- Anomaly: symmetry of classical theory
  $\neq$ symmetry of quantum theory

(1)

- $\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \gamma^\mu (\partial_\mu - iQA_\mu) \Psi$

- $\Psi \longmapsto e^{i\alpha} \Psi$

$$i \bar{\Psi} \gamma^\mu (\partial_\mu - iQA_\mu) \Psi$$

$$\longmapsto i \bar{\Psi} e^{-i\alpha} \gamma^\mu (\partial_\mu - iQA_\mu \Psi) e^{+i\alpha} \Psi$$

- $\Psi \longmapsto e^{i\beta\gamma^5} \Psi$

$$i \bar{\Psi} \gamma^\mu (\partial_\mu - iQA_\mu) \Psi$$

$$\longmapsto i \Psi^\dagger e^{-i\beta(\gamma^5)^\dagger} \gamma^0 \gamma^\mu (\partial_\mu - iQA_\mu) e^{+i\beta\gamma^5} \Psi \qquad \text{with } (\gamma^5)^\dagger = \gamma^5$$

$$= i \Psi^\dagger \gamma^0 e^{+i\beta\gamma^5} \gamma^\mu e^{+i\beta\gamma^5} (\partial_\mu - iQA_\mu) \Psi$$

$$= i \bar{\Psi} e^{+i\beta\gamma^5} e^{-i\beta\gamma^5} \gamma^\mu (\partial_\mu - iQA_\mu) \Psi$$

- $\bar{\Psi}\Psi \longmapsto \bar{\Psi} e^{+i\beta\gamma^5} e^{+i\beta\gamma^5} \Psi \neq \bar{\Psi}\Psi$

  $\rightarrow$ with mass term, this is no symmetry anymore

  $\rightarrow$ no conserved Noether current

- $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \delta\Psi = i \bar{\Psi} \gamma^\mu (i\alpha \Psi) \sim \bar{\Psi}\gamma^\mu\Psi$

$$J_5^\mu = i \bar{\Psi} \gamma^\mu (i\alpha \gamma^5 \Psi) \sim \bar{\Psi}\gamma^\mu\gamma^5\Psi$$

$\partial_\mu J_5^\mu = 0$ for $m = 0$.

$\partial_\mu J_5^\mu = 0$ for $m=0$.

We will show that this is not true quantum mechanically.

(2)

- $$H_{int} = -\int d^3x \; \bar\Psi \gamma^\mu Q A_\mu \Psi = -\int d^3x \; Q J^\mu A_\mu$$

$$\delta^{(4)}(p-q_1-q_2)\; \underset{\overset{\shortparallel}{\mathcal{M}^{\alpha\mu\nu}\varepsilon_\alpha \varepsilon^*_\mu \varepsilon^*_\nu}}{i\mathcal{M}} \sim \int d^4x\, d^4y\, d^4z \; \langle q_1, q_2 | \underset{\overset{\uparrow}{J^\mu \varepsilon^*_\mu \cdot J^\nu \varepsilon^*_\nu \; J_5^\alpha \varepsilon_\alpha}}{H_{int}^3} | p \rangle$$

- $$\mathcal{M}_5^{\alpha\mu\nu}(p, q_1, q_2)\, \delta^{(4)}(p-q_1-q_2)$$

$$= -i \int d^4x\, d^4y\, d^4z \; e^{-ipx} e^{iq_1 y} e^{iq_2 z} \langle J_5^\alpha(x) J^\mu(y) J^\nu(z) \rangle$$

$$= -i \iiint e^{\cdots} e^{\cdots} e^{\cdots} \langle 0 | T\left[\bar\Psi(x) \gamma^\alpha \gamma_5 \Psi(x) \bar\Psi(y) \gamma^\mu \Psi(y) \bar\Psi(z) \gamma^\nu \Psi(z) \right] | 0 \rangle$$

$$= -i \iiint e^{\cdots} e^{\cdots} e^{\cdots} \Big[\langle 0 | : \bar\Psi(x)_a \gamma^\alpha \gamma_{5\,ab} \Psi(x)_b \bar\Psi(y)_c \gamma^\mu_{cd} \Psi(y)_d \bar\Psi(z)_e \gamma^\nu_{ef} \Psi(z)_f : | 0 \rangle$$

$$+ \langle 0 | : \bar\Psi(x)_a \gamma^\alpha \gamma_{5\,ab} \Psi(x)_b \bar\Psi(y)_c \gamma^\mu_{cd} \Psi(y)_d \bar\Psi(z)_e \gamma^\nu_{ef} \Psi(z)_f : | 0 \rangle \Big]$$

