **Proof:** According to Sard's theorem the set of critical values of a C^∞-map is a set of full measure. Let p be a regular value, then f^{-1} is a codimension-n-submanifold, i.e. a finite set of points $q_1, \ldots, q_m$ (maybe empty). Around q_i there is a neighborhood U_i so that $f|_{U_i}$ is a diffeomorphism onto its image. We

may assume that the U_i are pairwise disjoint. Let ω be a bump form on N supported in the neighborhood $V = \cap_i f(U_i)$ of p. Then

$$\int_{U_i} (f|_{U_i})^* \omega = \int_V \omega \text{ if } D_{p_i} \text{ is orientation preserving}$$
$$\int_{U_i} (f|_{U_i})^* \omega = \int_V \omega \text{ if } D_{p_i} \text{ is not orientation preserving.}$$

This implies the claim.

22. Lecture on Jan. 23. – **Poincaré-duality, Riemannian metrics, Volume form**

- **Example:** It is not always possible to find a map of non-banishing degree between manifolds of the same dimension. For example, every smooth map $S^2 \longrightarrow T^2 = S^1 \times S^1$ has degree zero.
- **Observation:** The wedge product gives a well-defined product structure on cohomology. As volume form on $T^2 = S^1 \times S^1$ we choose $\omega = (\mathrm{pr}_1^* d\varphi) \wedge (\mathrm{pr}_2^* d\varphi)$ (where $\mathrm{pr}_i : T^2 \longrightarrow S^1, i = 1, 2$, denotes the projection on the first/second factor). Let $f : S^2 \longrightarrow T^2$. Then, for cohomology classes

$$f^*\omega = f^* \left(\mathrm{pr}_1^* d\varphi\right) \wedge f^* \left(\mathrm{pr}_2^* d\varphi\right)$$

  is a product of elements of $H^1(S^2) = 0$ (this follows form the last theorem of Lecture 14 applied to a cover of S^n by two copies of $\mathbb{R}^n$. Therefore, $\int_{S^2} f^*\omega = 0$.
- **Reminder:** On $\Omega^(M)$ there is the $\wedge$-product which is compatible with d and induces a product on cohomology groups.
- **Theorem: (Poincaré-duality):** Let M be a smooth,oriented connected manifold of dimension n. Then the bilinear pairing

$$H^k(M) \times H_c^{n-k}(M) \longrightarrow \mathbb{R}$$
$$([\omega], [\eta]) \longmapsto \int_M \underbrace{\omega \wedge \eta}_{\text{cpt. suppport}}$$

  is non-degenerate. The two spaces are duals of each other.
- **Notation:** The points $(x, x) \in M \times M$ form the *diagonal* Δ.
- **Definition:** Let M be a smooth manifold. A *Riemannian metric* g is a pairing

$$g : TM \times TM|_{\Delta \subset M \times M} \longrightarrow \mathbb{R}$$

  which is smooth, bilinear, symmetric and positive definite on fibers. A *Lorentzian metric* is indefinite, but non-degenerate.
- **Theorem:** Let M be a smooth manifold. Then there is a Riemannian metric on M.
- **Proof:** For notational reasons we assume that M is of dimension n. Let $(U_i, \varphi_i), i \in I$, be a covering of M by charts. Pick a locally finite refinement $V_j, j \in J$, and subordinate partition of unity $\lambda_j, j \in J$. On each of the V_j consider the standard metric g_j on $V_j \simeq \varphi_i(j)(V_j) \subset \mathbb{R}^n$. Then the sum $g = \sum_j \lambda_j g_j$ is a locally finite, convex combination of positive definite forms. As such it is positive definite.
- **Remark:** It is not true that every manifold admits a Lorentzian metric (e.g. S^{2n} does not).

- **Remark:** The proof of the above theorem made use of the second countability/paracompactness of manifolds. This is necessary. It can be shown, that a Riemannian metric on a connected manifold induces a metric (a distance) such that the induced topology coincides with the original topology. Since metrizable spaqces are paracompact (Theorem of Stone) every existence proof for Riemannian metrics has to somehow invoke paracompactness.
- **Remark:** A Lorentzian metric on a finite dimensional real vector space V extends canonically to a Lorentzian metric on forms. If $e_1, \ldots, e_n$ is an orthonormal basis for V (i.e. $g(e_i, e_i) = \varepsilon_i = \pm 1$), and δ^i is the dual basis, then for $1 \le \mu_1 < \ldots < \mu_k \le n$ and $1 \le \lambda_1 < \ldots < \lambda_k \le n$

$$\left\langle \delta^{\mu_1} \ldots, \delta^{\mu_k}, \delta^{\lambda_1} \ldots, \delta^{\lambda_k} \right\rangle = \begin{cases} 0 & \text{if } (\mu_1, \ldots, \mu_k) \neq (\lambda_1, \ldots, \lambda_k) \\ \varepsilon_{\mu_1} \ldots \varepsilon_{\mu_k} & \text{if } (\mu_1, \ldots, \mu_k) = (\lambda_1, \ldots, \lambda_k) \end{cases} \tag{10}$$

- After picking an orientation the *volume form* vol on M is a differential form of maximal degree such that $\mathrm{vol}(e_1, \cdots, e_n) = 1$ for every oriented orthonormal basis of T_pM. In terms of the coordinate basis the volume form takes the form

$$\mathrm{vol} = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \frac{\sqrt{|g|}}{n!}\epsilon_{i_1 \cdots i_n}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \tag{11}$$

where $\sqrt{|g|}$ is positive evaluation of the absolute value of the determinant of g.

