TMP-TC2: Cosmology

Solutions to Problem Set 1 23 & 25 April 2024

1. Covariant Derivative

1) Applying the coordinate transformation on the derivative and the vector gives

$$\begin{aligned}\frac{\partial V^\mu}{\partial x^\nu} \mapsto \frac{\partial}{\partial \bar{x}^\nu}\left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha}V^\alpha\right) &= \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial}{\partial x^\beta}\left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha}V^\alpha\right)\\ &= \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial^2 \bar{x}^\mu}{\partial x^\beta \partial x^\alpha}V^\alpha\end{aligned}$$

We can see that through the second term the derivative of a vector does not transform like a tensor. This leads to the fact that $\partial_\mu V^\mu = 0$ is not coordinate independent.

2) Remember that the Christoffel symbols transform as

$$\Gamma^\mu_{\nu\lambda} \mapsto \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\lambda}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu \partial \bar{x}^\lambda}\frac{\partial \bar{x}^\mu}{\partial x^\alpha} \tag{1}$$

By using the product rule, the second term can be rewritten to

$$\frac{\partial^2 x^\alpha}{\partial \bar{x}^\nu \partial \bar{x}^\lambda}\frac{\partial \bar{x}^\mu}{\partial x^\alpha} = -\frac{\partial x^\alpha}{\partial \bar{x}^\nu}\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial^2 \bar{x}^\mu}{\partial x^\beta \partial x^\alpha} \tag{2}$$

Now we have everything what we need to calculate the transformation of the covariant derivative of a vector :

$$\begin{aligned}\nabla_\nu V^\mu \mapsto\ & \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial^2 \bar{x}^\mu}{\partial x^\beta \partial x^\alpha}V^\alpha\\ &+ \left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\lambda}\Gamma^\alpha_{\beta\gamma} - \frac{\partial x^\alpha}{\partial \bar{x}^\nu}\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial^2 \bar{x}^\mu}{\partial x^\beta \partial x^\alpha}\right)\frac{\partial \bar{x}^\lambda}{\partial x^\delta}V^\delta\end{aligned}$$

Applying $\frac{\partial x^\alpha}{\partial \bar{x}^\lambda}\frac{\partial \bar{x}^\lambda}{\partial x^\beta} = \delta^\alpha_\beta$, the second and the last terms cancel. At the end we obtain

$$\nabla_\nu V^\mu \mapsto \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\nabla_\beta V^\alpha \tag{3}$$

We can observe that the covariant derivative of a vector transforms as a tensor. Therefore the expression $\nabla_\mu V^\mu = 0$ is coordinate independent.

As an example that will be relevant for us, you can take the energy-momentum tensor $T^{\mu\nu}$. We will use the fact that $\nabla_\mu T^{\mu\nu} = 0$ does not depend on the coordinate system.

3) Recall that ${V^\nu}_{;\mu} = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$ and $V_{\nu;\mu} = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha$. Then the idea is to consider expressions of the type $(A^\mu B^\nu)_{;\delta}$ to deduce the expression for ${T^{\mu\nu}}_{;\delta}$.
We have

$$
\begin{aligned}
(A^\mu B^\nu)_{;\delta} &= A^\mu_{;\delta} B^\nu + B^\nu_{;\delta} A^\mu \\
&= \partial_\delta A^\mu B^\nu + \partial_\delta B^\nu A^\nu + \Gamma^\mu_{\delta\alpha} A^\alpha B^\nu + \Gamma^\nu_{\delta\alpha} B^\alpha A^\mu \\
&= \partial_\delta (A^\mu B^\nu) + \Gamma^\mu_{\delta\alpha} A^\alpha B^\nu + \Gamma^\nu_{\delta\alpha} B^\alpha A^\mu
\end{aligned}
$$