(these two contraction possibilities correspond to the diagrams given on the sheet. All other terms we neglect, because they don't give triangle diagrams)

$\overbrace{\Psi(x)\bar\Psi(y)} = iD(x-y)$

$$= -i \iiint e^{\cdots} e^{\cdots} e^{\cdots} \Big[\underset{\overset{\uparrow}{\text{from Grassmann}}}{-} iD(y-x)_{da} \gamma^\alpha \gamma_{5\,ab}\, iD(x-z)_{be} \gamma^\mu_{cd} \gamma^\nu_{ef}\, iD(z-y)_{fc}$$

$$\overset{\downarrow}{-}\, iD(z-x)_{fa} \gamma^\alpha \gamma_{5\,ab}\, iD(x-y)_{bc} \gamma^\mu_{cd}\, D(y-z)_{de} \gamma^\nu_{ef} \Big]$$

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ik(x-y)}}{\not k}$$

$$\overset{\downarrow}{=} \int d^4x \int d^4y \int d^4z \; e^{-ipx} e^{iq_1 y} e^{iq_2 z} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4}$$

$$= \int d^4x \int d^4y \int d^4z \; e^{-ipx}\, e^{iq_1 y}\, e^{iq_2 z} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4}$$

$$\cdot \mathrm{Tr}\Big[\gamma^\alpha \gamma_5 \frac{i e^{ik_1(x-z)}}{\slashed{k}_1} \gamma^\nu \frac{i e^{ik_2(z-y)}}{\slashed{k}_2} \gamma^\mu \frac{i e^{ik_3(y-x)}}{\slashed{k}_3}$$

$$+ \gamma^\alpha \gamma_5 \frac{i e^{ik_1(x-y)}}{\slashed{k}_1} \gamma^\mu \frac{i e^{ik_2(y-z)}}{\slashed{k}_2} \gamma^\nu \frac{i e^{ik_3(z-x)}}{\slashed{k}_3} \Big]$$

$$= -i \int d^4k_1 \int d^4k_2 \int d^4k_3 \; \frac{1}{k_1^2 k_2^2 k_3^2}$$

$$\cdot \mathrm{Tr}\Big[\gamma^\alpha \gamma_5 \slashed{k}_1 \gamma^\nu \slashed{k}_2 \gamma^\mu \slashed{k}_3 \, \delta^{(4)}(-p+k_1-k_3)\, \delta^{(4)}(q_1-k_2+k_3)\, \delta^{(4)}(q_2-k_1+k_2)$$

$$+ \gamma^\alpha \gamma_5 \slashed{k}_1 \gamma^\mu \slashed{k}_2 \gamma^\nu \slashed{k}_3 \, \delta^{(4)}(-p+k_1-k_3)\, \delta^{(4)}(q_1-k_1+k_2)\, \delta^{(4)}(q_2-k_2+k_3) \Big]$$

$$= -i \int d^4k_2 \int d^4k_3$$

$$\cdot \mathrm{Tr}\Big[\frac{\gamma^\alpha \gamma_5 (\slashed{q}_2 + \slashed{k}_2) \gamma^\nu \slashed{k}_2 \gamma^\mu \slashed{k}_3}{(q_2+k_2)^2\, k_2^2\, k_3^2}$$

$$\delta^{(4)}(-p+q_2+k_2-k_3)\, \delta^{(4)}(q_1-k_2+k_3) + (\mu \leftrightarrow \nu,\ q_1 \leftrightarrow q_2) \Big]$$

$k := k_2$

$$= -i \int d^4k \; \mathrm{Tr}\Big[\frac{\gamma^\alpha \gamma_5 (\slashed{q}_2 + \slashed{k}) \gamma^\nu \slashed{k} \gamma^\mu (\slashed{k} - \slashed{q}_1)}{(q_2+k)^2\, k^2\, (k-q_1)^2} \Big] \delta^{(4)}(-p+q_1+q_2)$$

$$+ (\mu \leftrightarrow \nu,\ q_1 \leftrightarrow q_2)$$

(3)

