Problem 1:

$$E_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \tilde{H} = i\sigma_2 H^*$$

- Yukawa sector:

$$\mathcal{L}_Y = -\Lambda^{(e)}_{ij} \bar{E}^i_L H e^j_R - \Lambda^{(d)}_{ij} \bar{Q}^i_L H d^j_R - \Lambda^{(u)}_{ij} \bar{Q}^i_L \tilde{H} u^j_R + h.c.$$

family indices

(1)

- kinetic terms like $i\bar{u}^i_L \not{\partial} u^i_L$ need to be invariant, i.e. we can only apply unitary transformations on the fermions.

  → we need a hermitian matrix, because hermitian matrices can be diagonalized by unitary transformations

  → $\Lambda_u \Lambda_u^\dagger$ is hermitian

$$\Lambda_u \Lambda_u^\dagger = U_u D_u^2 U_u^\dagger$$

  where D_u^2 is diagonal.

  → $\Lambda_u = U_u D_u K_u^\dagger$ with $K_u^\dagger$ unitary

  In the same way we can get

$$\Lambda_d = U_d D_d K_d^\dagger$$

- Let us choose $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}$ → $\tilde{H} = i\sigma_2 H^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix}$

$$\rightarrow \mathcal{L}_Y \supset -\frac{1}{\sqrt{2}}(v+h)\,\bar{d}_L \Lambda_d d_R - \frac{1}{\sqrt{2}}(v+h)\,\bar{u}_L \Lambda_u u_R + h.c.$$

- $\bar{u}_L \Lambda_u u_R = \bar{u}_L U_u D_u K_u^+ u_R$

$$\longmapsto \bar{u}_L D_u u_R$$

where we applied

$$u_L \longmapsto U_u u_L \quad , \quad u_R \longmapsto K_u u_R$$

In the same way we can do it for the d-term and we find the Yukawa sector to be:

$$\mathcal{L}_Y \supset -\frac{1}{\sqrt{2}}(v+h)\,\bar{d}_L D_d d_R - \frac{1}{\sqrt{2}}(v+h)\,\bar{u}_L D_u u_R + h.c.$$

- In the neutral current sector there are only terms of form $A_\mu \bar{u}_R \gamma^\mu u_R$ that are flavour-diagonal, i.e. the above unitary transformations will not affect this term.

- The charged current sector is not invariant:

sheet 7

$$\mathcal{L}_{cc,quarks} = \frac{g}{\sqrt{2}}\bar{d}_L \gamma^\mu u_L W_\mu^- + \frac{g}{\sqrt{2}}\bar{u}_L \gamma^\mu d_L W_\mu^+$$

$$\longmapsto \frac{g}{\sqrt{2}}\bar{d}_L U_d^+ U_u \gamma^\mu u_L W_\mu^- + \frac{g}{\sqrt{2}}\bar{u}_L U_u^+ U_d \gamma^\mu d_L W_\mu^+$$

Mixing matrix $V_{CKM} := U_u^+ U_d$

Cabibbo - Kobayashi - Maskawa

- The charged current sector doesn't have right-handed quarks → rotations on right-handed quarks are unphysical.

(2)

- $V_{CKM} \in U(N)$ → dimensions N^2

If V_{CKM} would be real $\in O(N)$ it would have dimension $\frac{N\cdot(N-1)}{2}$

→ $\frac{N\cdot(N-1)}{2}$ angles

$$N^2 - \frac{N(N-1)}{2} = \frac{N(N+1)}{2} \quad \text{phases}$$

- But there are N quark families.

Hence, we can reabsorb $\underset{\uparrow \binom{u}{d}}{2N-1}$ parameters into the quark fields.

example $N=2$:

$$MM^+ = \begin{pmatrix} \cos\Theta\, e^{i\alpha} & \sin\Theta\, e^{i\beta} \\ -\sin\Theta\, e^{i\gamma} & \cos\Theta\, e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos\Theta\ e^{-i\alpha} & -\sin\Theta\, e^{-i\gamma} \\ \sin\Theta\ e^{-i\beta} & \cos\Theta\ e^{-i\delta} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \alpha - \gamma = \beta - \delta$$

$$\rightarrow M = \begin{pmatrix} \cos\Theta\ e^{i\alpha} & \sin\Theta\, e^{i\beta} \\ -\sin\Theta\ e^{i\gamma} & \cos\Theta\ e^{i(\beta+\gamma-\alpha)} \end{pmatrix}$$

$$\begin{pmatrix} \bar{u} \\ \bar{c} \end{pmatrix}^T M \begin{pmatrix} d \\ s \end{pmatrix} = \bar{u}\ \cos\Theta\, e^{i\alpha} d + \bar{u}\ \sin\Theta\, e^{i\beta} s$$
$$- \bar{c}\ \sin\Theta\ e^{i\gamma} d + \bar{c}\ \cos\Theta\, e^{i(\beta+\gamma-\alpha)} s$$

