LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik im SoSe 2024
T M1/TV: Advanced Statistical Physics
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Sheet 10:

Hand-out: Friday, Jun. 21, 2024; Hand-in: Sunday, Jun. 30, 2024, 11:59 pm

Problem 1 Duality of a $\mathbb{Z}_2$ lattice gauge theory & an Ising model – **(solution: Central Exercise)**

In this exercise, we discuss the simplest duality between a 2D lattice gauge theory (LGT) and a 1D Ising model. (There exist similar dualities between more complicated lattice gauge theories and higher-dimensional Ising models, which have led to the realization of phase transitions without local order parameters, but we won't discuss this here.)

We consider the simplest so-called Ising or $\mathbb{Z}_2$ gauge theory in two dimensions. To obtain its degrees of freedom, we start from Ising spins $s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} = \pm 1$ defined on the links $\langle \boldsymbol{i},\boldsymbol{j}\rangle$ of a square lattice with lattice sites $\boldsymbol{i},\boldsymbol{j}$. However, many different configurations $\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}$ can represent the *same* physical state $[\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}]$: Two configurations are called gauge-equivalent, $\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\} \simeq \{s'_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}$ iff $s'_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}$ can be obtained from $s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}$ (and vice versa) by applying gauge transformations $\prod_n G(\boldsymbol{r}_n)$ on lattice sites $\boldsymbol{r}_n$ defined as follows. A local gauge transformation at site $\boldsymbol{j}$ flips the sign of all link variables $s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}$ whose link contains site $\boldsymbol{j}$, i.e.

$$G(\boldsymbol{j}) s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} = -s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} \quad \text{and} \quad G(\boldsymbol{j}) s_{\langle \boldsymbol{r},\boldsymbol{x}\rangle} = s_{\langle \boldsymbol{r},\boldsymbol{x}\rangle} \ \text{ iff } \boldsymbol{x},\boldsymbol{r} \neq \boldsymbol{j}. \tag{1}$$

(1.a) (**4 Points**) Show that $\simeq$ defines an equivalence relation. The equivalence classes $A = [\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}]$ define the set of physically allowed states. As an example, determine all physical states, i.e. all equivalence classes, on a 2×2 lattice with four links (i.e. one plaquette).

(1.b) (**4 Points**) The Hamiltonian of the Ising LGT we consider is given by:

$$\mathcal{H} = -J \sum_P \prod_{\ell \in \partial P} s_\ell, \tag{2}$$

where $\sum_P$ denotes a sum over all plaquettes P of the lattice, and the product is taken over all links $\ell \in \partial P$ forming the edge of plaquette P. Show that $\mathcal{H}$ is invariant under local gauge transformations, i.e. $\mathcal{H}(\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}) = \mathcal{H}(G(\boldsymbol{r}_n)\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\})$ is the same for equivalent spin configurations. Hence $\mathcal{H}$ assigns one definite energy to each physical state (equivalence class) $[\{s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}]$. Does this result also hold for other 2D lattices (i.e. other than the square lattice)?

(1.c) (**4 Points**) Now we would like to make a particular gauge choice. We choose the so-called temporal gauge for which $s_{\langle \boldsymbol{i},\boldsymbol{j}\rangle_y} = +1$ for all bonds along the y-direction in the square lattice. Show that any given configuration of spins $\{s'_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\}$ can be transformed into a gauge-equivalent configuration which satisfies the temporal gauge condition.

(1.d) (**4 Points**) Show that the number of spin degrees of freedom in the temporal gauge, N_τ, is equal to the number of gauge equivalence classes, $N_{\mathbb{Z}_2}$, in the $\mathbb{Z}_2$ lattice gauge theory in the thermodynamic limit:

$$N_\tau / L^2 = N_{\mathbb{Z}_2} / L^2 + \mathcal{O}(1/L). \tag{3}$$

Hint: Use the result from (1.b)!

(1.e) (**4 Points**) Using the results above, calculate the partition function Z of the $\mathbb{Z}_2$ lattice gauge theory and show that it is equivalent to the 1D Ising model.

Problem 2 Spin-wave theory: Ferromagnetic case

Consider the isotropic Heisenberg Hamiltonian

$$\hat{\mathcal{H}} = -\sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} \left[\frac{J}{2} \left(\hat{S}^+_{\boldsymbol{i}} \hat{S}^-_{\boldsymbol{j}} + \hat{S}^+_{\boldsymbol{j}} \hat{S}^-_{\boldsymbol{i}} \right) + J_z \hat{S}^z_{\boldsymbol{i}} \hat{S}^z_{\boldsymbol{j}} \right], \tag{4}$$

with spin $S = 1/2$ and $J = J_z = 1$ (i.e. ferromagnetic case). Let us denote the state with all spins up by $|0\rangle$. Now consider the state:

$$|\Psi(\boldsymbol{k})\rangle = L^{-d/2} \sum_{\boldsymbol{j}} e^{i\boldsymbol{k}\cdot\boldsymbol{j}} \hat{S}^-_{\boldsymbol{j}} |0\rangle . \tag{5}$$

(2.a) (**4 Points**) Show that this state is an eigenstate of the Hamiltonian (4).

(2.b) (**4 Points**) Show that this state has a well-defined momentum $\boldsymbol{k}$.

(2.c) (**4 Points**) Show that this state is a ground state of the Hamiltonian (4) for $\boldsymbol{k} = 0$.

(2.d) (**4 Points**) Consider the Holstein-Primakov transformation:

$$\hat{S}^z_{\boldsymbol{j}} = S - \hat{a}^\dagger_{\boldsymbol{j}} \hat{a}_{\boldsymbol{j}}, \tag{6}$$

$$\hat{S}^+_{\boldsymbol{j}} = \sqrt{2S - \hat{a}^\dagger_{\boldsymbol{j}} \hat{a}_{\boldsymbol{j}}}\; \hat{a}_{\boldsymbol{j}}, \tag{7}$$

$$\hat{S}^-_{\boldsymbol{j}} = \hat{a}^\dagger_{\boldsymbol{j}} \sqrt{2S - \hat{a}^\dagger_{\boldsymbol{j}} \hat{a}_{\boldsymbol{j}}}. \tag{8}$$

Show that it yields the correct spin commutators for $[\hat{S}^+_{\boldsymbol{i}}, \hat{S}^-_{\boldsymbol{j}}] = 2\delta_{\boldsymbol{i},\boldsymbol{j}} \hat{S}^z_{\boldsymbol{j}}$.