From which we deduce

$$
{T^{\mu\nu}}_{;\delta} = \partial_\delta T^{\mu\nu} + \Gamma^\mu_{\delta\alpha} T^{\alpha\nu} + \Gamma^\nu_{\delta\alpha} T^{\mu\alpha}
$$

Similarly, we obtain

$$
\begin{aligned}
T_{\mu\nu;\delta} &= \partial_\delta T_{\mu\nu} - \Gamma^\alpha_{\delta\mu} T_{\alpha\nu} - \Gamma^\alpha_{\delta\nu} T_{\mu\alpha} \\
{T^\mu}_{\nu;\delta} &= \partial_\delta T^\mu_\nu + \Gamma^\mu_{\delta\alpha} {T^\alpha}_\nu - \Gamma^\alpha_{\delta\nu} {T^\mu}_\alpha
\end{aligned}
$$

2. Metric for a 3-sphere and a 4-dimensional hyperboloid

1) Take the derivative of the given constraint

$$
x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z + w\mathrm{d}w = 0 \tag{4}
$$

and use this to eliminate w in the metric :

$$
\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \frac{(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z)^2}{1 - x^2 - y^2 - z^2} \tag{5}
$$

2) First, let us calculate the differentials

$$
\begin{aligned}
\mathrm{d}x &= \cos\chi\cos\phi\sin\theta\mathrm{d}\chi - \sin\chi\sin\phi\sin\theta\mathrm{d}\phi + \sin\chi\cos\phi\cos\theta\mathrm{d}\theta \\
\mathrm{d}y &= \cos\chi\sin\phi\sin\theta\mathrm{d}\chi + \sin\chi\cos\phi\sin\theta\mathrm{d}\phi + \sin\chi\sin\phi\cos\theta\mathrm{d}\theta \\
\mathrm{d}z &= \cos\chi\cos\theta\mathrm{d}\chi - \sin\chi\sin\theta\mathrm{d}\theta
\end{aligned}
$$

Inserting this into the first part of the metric gives

$$
\begin{aligned}
&\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&= \left(\cos^2\chi\cos^2\phi\sin^2\theta + \cos^2\chi\sin^2\phi\sin^2\theta + \cos^2\chi\cos^2\theta\right)\mathrm{d}\chi^2 \\
&+ \left(\sin^2\chi\cos^2\phi\cos^2\theta + \sin^2\chi\sin^2\phi\cos^2\theta + \sin^2\chi\sin^2\theta\right)\mathrm{d}\theta^2 \\
&+ \left(\sin^2\chi\sin^2\phi\sin^2\theta + \sin^2\chi\cos^2\phi\sin^2\theta\right)\mathrm{d}\phi^2
\end{aligned}
$$

Note that all the off-diagonal terms cancelled. We can simplify this further :

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \cos^2\chi \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\theta^2 + \sin^2\chi \sin^2\theta \mathrm{d}\phi^2 \tag{6}$$

Furthermore, we can calculate

$$x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = \sin\chi\cos\chi\mathrm{d}\chi \tag{7}$$

and

$$1 - x^2 - y^2 - z^2 = \cos^2\chi. \tag{8}$$

Therefore, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{9}$$

3) In that case the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\chi^2 + \sinh^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{10}$$

The calculation works in the same way as in part 1) and part 2). Note that we used the identity $\cosh^2 - \sinh^2 = 1$.

3. Friedmann–Lemaître–Robertson–Walker (FLRW) metric

• $\mathbf{k = 0}$

First we consider the flat space case, with the line element given by

$$ds^2 = -(dx^0)^2 + a^2(x^0)\sum_i (dx^i)^2 , \tag{11}$$

where for later convenience we introduced the shorthand notation

$$\sum_i (dx^i)^2 = [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] .$$

1) The metric is

$$g_{\mu\nu} = \mathrm{diag}\big[-1, a^2, a^2, a^2\big] , \tag{12}$$

so

$$g^{\mu\nu} = \mathrm{diag}\big[-1, a^{-2}, a^{-2}, a^{-2}\big] . \tag{13}$$

2) The action for a classical particle with mass m is

$$S = m\int ds = m\int dp\ g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \int dp\ F(x, \dot{x}) , \tag{14}$$

where a dot denotes differentiation with respect to the affine parameter p. By introducing the explicit form of the metric we find

$$F(x,\dot{x}) = m\Big[-(\dot{x}^0)^2 + a^2 \sum_i (\dot{x}^i)^2 \Big] \ . \tag{15}$$