$d \longmapsto e^{i\alpha} d \,,\quad c \longmapsto e^{i\gamma - i\alpha} c \,,\quad s \longmapsto e^{-i\beta} s$

kills all phases ┘

- $\frac{N\cdot(N+1)}{2} - (2N-1) = \frac{N^2-3N+2}{2}$ phases that cannot be rotated away

- $N=2$: 1 real parameter
  0 imaginary parameters

  $N=3$: 3 real parameters
  1 imaginary parameter

(3)

- $\psi \xrightarrow{C} \psi^c \xrightarrow{P} \gamma^0 \psi^c \qquad (C = i\gamma_2\gamma_0 \,,\ C^{-1} = C^\dagger = C^T = -C)$

- $$\bar{Q}_L H \Lambda d_R + h.c. \longmapsto \overline{(\gamma^0 Q_L^c)}\, H\Lambda\, \gamma^0 d_R^c + h.c.$$
$$= (\gamma^0 C \bar{Q}_L^T)^\dagger \gamma^0 H\Lambda \gamma^0 C \bar{d}_R^T + h.c.$$
$$= \bar{Q}_L^* \underbrace{C^\dagger}_{\cdot(-1)} \gamma^0 H\Lambda \underbrace{\gamma^0\gamma^0}_{=1} C \gamma^0 d_R^* + h.c.$$
$$= -\bar{Q}_L^* H\Lambda d_R^* + h.c.$$
Grassmann math
$$\overset{!}{=} \left(-\bar{Q}_L^* H\Lambda d_R^* + h.c.\right)^*$$
$$= \bar{Q}_L H\Lambda^* d_R + h.c.$$

┌ $A^T B$ is scalar
$= (A^T B)^T = (B_i A_i)$
$= -A_i B_i = -A^T B$
↯ T has to give $-$ sign.
$A^\dagger A$ is real scalar
$= (A^\dagger A)^* = A^T A^* = (A^T A^*)^T$
$= -A^\dagger A$
↯ $*$ has to give $-$ sign ┘

$\rightarrow$ CP invariance requires $\Lambda = \Lambda^*$

- In $N=2$ this is fulfilled, i.e. there is

CP invariance.

From experiments we know that there is CP violation, which indicates that we need at least $N=3$.

(4)

- $\mathcal{L}_Y \supset -(v+h)\,\bar{e}_L \Lambda_e e_R$

$$= -(v+h)\,\bar{e}_L U_e D_e K_e^{\dagger} e_R$$

We can do the transformation

$$e_R \longmapsto K_e e_R \quad , \quad e_L \longmapsto U_e e_L$$

- This alone would lead to a mixing matrix. However, since neutrinos are massless we have the freedom to perform additionally

$$\nu_L \longmapsto U_e \nu_L$$

$\rightarrow$ No mixing matrix

Problem 2

(1)

- There are two different ways to write down a Lorentz invariant mass term for fermions

Dirac mass: $\quad m_D \bar{\Psi}_L \Psi_R + h.c.$

Majorana mass: $\quad m_M \bar{\Psi}_L^c \Psi_L + h.c.$

$\quad (\text{or } m_M \bar{\Psi}_R^c \Psi_R + h.c.)$

with $\Psi_L^c = C \bar{\Psi}_L^T = i\gamma^2\gamma^0\gamma^0 L \Psi^*$

$$= R\,\Psi^c$$

$$\rightarrow m_M \bar{\Psi}_L^c \Psi_L = m_M (\Psi_L^+ \gamma^0)^* \gamma^0 C \Psi_L = m_M \Psi_L^T C \Psi_L$$

- From the mass term we see that a Majorana particle has to carry zero charge otherwise the mass term violates charge conservation. The Dirac fermion can describe charged particles, but of course also chargeless particle. Therefore, the neutrino can be both, Majorana and/or Dirac.