Axial current:

- $p_\alpha M_5^{\alpha\mu\nu} = (q_{1\alpha} + q_{2\alpha}) M_5^{\alpha\mu\nu}$

add and subtracted

$$= -i \int d^4k \; \frac{\mathrm{Tr}\big((\slashed{q}_1 - \slashed{k} + \slashed{q}_2 + \slashed{k}) \gamma_5 (\slashed{q}_2 + \slashed{k}) \gamma^\nu \slashed{k} \gamma^\mu (\slashed{k} - \slashed{q}_1) \big)}{(q_2+k)^2 (k-q_1)^2 k^2} + \begin{pmatrix} \mu \leftrightarrow \nu \\ q_1 \leftrightarrow q_2 \end{pmatrix}$$

$$= +i \int d^4k \; \frac{\mathrm{Tr}\big(\gamma_5 (q_2+k)^2 \gamma^\nu \slashed{k} \gamma^\mu (\slashed{k} - \slashed{q}_1) \big) + \mathrm{Tr}\big(\gamma_5 (\slashed{q}_2 + \slashed{k}) \gamma^\nu \slashed{k} \gamma^\mu (k-q_1)^2 \big)}{(q_2+k)^2\, (k-q_1)^2\, k^2}$$

$$+ \begin{pmatrix} \mu \leftrightarrow \nu \\ q_1 \leftrightarrow q_2 \end{pmatrix}$$

$$+\begin{pmatrix}\mu\leftrightarrow\nu\\ q_1\leftrightarrow q_2\end{pmatrix}$$

$$= -4\int d^4k\left(\underbrace{\frac{\varepsilon^{\nu\alpha\mu\beta}\,k_\alpha\,(k_\beta-q_{1\beta})}{(k-q_1)^2\,k^2}}_{\text{no } q_2} + \underbrace{\frac{\varepsilon^{\alpha\nu\beta\mu}\,(q_{2\alpha}+k_\alpha)\,k_\beta}{(q_2+k)^2\,k^2}}_{\text{no } q_1} + \begin{pmatrix}\mu\leftrightarrow\nu\\ q_1\leftrightarrow q_2\end{pmatrix}\right)$$

- After integration we will get something of form

$$\varepsilon^{\nu\alpha\mu\beta}\left(A\,\eta_{\alpha\beta} + B\,q_{1\alpha}\,q_{1\beta}\right) = 0$$

for the first term.

$\rightarrow$ $p_\alpha\,\mathcal{M}_5^{\alpha\mu\nu} = 0$

$\rightarrow$ axial Ward-Takahashi identity not violated

Vector current

- $q^1_\mu\,\mathcal{M}_5^{\alpha\mu\nu}$

$$= -i\int d^4k\left(\frac{\mathrm{Tr}\left(\gamma^\alpha\gamma_5(\not{q}_2+\not{k})\gamma^\nu\not{k}\,\overset{=\not{k}+\not{q}_1-\not{k}}{\not{q}_1}\,(\not{k}-\not{q}_1)\right)}{(q_2+k)^2\,k^2\,(k-q_1)^2} + \frac{\mathrm{Tr}\left(\gamma^\alpha\gamma_5(\not{q}_1+\not{k})\,\overset{=\not{q}_1+\not{k}-\not{k}}{\not{q}_1}\,\not{k}\,\gamma^\nu(\not{k}-\not{q}_2)\right)}{(q_1+k)^2\,k^2\,(k-q_2)^2}\right)$$

$$= +i\int d^4k\left(\frac{\mathrm{Tr}\left(\gamma^\alpha\gamma_5(\not{q}_2+\not{k})\gamma^\nu(\not{q}_1-\not{k})\right)}{(q_2+k)^2\,(k-q_1)^2} + \underbrace{\frac{\mathrm{Tr}\left(\gamma^\alpha\gamma_5(\not{q}_2+\not{k})\gamma^\nu\not{k}\right)}{(q_2+k)^2\,k^2}}_{=0}\right.$$