Using the Euler-Lagrange equations

$$\frac{d}{dp}\frac{\partial F}{\partial \dot{x}^\mu} = \frac{\partial F}{\partial x^\mu} \ , \tag{16}$$

we find

$$\begin{aligned} \ddot{x}^0 &= -aa' \sum_i (\dot{x}^i)^2 \ , \qquad \text{for } \mu = 0 \ , \\ \ddot{x}^i &= -2\frac{a'}{a}\dot{x}^0\dot{x}^i \ , \qquad \text{for } \mu = 1,2,3 \ , \end{aligned} \tag{17}$$

where a prime denotes derivative with respect to x^0.

3) The Christoffel symbols are defined as

$$\ddot{x}^\lambda = -\Gamma^\lambda{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \ . \tag{18}$$

By identification, the non-zero Γs are

$$\Gamma^0{}_{ii} = aa' \quad \text{and} \quad \Gamma^i{}_{0i} = \Gamma^i{}_{i0} = \frac{a'}{a} \tag{19}$$

The $\Gamma^k{}_{ij}$, $i,j,k = 1,2,3$ are zero, because the spatial part of the metric is flat. Let us check the above results with the usual formula

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right) \ . \tag{20}$$

For $\Gamma^i{}_{0i}$, we find

$$\Gamma^i{}_{0i} = \frac{1}{2} g^{i\kappa} \left(\partial_0 g_{i\kappa} + \partial_i g_{0\kappa} - \partial_\kappa g_{0i} \right) \tag{21}$$

$$= \frac{1}{2} g^{ii} \partial_0 g_{ii} \tag{22}$$

$$= \frac{1}{2} a^{-2} \partial_0 a^2 = \frac{a'}{a} \ , \tag{23}$$

and for $\Gamma^0{}_{ii}$

$$\Gamma^0{}_{ii} = \frac{1}{2} g^{0\kappa} \left(\partial_i g_{i\kappa} + \partial_i g_{i\kappa} - \partial_\kappa g_{ii} \right) \tag{24}$$

$$= -\frac{1}{2} g^{00} \partial_0 g_{ii} \tag{25}$$

$$= -\frac{1}{2} \partial_0 a^2 = aa' \ . \tag{26}$$

4) Since $R^\mu{}_{\nu\rho\sigma}$ is antisymmetric in the last two indices, the only combinations we have to calculate are

μ	ν	ρ	σ	Results
0	0	0	i	0
0	0	i	j	0
0	i	j	k	0
0	i	0	j	$i=j$
i	0	0	j	$i=j$
i	0	j	k	0
i	j	0	k	0
i	j	k	l	$(k,l)=(i,j), i\neq j$

Then

$$R^0{}_{i0i} = \partial_0 \Gamma^0_{ii} - \Gamma^0{}_{\kappa i}\Gamma^\kappa{}_{i0} = aa'' , \tag{27}$$

and

$$R^i{}_{00i} = \frac{a''}{a} \ , \qquad R^i{}_{jij} = (a')^2 \ . \tag{28}$$

5) The components of the Ricci tensor are

$$R_{00} = R^\kappa{}_{0\kappa 0} = -3\frac{a''}{a} \quad \text{and} \quad R_{ii} = R^\kappa{}_{i\kappa i} = aa'' + 2(a')^2 \ . \tag{29}$$

6) Using the above, we see that the scalar curvature is

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right] \ . \tag{30}$$

7) Finally, the non zero components of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ , \tag{31}$$

are

$$G_{00} = -3\frac{a''}{a} + 3\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right] = 3\left(\frac{a'}{a}\right)^2 \ , \tag{32}$$

$$G_{ii} = aa'' + 2(a')^2 - 3a^2\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right] = -2aa'' - (a')^2 . \tag{33}$$

We extracted all the information contained in the metric. The tensor G contains the geometric part of the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$, where T is the energy momentum tensor and Λ is the cosmological constant.