(2)

$$Y = \tfrac{1}{2} \quad Y = -\tfrac{1}{2} \quad Y = 0$$

- Dirac mass: $-\Lambda_{ij}^{(\nu)} \bar{E}_L^i \tilde{H} \nu_R^j + h.c.$

- Majorana mass: $-M_{ij} \bar{\nu}_R^{c\,i} \nu_R^j + h.c.$

Notice that we cannot write down a Majorana mass term for ν_L, because ν_L is charged under $SU(2)_L \times U(1)_Y$.

$$\rightarrow \mathcal{L}_{Y,leptons} = -\Lambda_{ij}^{(e)} \bar{E}_L^i H e_R^j - \Lambda_{ij}^{(\nu)} \bar{E}_L^i \tilde{H} \nu_R^j - M_{ij} \bar{\nu}_R^{c\,i} \nu_R^j + h.c.$$

(3)

- $\Lambda_e = U_e D_e K_e^+$

  $\Lambda_\nu = U_\nu D_\nu K_\nu^+$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$\rightarrow \mathcal{L}_Y \supset -\bar{E}_L U_e D_e K_e^+ H e_R - \bar{E}_L U_\nu D_\nu K_\nu^+ \tilde{H} \nu_R + h.c.$$

$$\rightarrow \mathcal{L}_Y \supset -\bar{E}_L U_e D_e K_e^\dagger H e_R - \bar{E}_L U_\nu D_\nu K_\nu^\dagger \tilde{H} \nu_R + h.c.$$

$$= -\frac{1}{\sqrt{2}}(v+h)\,\bar{e}_L U_e D_e K_e^\dagger e_R - \frac{1}{\sqrt{2}}(v+h)\,\bar{\nu}_L U_\nu D_\nu K_\nu^\dagger \nu_R + h.c.$$

- Apply the transformation

$$e_R \longmapsto K_e e_R \qquad \nu_R \longmapsto K_\nu \nu_R$$

$$e_L \longmapsto U_e e_L \qquad \nu_L \longmapsto U_\nu \nu_L$$

- In the electroweak current sector we get

$$\mathcal{L} \supset \frac{g}{\sqrt{2}}\left(\bar{\nu}_L W_\mu^+ \gamma^\mu e_L + \bar{e}_L W_\mu^- \gamma^\mu \nu_L\right)$$

$$\longmapsto \frac{g}{\sqrt{2}}\left(\bar{\nu}_L \underbrace{U_\nu^\dagger U_e}_{=P} W_\mu^+ \gamma^\mu e_L + \bar{e}_L \underbrace{U_e^\dagger U_\nu}_{=P^\dagger} W_\mu^- \gamma^\mu \nu_L\right)$$

$\rightarrow$ We obtained a mixing matrix (PMNS matrix)

(4)

- flavor eigenstates $|\nu_e\rangle$, $|\nu_\mu\rangle$

mass eigenstates $|\nu_1\rangle$, $|\nu_2\rangle$

- The mass eigenstates evolve with time as

$$|\nu_1(t)\rangle = e^{-iE_1 t}\left(\cos\Theta\,|\nu_e\rangle + \sin\Theta\,|\nu_\mu\rangle\right)$$

$$|\nu_2(t)\rangle = e^{-iE_2 t}\left(\cos\Theta\,|\nu_\mu\rangle - \sin\Theta\,|\nu_e\rangle\right)$$

- $e^{+iE_1 t}\cos\Theta\,|\nu_1\rangle - e^{+iE_2 t}\sin\Theta\,|\nu_2\rangle$

$$= \cos^2\Theta\,|\nu_e\rangle + \sin^2\Theta\,|\nu_e\rangle$$

$$\rightarrow |\nu_e(t)\rangle = e^{+iE_1 t}\cos\Theta\,|\nu_1\rangle - e^{+iE_2 t}\sin\Theta\,|\nu_2\rangle$$

$$\rightarrow |\nu_e(t)\rangle = e^{+iE_1 t} \cos\Theta |\nu_1\rangle - e^{+iE_2 t} \sin\Theta |\nu_2\rangle$$