$$\left. - \underbrace{\frac{\mathrm{Tr}\left(\gamma^\alpha\gamma_5\not{k}\gamma^\nu(\not{k}-\not{q}_2)\right)}{k^2\,(k-q_2)^2}}_{\substack{=0 \text{ after integration}\\ \text{for same reason as above}}} + \frac{\mathrm{Tr}\left(\gamma^\alpha\gamma_5(\not{q}_1+\not{k})\gamma^\nu(\not{k}-\not{q}_2)\right)}{(q_1+k)^2\,(k-q_2)^2}\right)$$

$$= 4\varepsilon^{\alpha\beta\nu\mu}\int d^4k\left(\frac{(q_{2\beta}+k_\beta)(q_{1\mu}-k_\mu)}{(q_2+k)^2\,(q_1-k)^2} + \frac{(q_{1\beta}+k_\beta)(k_\mu-q_{2\mu})}{(q_1+k)^2\,(k-q_2)^2}\right)$$

- In the first, we could apply the shift $k\longmapsto k+q_1$

- In the first, we could apply the shift $k \longmapsto k+q_1$ and in the second term, $k \longmapsto k+q_2$.

$\rightarrow q^1_\mu M_5^{\alpha\mu\nu}$

$$= 4\varepsilon^{\alpha\beta\nu\mu}\int d^4k\left(-\frac{(q_{2\beta}+k_\beta+q_{1\beta})\,k_\mu}{(q_2+k+q_1)^2\,k^2}+\frac{(q_{1\beta}+k_\beta+q_{2\beta})\,k_\mu}{(q_1+k+q_1)^2\,k^2}\right)=0$$

BUT: Shift is not allowed, because integral goes like $\int \frac{d^4k}{k^3}$ (k^2 term vanish by symmetry of indices)

$\rightarrow$ linear divergent

- $q^1_\mu M_5^{\alpha\mu\nu}$

$$= 4\varepsilon^{\alpha\beta\nu\mu}\int d^4k\left(\frac{(q_{2\beta}+k_\beta)(q_{1\mu}-k_\mu)}{(q_2+k)^2(q_1-k)^2}+\frac{(q_{1\beta}+k_\beta)(k_\mu-q_{2\mu})}{(q_1+k)^2(k-q_2)^2}\right)$$

$$= \int d^4k\left(f^{\alpha\nu}(k+q_1-q_2)-f^{\alpha\nu}(k)\right)$$

where $f^{\alpha\nu}(k) = 4\varepsilon^{\alpha\beta\nu\mu}\frac{(q_{2\beta}+k_\beta)(k_\mu-q_{1\mu})}{(q_2+k)^2(q_1-k)^2}$

- Solving this integral in general:

$$I = \int d^4k\left[f_\alpha(k+a)-f_\alpha(k)\right]$$

Applying Wick rotation $\vec{k}_E = i\vec{k}$

$$\rightarrow k^2 = (k^0)^2-\vec{k}^2 = (k_E^0)^2+\vec{k}_E^2$$

$$I = i\int d^4k_E\left[f_\alpha(k_E+a)-f_\alpha(k_E)\right]$$

Taylor around $a=0$

$$= i\int d^4k_E\left[a^\mu\frac{\partial}{\partial k_E^\mu}f_\alpha(k_E)+\mathcal{O}(a^2)\right]$$

Ground $a=0$

$$= i\int d^4k_E \left[a^\mu \underbrace{\frac{\partial}{\partial k_E^\mu} f_\alpha(k_E)}_{\sim \frac{1}{k^4}} + \mathcal{O}(a^2)\right]$$

$f_\alpha(k_E) \sim \frac{1}{k^3}$

$\mathcal{O}(a^2) \sim \mathcal{O}\left(\frac{1}{k^5}\right)$

In the high momentum limit we can neglect the higher order terms.

