Problem 1:

- $$\mathcal{L} = -\frac{1}{4} W_{\mu\nu}^{a} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu H)^\dagger (D^\mu H) - \lambda \left(H^\dagger H - \frac{v^2}{2}\right)^2$$

where

$$W_{\mu\nu}^{a} = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\, \varepsilon_{abc} W_\mu^b W_\nu^c$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

- $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ with $H_{1,2} \in \mathbb{C}$

$$D_\mu H = \partial_\mu H - i g W_\mu^a \underbrace{\tau^a}_{SU(2)\ \text{generators}} H - i \frac{g'}{2} B_\mu H$$

(1)
- $V = \lambda \left(H^\dagger H - \frac{v^2}{2}\right)^2$

$$\frac{\partial V}{\partial H} = 2\lambda \left(H^\dagger H - \frac{v^2}{2}\right) H^\dagger \overset{!}{=} 0$$

$$\to H = 0 \quad \text{or} \quad H^\dagger H = \frac{v^2}{2}$$

$$\frac{\partial^2 V}{\partial H^\dagger \partial H} = 2\lambda \left(H^\dagger H - \frac{v^2}{2}\right) + 2\lambda H H^\dagger$$

eigenvalues ≤ 0 for $H = 0$

≥ 0 for $H^\dagger H = \frac{v^2}{2}$ $\to$ minima

- The vacuum manifold is

$$\mathcal{M} = \{ H : H^\dagger H = \frac{v^2}{2} \} \simeq S^3$$

(2)

- Let's write down how the generators act on the vacuum

$$\tau^1 \langle H \rangle = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \neq 0$$

$$\tau^2 \langle H \rangle = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} -iv \\ 0 \end{pmatrix} \neq 0$$

$$\tau^3 \langle H \rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 \\ -v \end{pmatrix} \neq 0$$

generator of U(1)

$$Y \langle H \rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0$$

- At first glance it seems that all generators are broken. However, also linear combinations have to be taken into account.

$Q := \tau^3 + Y$ is unbroken, because $Q \langle H \rangle = 0$.

→ We have one unbroken generator and thus, the unbroken symmetry is $U(1)$.

- We have three broken generators and so we expect three massless Goldstone bosons that get 'eaten' by three gauge fields. → Three gauge fields will be massive.

(3)

- $$V = \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2$$

$$= \lambda \left(\frac{1}{2} (v+h)^2 - \frac{v^2}{2} \right)^2$$

$$= \lambda\left(\frac{1}{2}(v+h)^2 - \frac{v^2}{2}\right)^2$$

$$= \frac{\lambda}{4}\left(v^2 + 2vh + h^2 - v^2\right)^2$$

$$= \frac{\lambda}{4}\left(4v^2h^2 + 4vh^3 + h^4\right)$$

$$= \underbrace{\lambda v^2}_{=\frac{1}{2}m_h^2} h^2 + \lambda v h^3 + \frac{1}{4}\lambda h^4$$

$$\rightarrow m_h = \sqrt{2\lambda}\, v \quad \rightarrow \quad \lambda = \frac{m_h^2}{2v^2}$$

- $$V = \frac{1}{2} m_h^2 h^2 + \frac{m_h^2}{2v} h^3 + \frac{m_h^2}{8v^2} h^4$$

(4)

- The only term that will give a mass contribution to the gauge fields is

$$(D_\mu \langle H \rangle)^\dagger (D^\mu \langle H \rangle)$$

- $$D_\mu \langle H \rangle = -ig\, W_\mu^a \tau^a \langle H \rangle - i\frac{g'}{2} B_\mu \langle H \rangle$$

$$= -\frac{1}{2} i \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$= -\frac{iv}{2\sqrt{2}} \begin{pmatrix} g(W_\mu^1 - iW_\mu^2) \\ -gW_\mu^3 + g'B_\mu \end{pmatrix}$$

- $$\mathcal{L} \supset (D_\mu \langle H \rangle)^\dagger (D^\mu \langle H \rangle)$$

$$= \frac{v^2}{8} \begin{pmatrix} g(W_\mu^1 + iW_\mu^2) \\ -gW_\mu^3 + g'B_\mu \end{pmatrix}^T \begin{pmatrix} g(W_1^\mu - iW_2^\mu) \\ -gW_3^\mu + g'B^\mu \end{pmatrix}$$

$g^2 v^2$

$$= \frac{g^2 v^2}{8} (W_\mu^1 + i W_\mu^2)(W_1^\mu - i W_2^\mu)$$

$$+ \frac{v^2}{8} (g W_\mu^3 - g' B_\mu)^2$$

$$= \frac{g^2 v^2}{4} W_\mu^+ W_-^\mu + \frac{v^2}{8} (g^2 + g'^2) Z_\mu Z^\mu$$

where we used

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp W_\mu^2)$$

$$Z_\mu = \frac{g W_\mu^3 - g' B_\mu}{\sqrt{g^2 + g'^2}} \quad , \quad A_\mu = \frac{g B_\mu + g' W_\mu^3}{\sqrt{g^2 + g'^2}}$$

(These four fields are linear independent)

- We have three massive gauge fields and one massless gauge field that agrees with our expectations from part 2.