23. Lecture on Jan. 28. – **Hodge-$*$-operator, codifferential, harmonic forms, Hodge-theorem**

- **Definition:** Let V be an oriented vector space of dimension n with a non-degenerate, symmetric, bilinear form (a Riemannian metric or an indefinite metric). The volume form on V is ω. Then, for each $k = 0, \ldots, n$, there is a unique linear operator

$$* : \Lambda^k V^* \longrightarrow \Lambda^{n-k} V^*$$

such that $\eta \wedge *\zeta = \langle \eta, \zeta \rangle \omega$ for all $\eta, \zeta \in \Lambda^k V^*$.
- **Proof:** If $*_1, *_2$ are two operators satisfying the defining property of the $*$-operator, then $\eta \wedge (*_1\zeta - *_2\zeta) = 0$. If the second factor is non-zero for some ζ, then one can easily find η so that the product is a mulitple of the volume form. This proves uniqueness.

  To show existence, let $e_1, \ldots, e_n$ be an orthonormal basis and $\delta^1, \ldots, \delta^n$ the dual basis. Using (10) one sees that for $1 \le \lambda_1 < \ldots < \lambda_k$ and $1 \le \mu_1 < \ldots < \mu_k \le n$ one has to have

$$\left(\delta^{\lambda_1} \wedge \ldots \wedge \delta^{\lambda_k}\right) \wedge * \left(\delta^{\mu_1} \wedge \ldots \wedge \delta^{\mu_k}\right) = \begin{cases} \varepsilon_{\mu_1} \ldots \varepsilon_{\mu_k}\omega & \text{if } (\mu_1, \ldots, \mu_k) = (\lambda_1, \ldots, \lambda_k) \\ 0 & \text{if not.} \end{cases}$$

  This shows that only one of the coefficients of $*\left(\delta^{\mu_1} \wedge \ldots \wedge \delta^{\mu_k}\right)$ can be non-zero, namely

$$* \left(\delta^{\mu_1} \wedge \ldots \wedge \delta^{\mu_k}\right) = \varepsilon_{\mu_1} \ldots \varepsilon_{\mu_k} \mathrm{sgn} \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ \mu_1 & \cdots & \mu_k & \nu_1 & \cdots & \nu_{n-k} \end{pmatrix} \delta^{\nu_1} \wedge \ldots \wedge \delta^{\nu_{n-k}}$$

  This defines an operator with the desired property. The previous formula holds without the requirement that $\mu_1 < \ldots < \mu_k, \nu_1 < \ldots < \nu_{n-1}$ (the sign of τ takes care of this).
- **Lemma:** The Hodge-$*$-operator has the following properties:

a) If $e_1, \dots, e_n$ is an oriented ONB, then

$$*\eta(e_{\tau(k+1)}, \dots, e_{\tau(n)}) = \varepsilon_{\tau(1)} \dots \varepsilon_{\tau(k)} \mathrm{sgn}(\tau) \cdot \delta^{\tau(k+1)} \wedge \dots \wedge \delta^{\tau(n)}$$

b) $** = \varepsilon_1 \dots \varepsilon_n (-1)^{k(n-k)} \mathrm{id}$ on $\Lambda^k(V^*)$.

c) $\eta \wedge \zeta = \varepsilon_1 \dots \varepsilon_n (-1)^{k(n-k)} \langle \eta, *\zeta \rangle \omega_V$ for $\eta \in \Lambda^k(V^*)$ and $\zeta \in \Lambda^{n-k}(V^*)$.

d) $\langle *\eta, *\zeta \rangle = \varepsilon_1 \dots \varepsilon_n \langle \eta, \zeta \rangle$

e) When the orientation of M is replaced by the opposite orientation, then the Hodge-$*$-operator changes its sign.

- **Definition/Lemma:** Given Riemannian metric on the oriented closed manifold M, there is a scalar product on the alternating k-forms

$$\Omega^k(M) \times \Omega^k(M) \longrightarrow \mathbb{R}$$

$$(\eta, \zeta) \longmapsto \langle \eta, \zeta \rangle_M = \int_M \langle \eta, \zeta \rangle \omega = \int_M \eta \wedge *\zeta$$

where ω is the volume form on M.