3D-surface

- $$I = i a^\mu \int dS_\mu \, f_\alpha(k_E) \Big|_{|k_E| \to \infty}$$

$$dS_\mu = |k_E|^3 \frac{k_{E\mu}}{|k_E|} d\Omega_3$$

$$= i a^\mu \lim_{|k_E| \to \infty} \int d\Omega_3 \, |k_E|^2 \, k_{E\mu} \, f_\alpha(k_E)$$

$$\lim_{|k_E| \to \infty} f_\alpha(k_E) = A \cdot \frac{k_{E\alpha}}{k_E^4}$$

int. is linear divergent

- Before we continue we have to prepare some identities

$$\int d^Dx \; f(x^2)\, x_\mu x_\nu = 0 \quad \text{for} \quad \mu \neq \nu$$

$$\rightarrow \int d^Dx \; f(x^2)\, x_\mu x_\nu = \delta_{\mu\nu} \cdot B$$

Euclidean coordinates

Taking trace: $\int d^Dx \; f(x^2)\, x^2 = D B \quad | \cdot \delta_{\mu\nu}$

$$\int d^Dx \; f(x^2)\, x^2 \delta_{\mu\nu} = D \, \delta_{\mu\nu} B$$

$$\rightarrow x_\mu x_\nu = \frac{1}{D} x^2 \delta_{\mu\nu} \quad \text{(only true within integral)}$$

In our case: $k_{E\mu} k_{E\nu} = \frac{1}{4} |k_E|^2 \delta_{\mu\nu}$

- $\int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$

$\rightarrow \int d\Omega_3 = \frac{2\pi^{4/2}}{\Gamma(2)} = 2\pi^2$

$$\int d^n x \; e^{-\vec{x}^2} = \int d\Omega_{n-1} \int dr \; r^{n-1} e^{-r^2}$$

$y = r^2 \rightarrow dy = 2r\,dr$

$$\sqrt{\pi}^n = \int d^n x \, e^{-\vec{x}^2} = \int d\Omega_{n-1} \int dy \; \frac{1}{2} \, y^{\frac{n}{2}-1} \cdot e^{-y}$$

$$= \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \int d\Omega_{n-1}$$

$$\rightarrow \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

- Finally we obtain for I:

$$I = i a^\mu \int d\Omega_3 \; |k_E|^2 \frac{1}{|k_E|^4} |k_E|^2 A \, \delta_{\mu\alpha}$$

$$= 2 i \pi^2 a_\alpha A$$

$$\rightarrow \int d^4 k \left[f_\alpha(k+a) - f_\alpha(k) \right] = 2 i \pi^2 a_\alpha A$$

- $f^{\alpha\nu} = 4 \varepsilon^{\alpha\beta\nu\mu} \frac{(q_{2\beta} + k_\beta)(k_\mu - q_{1\mu})}{(q_2 + k)^2 (q_1 - k)^2}$

$$\xrightarrow{k \to \infty} 4 \varepsilon^{\alpha\beta\nu\mu} \frac{1}{k^4} (k_\mu q_{2\beta} - k_\beta q_{1\mu})$$

$$= 4 \varepsilon^{\alpha\beta\nu\mu} \frac{k_\mu}{k^4} (q_2 + q_1)_\beta = A^{\alpha\nu\mu} \frac{k_\mu}{k^4}$$

$$\Rightarrow q^1_\mu M_5^{\alpha\mu\nu} = 2i\pi^2 (q_1 - q_2)_\mu \, 4\varepsilon^{\alpha\beta\nu\mu} (q_2 + q_1)_\beta$$

$$= -16 i\pi^2 \varepsilon^{\alpha\beta\nu\mu} q_{1\mu} q_{2\beta} \neq 0$$

$$\Rightarrow p_\alpha M_5^{\alpha\mu\nu} = 0, \quad \text{but} \quad q^1_\mu M_5^{\alpha\mu\nu} \neq 0$$

→ We have an anomaly, because classically both is zero

(4)

- In first diagram:

$$k^\mu \longmapsto k^\mu + b_1 q_1^\mu + b_2 q_2^\mu$$

$$M_{\text{first diagram}}(q_1, q_2) \overset{!}{=} M_{\text{second diagram}}(q_2, q_1)$$