- The masses are

$$m_W = \frac{gv}{2}$$

$$m_Z = \frac{v}{2} \sqrt{g^2 + g'^2}$$

$$m_A = 0$$

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- $\tan \Theta = \frac{g'}{g}$

$$Z = (W^3 - \tan\Theta\, B) \cdot \frac{1}{\;\;}$$

$$Z_\mu = (W_\mu^3 - \tan\Theta\, B_\mu)\cdot\frac{1}{\sqrt{1+\tan^2\Theta}}$$

$$= \cos\Theta\, W_\mu^3 - \sin\Theta\, B_\mu$$

$$A_\mu = \sin\Theta\, W_\mu^3 + \cos\Theta\, B_\mu$$

Inverted this gives

$$B_\mu = \cos\Theta\, A_\mu - \sin\Theta\, Z_\mu$$

$$W_\mu^3 = \sin\Theta\, A_\mu + \cos\Theta\, Z_\mu$$

- $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$

Inverted this gives

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-)$$

$$W_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-)$$

- $D_\mu = \partial_\mu - i g W_\mu^a \hat{\tau}^a - i g' \underset{= \frac{1}{2}\hat{\mathbb{1}}}{\hat{Y}} B_\mu$

$$= \partial_\mu - ig(W_\mu^1 \hat{\tau}^1 + W_\mu^2 \hat{\tau}^2) - ig W_\mu^3 \hat{\tau}^3 - ig' \hat{Y} B_\mu$$

$$= \partial_\mu - ig\left(W_\mu^+ \frac{1}{\sqrt{2}}(\hat{\tau}^1 + i\hat{\tau}^2) + W_\mu^-(\hat{\tau}^1 - i\hat{\tau}^2)\right)$$

$$- i(g\sin\Theta\, \hat{\tau}^3 + g'\cos\Theta\, \hat{Y})A_\mu$$

$$- i(g\cos\Theta\, \hat{\tau}^3 - g'\sin\Theta\, \hat{Y})Z_\mu$$

$$= \partial_\mu - ig W_\mu^+ \hat{\tau}^+ - ig W_\mu^- \hat{\tau}^-$$

$$- ig(\cos\Theta\, \hat{\tau}^3 - \tan\Theta \sin\Theta\, \hat{Y})Z_\mu$$

$$- i g \sin\Theta\, \hat{Q} A_\mu$$

$$- i g \sin\Theta \, \hat{Q} A_\mu$$

where $\hat{\tau}^\pm = \frac{1}{\sqrt{2}} (\hat{\tau}^1 \pm i \hat{\tau}^2)$

$\hat{Q} = \hat{\tau}^3 + \hat{Y}$

$\rightarrow e = g \sin\Theta$

- $\hat{Q} Z_\mu \sim [\tau^3 + \frac{1}{2}, \cos\Theta \tau^3 - \tan\Theta \sin\Theta \frac{1}{2}] = 0 \rightarrow$ charge 0

  $\hat{Q} W_\mu^\pm \sim [\tau^3 + \frac{1}{2}, \frac{1}{\sqrt{2}} (\tau^1 \pm i \tau^2)]$

  $= \frac{1}{\sqrt{2}} i \tau^2 \pm \frac{i}{\sqrt{2}} (-i) \tau^1 = \pm \tau^\pm \rightarrow$ charge $\pm e$

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- The relevant interaction terms are coming from $(D_\mu H)^\dagger (D^\mu \langle H \rangle)$

  Therefore in the interaction part of part 4, we can just replace one v by h:

$$\mathcal{L} \supset \frac{g^2 v}{4} W_\mu^+ W_-^\mu h + \frac{v}{8} (g^2 + g'^2) Z_\mu Z^\mu h$$

- $h \longrightarrow W_\mu^+ + W_\mu^-$

  $h \longrightarrow 2 Z_\mu$

BUT: We know that $m_h < 2 m_W$, $m_h < 2 m_Z$

$\rightarrow$ processes are kinematically not allowed