- **Definition:** The map $\delta := (-1)^k * d *^{-1} : \Omega^{n-k}(M) \longrightarrow \Omega^{n-k-1}(M)$ is the *codifferential.*
- **Remark:** This independent form the orientation (not the orientability). The following computation establish an expected relationship between $d*$ and $*\delta$: For $\zeta \in \Omega^{k+1}(M)$

$$\begin{aligned} d * \zeta &= (-1)^{k(n-k)+} * *d * \zeta \\ &= (-1)^{k(n-k)+} * *d * * \underbrace{*^{-1}\zeta}_{\in \Omega^{n-k-1}} \\ &= (-1)^{k(n-k)+2+(n-k-1)(k+1)} * *d *^{-1} \zeta \\ &= \dots = (-1)^k * \delta\zeta. \end{aligned}$$

- **Lemma:** The product rule for the exterior differential of a wedge-product yield

$$d(\eta \wedge *\zeta) = d\eta \wedge *\zeta + \eta \wedge *\delta\zeta \tag{12}$$

for $\eta \in \Omega^k(M)$ and $\zeta \in \Omega^{k+1}(M)$. If the (intersection of the) supports of η and ζ are compact and contained in $M \setminus \partial M$, then Stokes theorem implies $\int_M d(\eta \wedge *\zeta) = 0$.

- **Proposition:** Under all assumptions from the previous Lemma

$$\langle d\eta, \zeta \rangle_M + \langle \eta, \delta\zeta \rangle_M = 0$$

Thus, $-\delta$ is the formal adjoint of d

- **Warning:** Other sign conventions are common.
- **Notation:** We use the notations

$$d_k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$
$$\delta_k : \Omega^{n-k}(M) \longrightarrow \Omega^{n-k-1}(M)$$

Then $(-1)^k \delta_k * = *d$. Then $-\delta_k$ is the formal adjoint of d_{n-k-1} (via the scalar product) and $(-1)^k \delta_k$ is conjugate to of d_k (via $*$).

- **Convention:** From now on we consider only Riemannian metrics on closed, oriented manifolds. Then the scalar product on alternating forms is positive definite and

$$\langle d\eta, \zeta \rangle_M + \langle \eta, \delta\zeta \rangle_M = 0$$

for all $\eta \in \Omega^k(M)$ and $\zeta \in \Omega^{k+1}(M)$.

- **Fact:** An obvious consequence of the the adjointness is

$$\ker d_k = (\operatorname{image} \delta_{n-k-1})^{\perp}$$
$$\ker \delta_{n-k} = (\operatorname{image} d_{k-1})^{\perp}$$

In a vector space of finite dimension, this would imply an direct, orthogonal (with respect to $\langle \cdot, \cdot \rangle_M$) sum decomposition

$$\begin{aligned} \Omega^k(M) &= \ker d_k \oplus \operatorname{image} \delta_{n-k-1} \\ &= \ker \delta_{n-k} \oplus \operatorname{image} d_{k-1}. \end{aligned} \tag{13}$$

- **Theorem:** The identities (13) hold.
- The proof of the theorem above (in a more general form) is probably the content of a standard course with the title PDE 2 (elliptic PDE's), c.f. [W].

  From the view point of deRham cohomology, we are interested in image$(d_{k-1}) \subset$ ker(d_k). Intersecting the second part of (13) with ker(d_k) (and the first equation with ker(δ_{n-1})). Note that ker$(d_k) \supset$ image(d_{k-1}) etc.
- **Corollary:** For M closed, oriented, Riemannian manifold

$$\ker(d_k) = \operatorname{image}(d_{k-1}) \oplus (\ker(d_k) \cap \ker(\delta_{n-k}))$$
$$\ker(\delta_{n-k}) = \operatorname{image}(\delta_{n-k-1}) \oplus (\ker(d_k) \cap \ker(\delta_{n-k}))$$

- **Definition/Lemma:** Elements of $\mathcal{H}^k(M) = \ker(d_k) \cap \ker(\delta_{n-k})$ are *harmonic*. The operator $\Delta = \delta \circ d + d \circ \delta$ is the *Laplace operator*. Then

$$\mathcal{H}^*(M) = \{\omega \in \Omega^*(M) \,|\, \Delta\omega = 0\}.$$

This follows from $\langle \Delta\eta, \zeta \rangle_M = -\langle \delta\eta, \delta\zeta \rangle_M - \langle d\eta, d\zeta \rangle_M$. This shows that Δ is negative-semidefinite. The following is a corollary of this observation and the corollary preceding it.
- **Hodge-Theorem**: $\mathcal{H}^k(M) \hookrightarrow H^k(M)$ is an isomorphism.
- **Corollary (special case of Poincaré-duality):** The Hodge-$*$-operator is an isomorphism

$$* : \mathcal{H}^k(M) = H^k(M) \longrightarrow \mathcal{H}^{n-k}(M) = H^{n-k}(M).$$

- **Fact:** For closed M, the space $\mathcal{H}^k(M)$ has finite dimension. Thus the Hodge-theorem implies that the cohomology of a closed manifold has finite dimension. This could have been deduced from the zigzag-Lemma.
- **Hodge-decomposition:** For a closed, oriented, Riemannian manifold

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \operatorname{image}(d_{k-1}) \oplus \operatorname{image}(\delta_{n-k-1})$$

24. Lecture on Jan. 30. – **Coordinates, applications of Hodge decomposition**

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