→ we need to exchange b_1 and b_2 for second diagram:

$$k^\mu \longmapsto k^\mu + b_2 q_1^\mu + b_1 q_2^\mu$$

- For the vector case:

$$\int d^4k \, [\underbrace{f_\alpha(k+a)}_{\text{from 1st diagram}} - \underbrace{f_\alpha(k)}_{\text{from 2nd diagram}}] = 2i\pi^2 a_\alpha A$$

$$\longmapsto \int d^4k \, [f_\alpha(k + b_1 q_1 + b_2 q_2 + a) - f_\alpha(k + b_2 q_1 + b_1 q_2)]$$

shift integral

$$= \int d^4k \, [f_\alpha(k + b_1 q_1 + b_2 q_2 - b_2 q_1 - b_1 q_2 + a) - f_\alpha(k)]$$

$$= \int d^4k \, [f_\alpha(k + \underbrace{(b_1 - b_2)(q_1 - q_2)}_{\text{absorb in } a} + a) - f_\alpha(k)]$$

$q_1 - q_2$

$$\rightarrow q^1_\mu \mathcal{M}_5^{\alpha\mu\nu} = 2i\pi^2 \, (q_1 - q_2)_\mu \, (1 + b_1 - b_2) \quad 4\,\varepsilon^{\alpha\beta\nu\mu} \, (q_2 + q_1)_\beta$$

$$= -16 i\pi^2 \, \varepsilon^{\alpha\beta\nu\mu} \, q_{1\mu} q_{2\beta} \, (1 + b_1 - b_2)$$

- In the axial calculation we have to do the same replacement.

$$p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = -4 \int d^4k \Bigg(\overbrace{\frac{\varepsilon^{\nu\alpha\mu\beta} k_\alpha (k_\beta - q_{1\beta})}{(k-q_1)^2 k^2}}^{f_1(k-q_1)} + \overbrace{\frac{\varepsilon^{\alpha\nu\beta\mu} (q_{2\alpha} + k_\alpha) k_\beta}{(q_2+k)^2 k^2}}^{f_2(k-q_2)} \quad \text{1st diagram}$$

$$- \underbrace{\frac{\varepsilon^{\nu\alpha\mu\beta} k_\alpha (k_\beta - q_{2\beta})}{(k-q_2)^2 k^2}}_{f_2(k)} - \underbrace{\frac{\varepsilon^{\alpha\nu\beta\mu} (q_{1\alpha} + k_\alpha) k_\beta}{(q_1+k)^2 k^2}}_{f_1(k)} \Bigg) \quad \text{2nd diagram}$$

- $\int d^4k \, \big(f_1(k-q_1) - f_1(k)\big)$

$$\longrightarrow \int d^4k \, \big(f_1(k + b_1 q_1 + b_2 q_2 - q_1) - f_1(k + b_2 q_1 + b_1 q_2)\big)$$

$$= \int d^4k \, \big(f_1(k+a) - f(k)\big)$$

with $a = (b_1 - b_2)(q_1 - q_2) - q_1$

$$f_1(k) \xrightarrow{k\to\infty} -\varepsilon^{\alpha\nu\beta\mu} \frac{k_\alpha}{k^4} q_{1\beta}$$

- $\int d^4k \, \big(f_2(k-q_2) - f_2(k)\big) \longrightarrow \int d^4k \, \big(f_2(k+a) - f_2(k)\big)$

with $a = (b_1 - b_2)(q_1 - q_2) - q_2$

$$f_2(k) \xrightarrow{k\to\infty} -\varepsilon^{\alpha\nu\beta\mu} \frac{k_\alpha}{k^4} q_{2\beta}$$

- $p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = +4 \cdot 2i\pi^2 \Big[\big((b_1 - b_2)(q_{1\alpha} - q_{2\alpha}) - q_{1\alpha}\big) \, \varepsilon^{\alpha\nu\beta\mu} q_{1\beta}$

- $$p_\alpha M_5^{\alpha\mu\nu} = +4\cdot 2i\pi^2 \Big[\big((b_1-b_2)(q_{1\alpha}-q_{2\alpha})-q_{1\alpha}\big)\,\varepsilon^{\alpha\nu\beta\mu}\, q_{1\beta} + \big((b_1-b_2)(q_{1\alpha}-q_{2\alpha})-q_{2\alpha}\big)\,\varepsilon^{\alpha\nu\beta\mu}\, q_{2\beta} \Big]$$

$$= -16 i\pi^2\, \varepsilon^{\alpha\nu\beta\mu}\, (b_1-b_2)\, q_{2\alpha} q_{1\beta}$$

- Summarized:

vector: $q_{1\mu} M^{\alpha\mu\nu} \sim (1+b_1-b_2)$

axial: $p_\alpha M_5^{\alpha\mu\nu} \sim (b_1-b_2)$

→ for $b_1-b_2=-1$ we get conservation of vector current but not of axial current anymore.

