Fakultät für Physik im SoSe 2024
T M1/TV: Advanced Statistical Physics
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https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_24/t_m1_advanced-statistical-physics/index.html

Sheet 2:

Hand-out: Friday, Apr. 26, 2024; Hand-in: Sunday, May. 05, 2024, 11:59 pm

Problem 1 Critical exponents of the van der Waals gas – **(solution: Central Exercise)**

(1.a) (**3 Points**) Starting from the free energy of the van der Waals gas

$$F(T,V,N) = -k_B T N \ln\left(\frac{V-bN}{N\lambda_T^3}\right) - k_B T N - \frac{aN^2}{V}\,, \tag{1}$$

where $\lambda_T = h/\sqrt{2\pi m k_B T}$, calculate an expression for the pressure P in terms of the particle number N, volume V, temperature T and the constants a (attractive force between particles) and b (hard-sphere volume of one particle). Rewrite this expression as an equation of state.

(1.b) (**2 Points**) Calculate the heat capacity at constant volume C_V.

(1.c) (**4 Points**) The heat capacities at constant volume and pressure, C_V and C_P respectively, are connected via

$$C_P - C_V = -T\left(\frac{\partial P}{\partial T}\right)_V^2 \left(\frac{\partial V}{\partial P}\right)_T. \tag{2}$$

Let $V = V_c$ and $T \to T_c$ from above to arrive at a pair of scaling relations of the form

$$C_V \sim (T-T_c)^{-\alpha}, \quad C_P \sim (T-T_c)^{-\mu} \tag{3}$$

and determine the critical exponents α and μ.

(1.d) (**4 Points**) Define the reduced variables

$$t = \frac{T-T_c}{T_c}, \quad \phi = \frac{V-V_c}{V_c} \tag{4}$$

with $t \leqslant 0$ and rewrite the equation of state obtained in (1.a) into the form

$$P/P_c = c_1 + c_2 t + c_3 t\phi + c_4\phi^3 + \mathcal{O}(t\phi^2, \phi^4), \tag{5}$$

valid close to the critical point (i.e. $t \approx 0$, $\phi \approx 0$). Determine the constants c_i. Let ϕ_{liq}, ϕ_{gas} denote the ϕ corresponding to V_{liq} and V_{gas} respectively. Use the Maxwell construction

$$\oint_C V dP = 0 \tag{6}$$

to determine the coefficients c_5 and c_6 in

$$\phi_{\text{liq}} = c_5\phi_{\text{gas}} + c_6. \tag{7}$$

Observe that Eq. (5) holds independently for both ϕ_{liq} and ϕ_{gas}. Solve for either ϕ_i and determine the critical exponent β in

$$\phi_i \sim |t|^\beta. \tag{8}$$

(1.e) (**3 Points**) Set $t = 0$ in Eq. (5) such that the system is on the critical isotherm and determine the critical exponent δ in

$$\frac{P - P_c}{P_c} \sim \left(\frac{V - V_c}{V_c}\right)^{\delta}. \tag{9}$$

Problem 2 Some details from the lecture – (solution: Tutorials)

(2.a) (**5 Points**) Show, in analogy to the lecture, that for a Hamiltonian with added source term

$$\mathcal{H} \to \mathcal{H}[H(\mathbf{x})] = \mathcal{H} - \int \mathrm{d}^d\mathbf{x}\, m(\mathbf{x}) \cdot H(\mathbf{x}) \tag{10}$$

we can calculate the (unconnected) two-point correlator

$$G^{(2)}(\mathbf{r}) = \langle m(0)m(\mathbf{r})\rangle \tag{11}$$

as

$$\beta^2 G^{(2)}(\mathbf{r}) = \frac{1}{Z}\, \frac{\partial^2 Z[H(\mathbf{x})]}{\partial H(0)\partial H(\mathbf{r})}\bigg|_{H(\mathbf{x})=0}. \tag{12}$$

(2.b) (**5 Points**) Show that for a function $G(r)$ with

$$G(br) = \phi(b)G(r)$$

and $b \geq 1$, $G(r)$ must be homogeneous, i.e. $G(br) = b^{-y}G(r)$, where the exponent y is not specified further.

Problem 3 Existence of the thermodynamic limit – (solution: Tutorials)

A key concept of statistical physics is given by the thermodynamic limit of an infinite system; one postulate that is always made for the thermodynamic limit to exist is that energy density (per volume or per particle) does not diverge in the limit of particle number $N \to \infty$, $V \to \infty$, with N/V constant. Now consider a density-density interaction with a power-law decay: $V(\mathbf{x} - \mathbf{x}') = A\rho(\mathbf{x})\rho(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^{\sigma}$. Both for the Coulomb interaction and for gravitation, $\sigma = 1$, and the densities are charge or mass densities respectively.

(3.a) (**8 Points**) Considering a d-dimensional hypersphere of radius R and constant (mass or charge) density ρ, calculate the behavior of the energy density in the thermodynamic limit $R \to \infty$ (keeping density constant!). Show that the thermodynamic limit only exists in the sense above if $\sigma > d$.
Advice: one can extract the dependence on R without explicit evaluation of the integral; also ignore worries that the interaction V might lead to a divergence in the energy due to interactions on very short distances!

(3.b) (**2 Points**) It seems that both the Coulomb and gravitational interaction do not admit the existence of a thermodynamic limit. Why can we nevertheless do thermodynamics?

(Bonus) What happens at $\sigma = d$? Here, the integral must be evaluated; to get around the problem of a divergence due to short distances, set the minimal distance between interacting points to some small constant a (that might be a crystalline lattice constant, for example).