01 Problem Sheet

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Problem 1:

$$U \in SU(N): \qquad U = e^{i \overset{\substack{\text{real parameters}\\ \downarrow}}{\lambda^a} \underset{\substack{\uparrow \\ \text{complex matrices}}}{T^a}}, \quad U^\dagger U = 1, \quad \det U = 1$$

$$(a = 1, \dots, k)$$

(1)

- Let λ^a be small:

$$U^\dagger U = e^{-i\lambda^a T^{a\dagger}} e^{i\lambda^a T^a}$$

$$\approx (1 - i\lambda^a T^{a\dagger})(1 + i\lambda^b T^b)$$

$$\approx 1 + i\lambda^a T^a - i\lambda^a T^{a\dagger} \overset{!}{=} 1$$

$$\rightarrow T^a = T^{a\dagger}$$

- $\det U = \det e^{i\lambda^a T^a} = e^{i\lambda^a \mathrm{Tr}\, T^a} \overset{!}{=} 1$

$$\rightarrow \mathrm{Tr}\, T^a = 0$$

$\ulcorner$

Schur decomposition:

$$A = \underset{\substack{\uparrow \\ \text{unitary}}}{Q^\dagger} \underset{\substack{\uparrow \\ \text{upper triangular}}}{U} Q$$

$$e^A = e^{Q^\dagger U Q} = Q^\dagger U Q + \frac{1}{2} Q^\dagger U Q^\dagger Q U Q + \dots$$

$$= Q^\dagger e^U Q$$

$$\det(e^A) = \det(Q^\dagger e^U Q) = \det e^U$$

$$= e^{\lambda_1} \cdot e^{\lambda_2} \cdot \dots \cdot e^{\lambda_N} = e^{\lambda_1 + \lambda_2 + \dots + \lambda_N}$$

$$= e^{\mathrm{Tr}\, U} = e^{\mathrm{Tr}\, Q A Q^\dagger} = e^{\mathrm{Tr}\, A}$$

$$M = e^A \rightarrow A = \ln M$$

$$M = e^{A} \to A = \ln M$$

$$\det M = e^{\mathrm{Tr} \ln M}$$

$\lrcorner$

- Number of generators:

$$T^a = T^{a\dagger} \to \left.\begin{matrix} N \text{ diagonal matrices} \\ \frac{N^2-N}{2} \text{ real off-diagonal} \\ \frac{N^2-N}{2} \text{ imaginary off-diagonal} \end{matrix}\right\} = N^2$$

$\mathrm{Tr}\, T^a = 0 \to$ constraint on diagonal matrices $\to -1$

$\Rightarrow$ # generators $= N^2 - 1$

(2)

SU(2)

- $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \overset{!}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$b = c^* \to \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$$

$$\mathrm{Tr} \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \overset{!}{=} 0 \to a = -d$$

$$\to \begin{pmatrix} a & b \\ b^* & -a \end{pmatrix} = \begin{pmatrix} a & b_1 - i b_2 \\ b_1 + i b_2 & -a \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b_1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

- these matrices are already orthogonal to each other (scalar product $\langle \cdot, \cdot \rangle = 2\,\mathrm{Tr}(\cdot\,\cdot)$).

each other (scalar product $\langle \cdot , \cdot \rangle = 2 \mathrm{Tr}(\cdot \, \cdot)$).

With right normalisation we obtain:

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SU(3)

- $T^a = T^{a\dagger}$, $\mathrm{Tr}\, T^a = 0$

$$\rightarrow \begin{pmatrix} a & b_1 - ib_2 & c_1 - ic_2 \\ b_1 + ib_2 & d & e_1 - ie_2 \\ c_1 + ic_2 & e_1 + ie_2 & -a-d \end{pmatrix}$$

$$= a \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=A} + b_1 \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b_2 \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ c_1 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} + d \cdot \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=B}$$

$$+ e_1 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + e_2 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

- $\langle A, B \rangle \neq 0$

Due to convention we choose $\tilde{T}_8 = A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Normalisation gives $T_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Gram-Schmidt:

$$\tilde{T}_3 = A - \langle T_8 , A \rangle \cdot T_8$$

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$$= \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} - \frac{1}{6} \, \mathrm{Tr}\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \right) \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix} = T_3$$

- It is straightforward to check that now all matrices are orthogonal to each other.

With the right normalisation we have:

$$T_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad T_5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$T_6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad T_7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad T_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(3)

- rank (G) = dim (Cartan subalgebra of G)

Cartan subalgebra: maximal Abelian subalgebra

$\Leftrightarrow$ maximal set of commuting generators

practical tip: Cartan subalgebra of $SU(N)$

$\hat{=}$ set of diagonal matrices

→ rank $(SU(N)) = N-1$ (= # Casimir operators)

$a \in \{1, \dots, N^2-1\}$

- Casimir operator: linear combination of symmetric homogeneous polynomials in the generators that commutes with all elements of the algebra.

$$C_m = \underbrace{x_{a_1 \dots a_m}}_{\text{sym.}} T^{a_1} \dots T^{a_m} \sim \mathbb{1}$$

↑ to commute with all elements

- $SU(2)$: $x_{ab} = \overset{\text{convention}}{2} \mathrm{Tr}(T^a T^b) = \delta_{ab}$

$$\to C_2 = T^a T^a = \frac{3}{4} \mathbb{1}$$

- $SU(3)$: $C_2 = T^a T^a$

rank $(SU(3)) = 2$ → there is one more Casimir

→ need to construct symmetric tensor out of three generators:

$$\mathrm{Tr}(\{T^a, T^b\} T^c) =: d^{abc}$$

$$\to C_3 = d_{abc} T^a T^b T^c$$

(4)

- fundamental representation:

smallest dimensional rep.