→ anomaly still there

(5)

- In (2) we found:

$$M_5^{\alpha\mu\nu} = -i\int d^4k\, \mathrm{Tr}\left[\frac{\gamma^\alpha\gamma_5(\not q_2+\not k)\gamma^\nu \not k \gamma^\mu(\not k-\not q_1)}{(q_2+k)^2\, k^2\, (k-q_1)^2}\right] + \begin{pmatrix}\mu\leftrightarrow\nu\\ q_1\leftrightarrow q_2\end{pmatrix}$$

- The calculation is similar with J_L.

We just need to replace

$$\gamma^\alpha\gamma_5,\ \gamma^\nu,\ \gamma^\mu \longrightarrow \gamma^\alpha P_L,\ \gamma^\nu P_L,\ \gamma^\mu P_L$$

$$\to M_L^{\alpha\mu\nu} = -i\int d^4k\, \mathrm{Tr}\left[\frac{\gamma^\alpha P_L(\not q_2+\not k)\gamma^\nu P_L \not k\gamma^\mu P_L(\not k-\not q_1)}{(q_2+k)^2\, k^2\, (k-q_1)^2}\right] + \begin{pmatrix}\mu\leftrightarrow\nu\\ q_1\leftrightarrow q_2\end{pmatrix}$$

$$= -\frac{1}{2} i\int d^4k\, \mathrm{Tr}\left[\frac{\gamma^\alpha(1+\gamma_5)(\not q_2+\not k)\gamma^\nu \not k\gamma^\mu(\not k-\not q_1)}{(q_2+k)^2\, k^2\, (k-q_1)^2}\right] + \begin{pmatrix}\mu\leftrightarrow\nu\\ q_1\leftrightarrow q_2\end{pmatrix}$$

$$= \frac{1}{2} M_V^{\alpha\mu\nu} + \frac{1}{2} M_5^{\alpha\mu\nu}$$

$\rightarrow \mathcal{M}_L^{\alpha\mu\nu}$ contains $\mathcal{M}_S^{\alpha\mu\nu}$

$\rightarrow J_L^\mu$ is anomalous

(6)

- $J^\mu_{mix} = J_L^\mu + J_R^\mu$

$$= Q_L \bar{\Psi}\gamma^\mu P_L \Psi + Q_R \bar{\Psi}\gamma^\mu P_R \Psi$$

- To find $\mathcal{M}_{mix}^{\alpha\mu\nu}$ we have to replace all P_L with $Q_L P_L + Q_R P_R$ in $\mathcal{M}_L^{\alpha\mu\nu}$.
  All terms with mixing chirality projectors will be zero, because $P_L P_R = 0$.

$$\rightarrow \mathcal{M}_{mix}^{\alpha\mu\nu} = Q_L^3 \mathcal{M}_L^{\alpha\mu\nu} + Q_R^3 \mathcal{M}_R^{\alpha\mu\nu}$$

$$= \frac{1}{2} Q_L^3 (\mathcal{M}_V^{\alpha\mu\nu} + \mathcal{M}_S^{\alpha\mu\nu}) + \frac{1}{2} Q_R^3 (\mathcal{M}_V^{\alpha\mu\nu} - \mathcal{M}_S^{\alpha\mu\nu})$$

$$= \frac{1}{2} (Q_L^3 - Q_R^3) \mathcal{M}_S^{\alpha\mu\nu} + \frac{1}{2} (Q_L^3 + Q_R^3) \mathcal{M}_V^{\alpha\mu\nu}$$

$\rightarrow$ Anomaly free if $Q_L = Q_R$

- Comment:

  global sym. $\downarrow$ in SM

  In the SM all currents $(SU(3), SU(2)_L, U(1)_Y, U(1)_{em}, U(1)_{B-L})$ are anomaly free.