- $\langle \nu_e(T) | \nu_e(0)\rangle = e^{-iE_1 T} \cos^2\Theta + e^{-iE_2 T} \sin^2\Theta$

$$P \sim |\langle \nu_e(T) | \nu_e(0)\rangle|^2$$

$$= \cos^4\Theta + \sin^4\Theta + \sin^2\Theta \cos^2\Theta \left(e^{i(E_1-E_2)T} + e^{-i(E_1-E_2)T} \right)$$

$$= (\cos^2\Theta + \sin^2\Theta)^2 - 2\sin^2\Theta\cos^2\Theta + 2\sin^2\Theta\cos^2\Theta \cos((E_1-E_2)T)$$

$$= 1 - 2\sin^2\Theta\cos^2\Theta \left(1 - \cos^2\left(\frac{(E_1-E_2)T}{2}\right) + \sin^2\left(\frac{(E_1-E_2)T}{2}\right)\right)$$

$$= 1 - \sin^2(2\Theta) \sin^2\left(\frac{(E_1-E_2)T}{2}\right)$$

- *Non-relativistic limit:*

$$E_1 = \sqrt{|\vec{p}_1|^2 + m_1^2} \approx |\vec{p}_1|^2 \cdot \left(1 + \frac{1}{2} \frac{m_1^2}{|\vec{p}_1|^2}\right)$$

$$E_1 - E_2 \overset{|\vec{p}_1| = |\vec{p}_2|}{\approx} \frac{1}{2} \frac{(m_1^2 - m_2^2)}{|\vec{p}_1|} = \frac{1}{2} \frac{\Delta m^2}{E}$$

$$P \sim 1 - \sin^2(2\Theta) \sin^2\left(\frac{\Delta m^2 T}{4E}\right)$$

$\rightarrow$ $P \neq 1$ means that neutrino flavour will change if $\Delta m \neq 0$, i.e. at least one neutrino is massive.

(5)

- Dirac for left- and righthanded neutrino, but Majorana only for righthanded neutrino.

$$\mathcal{L}_{mass} = -m_D \bar{\nu}_R \nu_L - m_M \nu_R^T C \nu_R + h.c.$$

- $\bar{\nu}_R \nu_L = \nu_R^\dagger \gamma^0 \nu_L$ with $\nu_R = C\,\overline{(\nu_R^c)}^T = C\gamma^0 (\nu_R^c)^*$

$$= (\nu_R^c)^T \gamma^0 C^\dagger \gamma^0 \nu_L$$
$$= N_L^T C \nu_L$$

- $\nu_R^T C \nu_R + h.c. = (\nu_R^c)^\dagger \gamma^0 C^T C\, C \gamma^0 (\nu_R^c)^* + h.c.$

$$= -(\nu_R^c)^\dagger C (\nu_R^c)^* + h.c.$$
$$= \left(-N_L^\dagger C \; N_L^* + h.c.\right)^*$$
$$= + N_L^T C^* N_L + h.c.$$
$$= N_L^T C N_L$$

$$\rightarrow \mathcal{L}_{mass} = -m_D N_L^T C \nu_L - m_M N_L^T C N_L + h.c.$$
$$= -\frac{m_D}{2} N_L^T C \nu_L - \frac{m_D}{2} \nu_L^T C N_L - m_M N_L^T C N_L + h.c.$$

- $\mathcal{L}_{mass} = -\begin{pmatrix} N_L^T & \nu_L^T \end{pmatrix} \begin{pmatrix} m_M & \frac{m_D}{2} \\ \frac{m_D}{2} & 0 \end{pmatrix} \begin{pmatrix} N_L \\ \nu_L \end{pmatrix} + h.c.$

$$\det \begin{pmatrix} m_M - \lambda & m_D/2 \\ \frac{m_D}{2} & -\lambda \end{pmatrix} = -\lambda (m_M - \lambda) - \frac{m_D^2}{4}$$
$$= \lambda^2 - \lambda m_M - \frac{m_D^2}{4}$$

$$\rightarrow \lambda_{1,2} = \frac{1}{2}\left(m_M \pm \sqrt{m_M^2 + m_D^2}\right)$$

- We can assume that $m_D \ll m_M$, because from experiments we know that the left-handed neutrino is very light.

$\lambda \approx m_M + \frac{1}{\cdot}\frac{m_D^2}{\cdot}$ → heavy neutrino

$$\lambda_1 \approx m_M + \frac{1}{4} \frac{m_D^2}{m_M} \quad \rightarrow \text{heavy neutrino}$$

$$\lambda_2 \approx -\frac{1}{4} \frac{m_D^2}{m_M} \quad \rightarrow \text{light neutrino}$$

General idea:

One adds a right-handed neutrino that has a different (way higher) mass than the left-handed neutrino. Diagonalization leads to a very low mass for the light neutrino.