• $\mathbf{k \neq 0}$

Now we move to the curved space,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] \ . \tag{34}$$

1) The metric is

$$g_{\mu\nu} = \text{diag}\left[-1, \frac{a^2}{1-kr^2}, a^2r^2, a^2r^2\sin^2\theta\right], \tag{35}$$

so

$$g^{\mu\nu} = \text{diag}\left[-1, \frac{1-kr^2}{a^2}, \frac{1}{a^2r^2}, \frac{1}{a^2r^2\sin^2\theta}\right] . \tag{36}$$

2)The Lagrangian is given by

$$F(x,\dot{x}) = m\left(-\dot{t}^2 + \frac{a^2}{1-kr^2}\dot{r}^2 + a^2r^2\dot{\theta}^2 + a^2r^2\sin^2\theta\dot{\phi}^2\right). \tag{37}$$

Thus, the equations of motion are

$$\begin{aligned}
\ddot{t} &= -aa'\left(\frac{\dot{r}^2}{1-kr^2} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)\\
\ddot{r} &= r(1-kr^2)\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] - k\frac{r\dot{r}^2}{1-kr^2} - 2\frac{a'}{a}\dot{t}\dot{r}\\
\ddot{\theta} &= \sin\theta\cos\theta\dot{\phi}^2 - 2\frac{\dot{r}}{r}\dot{\theta} - 2\frac{a'}{a}\dot{t}\dot{\theta}\\
\ddot{\phi} &= -2\frac{\dot{r}}{r}\dot{\phi} - 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} - 2\frac{a'}{a}\dot{t}\dot{\phi}
\end{aligned}$$

3) The non zero Christoffel symbols are

$$\begin{aligned}
\Gamma^t{}_{rr} &= \frac{aa'}{1-kr^2}\\
\Gamma^t{}_{\theta\theta} &= aa'r^2\\
\Gamma^t{}_{\phi\phi} &= aa'r^2\sin^2\theta\\
\Gamma^r{}_{rt} &= \frac{a'}{a}\\
\Gamma^r{}_{rr} &= \frac{kr}{1-kr^2}\\
\Gamma^r{}_{\theta\theta} &= -r\left(1-kr^2\right)\\
\Gamma^r{}_{\phi\phi} &= -r\left(1-kr^2\right)\sin^2\theta\\
\Gamma^\theta{}_{\theta t} &= \frac{a'}{a}\\
\Gamma^\theta{}_{\theta r} &= r^{-1}\\
\Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta\\
\Gamma^\phi{}_{\phi t} &= \frac{a'}{a}\\
\Gamma^\phi{}_{\phi r} &= r^{-1}\\
\Gamma^\phi{}_{\phi\theta} &= \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

4) To calculate the non-zero components of the Riemann tensor, it is very useful to remind some of its properties. The Riemann tensor is antisymmetric in the last two indices, so $R^{\mu}{}_{\nu\rho\rho} = 0 \; \forall \; \mu, \nu$. We can also show that $R^{\mu}{}_{\mu\rho\sigma} = 0 \; \forall \; \rho, \sigma$, since $R_{\mu\nu\rho\sigma}$ is antisymmetric in the first two indices. Also, the Christoffel symbols have always a repeated index, and as a consequence $R^{\mu}{}_{\nu\rho\sigma} = 0$, if the four indices are different. Taking the above considerations into account, we easily see that the only non-zero combinations are

$$R^{\mu}{}_{\nu\mu\sigma} = -R^{\mu}{}_{\nu\sigma\mu} \quad \text{and} \quad R^{\mu}{}_{\nu\nu\sigma} = -R^{\mu}{}_{\nu\sigma\nu} \; . \tag{38}$$