$SU(N)$: $\psi \longmapsto U\psi$ for some $U \in SU(N)$, $\psi \in \mathbb{C}^N$

→ dim = N

- adjoint representation:

- adjoint representation:

rep. that acts on the Lie-algebra itself

$SU(N)$: $\quad \phi \longmapsto U \phi U^{+}$ for some $U \in SU(N)$, $\phi \in su(N)$

$\rightarrow$ dim $= N^2 - 1$

(5)

- $SU(2)$:

$$[T^1, T^2] = i T^3$$
$$[T^2, T^3] = i T^1$$
$$[T^1, T^3] = -i T^2$$

$\rightarrow$ algebra closes and the structure constants

are $f_{abc} = \varepsilon_{abc}$

- $SU(3)$:

same procedure: calculate all possible commutators
and check if they are elements
of the algebra.

at the end this gives:

$$f^{123} = 1$$
$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$$
$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

(others zero or related to permutations)

- relation between structure constants and adjoint rep.:

$$\Phi \longmapsto U^{+} \Phi U \approx (1 - i\alpha T^a)\, \Phi^b T^b (1 + i\alpha T^c)$$

- relation between structure constants and adjoint rep.:

$$\Phi \longmapsto U^{+}\Phi U \approx (1 - i\alpha_a T^a)\,\Phi^b T^b (1 + i\alpha_c T^c)$$
$$\approx \Phi + i\,\alpha_a\,\Phi^b\,[T^b, T^a]$$
$$= \Phi + \alpha_a\, f_{abc}\,\Phi^b\,T^c$$

$$\Phi^b \longmapsto \Phi^b + \alpha_a\, f_{abc}\,\Phi^c$$

$\rightarrow$ adjoint rep.: $(T^a_{adj})_{bc} \sim f_{abc}$

Problem 2:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi$$

(1)

- $\mathcal{L} \longmapsto \bar{\Psi}\, e^{ie\alpha}\,(i\not{\partial} - m)\,e^{-ie\alpha}\,\Psi = \bar{\Psi}\,(i\not{\partial} - m)\Psi = \mathcal{L}$

- Noether current:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi}\,\delta\Psi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi^{+}}\,\delta\Psi^{+}$$

$$\delta\Psi = ie\alpha\,\Psi$$

$$j^\mu \sim (\bar{\Psi}\;\; i\gamma^\mu)(-ie\alpha\Psi)$$

$$\rightarrow j^\mu = e\,\bar{\Psi}\gamma^\mu\Psi$$

- equations of motion:

$$(i\not{\partial} - m)\Psi = 0$$

or equivalently

$$i(\partial_\mu \bar{\Psi})\gamma^\mu + m\bar{\Psi} = 0$$

- $\partial_\mu j^\mu = e(\partial_\mu \bar\psi)\gamma^\mu\psi + e\bar\psi\gamma^\mu(\partial_\mu\psi)$

  $= e\,(im\bar\psi\gamma^\mu)\psi + e\bar\psi\,(-im\gamma^\mu\psi) = 0$

- $Q = \int d^3x\; j^0 = e\int d^3x\;\psi^\dagger\psi$

(2)

- $\mathcal{L} \longmapsto \bar\psi\, e^{ie\alpha(x)}(i\not\partial - m)\, e^{-ie\alpha(x)}\psi$

  $\overset{\text{product rule}}{=} \bar\psi\, e^{ie\alpha(x)}\; e^{-ie\alpha(x)}(i\not\partial + i(-ie\,\partial_\mu\alpha(x))\gamma^\mu - m)\,\psi$

  $= \bar\psi\,(i\not\partial - m + e\not\partial\alpha(x))\,\psi$

  → not invariant

(3)

- $\delta\mathcal{L} = j^\mu A_\mu = e\bar\psi\gamma^\mu\psi A_\mu$

  $\longmapsto e\bar\psi\gamma^\mu\psi A_\mu + e\bar\psi\gamma^\mu\psi\,\partial_\mu\alpha$

  → $\mathcal{L} = \bar\psi(i\not\partial - m)\psi - j^\mu A_\mu$ is invariant

(4)

- $\bar\psi\, i\not\partial\psi - j^\mu A_\mu$

  $= \bar\psi\,(i\gamma^\mu(\partial_\mu + ie A_\mu))\,\psi$

  $= \bar\psi\, i\not D\psi$ where $D_\mu = \partial_\mu + ie A_\mu$

(5)

- $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi\,(i\not D - m)\,\psi$

- equations of motion for ψ

  $(i\not D - m)\psi = 0$

- equations of motion for A_μ

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2} F^{\alpha\beta} \frac{\partial}{\partial(\partial_\mu A_\nu)} F_{\alpha\beta}$$

$$= -F^{\mu\nu}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu}$$

$$-\partial_\mu F^{\mu\nu} = -j^\mu$$

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu \psi$$

(6)

- $$Q = e\int_V d^3x \; \psi^+\psi = e\int_V d^3x \; \bar{\psi}\gamma^0\psi$$

$$\overset{\text{e.o.m.}}{=} \int_V d^3x \; \partial_i F^{i0}$$

$$= \int_{\partial V} dS_i \; E^i \qquad \text{(Gauß law)}$$