Look at the first case with $\nu \neq \sigma$. Since $\Gamma^{\mu}{}_{\mu\nu}$ only depends of ν, we deduce that $R^{\mu}{}_{\nu\mu\sigma} = 0$. This means that necessarily $\nu = \sigma$. We use the same procedure in the second case. The non-zero components of the Riemann tensor are

$$\begin{aligned}
R^{t}{}_{rtr} &= \frac{aa''}{1-kr^2} \\
R^{t}{}_{\theta t\theta} &= r^2 aa'' \\
R^{t}{}_{\phi t\phi} &= r^2 \sin^2\theta aa'' \\
R^{r}{}_{trt} &= -\frac{a''}{a} \\
R^{r}{}_{\theta r\theta} &= r^2 \left(k + (a')^2\right) \\
R^{r}{}_{\phi r\phi} &= r^2 \sin^2\theta \left(k + (a')^2\right) \\
R^{\theta}{}_{t\theta t} &= -\frac{a''}{a} \\
R^{\theta}{}_{r\theta r} &= \frac{k + (a')^2}{1-kr^2} \\
R^{\theta}{}_{\phi\theta\phi} &= r^2 \sin^2\theta \left(k + (a')^2\right) \\
R^{\phi}{}_{t\phi t} &= -\frac{a''}{a} \\
R^{\phi}{}_{r\phi r} &= \frac{k + (a')^2}{1-kr^2} \\
R^{\phi}{}_{\theta\phi\theta} &= r^2 \left(k + (a')^2\right)
\end{aligned}$$

5) The Ricci tensor components are

$$\begin{aligned}
R_{tt} &= -3\frac{a''}{a} \\
R_{rr} &= \frac{aa'' + 2k + 2(a')^2}{1-kr^2} \\
R_{\theta\theta} &= r^2 \left(aa'' + 2k + 2(a')^2\right) \\
R_{\phi\phi} &= r^2 \sin^2\theta \left(aa'' + 2k + 2(a')^2\right)
\end{aligned}$$

6) The scalar curvature is

$$R = g^{\mu\nu} R_{\mu\nu} = 6\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right] \tag{39}$$

Remark that the *spatial* curvature modifies the *space-time* curvature by introducing the last term.

7) The Einstein tensor components are

$$\begin{aligned}
G_{tt} &= 3\left[\left(\frac{a'}{a}\right)^2 + \frac{k}{a^2}\right] \\
G_{rr} &= -\frac{2aa'' + (a')^2 + k}{1 - kr^2} \\
G_{\theta\theta} &= -r^2\left(2aa'' + (a')^2 + k\right) \\
G_{\phi\phi} &= -r^2 \sin^2\theta\left(2aa'' + (a')^2 + k\right)
\end{aligned}$$

4. Volume in curved spacetime

The volume is given by

$$V = \int d^3x \sqrt{\gamma}\ ,$$

with

$$\gamma = \det\begin{pmatrix} \frac{1}{1-r^2/R^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} = \frac{r^4 \sin^2\theta}{1 - r^2/R^2}\ .$$

Therefore, we have

$$\begin{aligned}
V &= 2\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{1 - r^2/R^2}} \\
&= 8\pi \int_0^R dr \frac{r^2}{\sqrt{1 - r^2/R^2}} \\
&= 8\pi R^3 \int_0^{\pi/2} d\chi \sin^2\chi = 2\pi^2 R^3\ .
\end{aligned}$$

The appearance of the factor 2 in the above calculation is because we have to account twice for the interval of R (r increases from 0 to 1, then it decreases from 1 to 0).