Problem set : Operator identities useful for Bosonization

H1

1. Baker-Hausdorff: Define $[A,B]_{n+1} = [[A,B]_n, B]$, and $[A,B]_0 = A$.

Show
$$e^{-B} A e^{B} = \sum_{n=0}^{\infty} \frac{1}{n!} [A,B]_n \qquad (1)$$

Hint: Expand the function $\mathcal{A}(s) = e^{-sB} A e^{sB}$ $(s \in \mathbb{R})$ as a Taylor series in s, and evaluate this series at $s=1$.

2. Suppose $C = [A,B]$ satisfies $[A,C] = [B,C] = 0$. Show that

2(i) $e^{-B} A e^{B} = A + C$

2(ii) $e^{-B} f(A) e^{B} = f(A+C)$

2.(iii) $e^{A} e^{B} = e^{A+B} e^{C/2}$

2.(iv) $e^{A} e^{B} = e^{B} e^{A} e^{C}$

H2

Hints: (i) Baker-Hausdorff

(ii) Taylor-expand $f(A)$, find n-th term by induction, starting from 2(i).

(iii) Define $T(s) = e^{sA} e^{sB}$ $(s \in \mathbb{R})$,

calculate $\frac{dT(s)}{ds} = ?$, and show that the solution of this diff. equation is $T(s) = e^{s(A+B)} e^{s^2 C/2}$

3. Suppose $[A,B] = DB$ and $[A,D] = [B,D] = 0$

3(i) Show that $f(A) B = B f(A+D)$.

Hint: Taylor-expand, induction!

Use 3(i) to show that

H3

3.(ii) $$e^{A} B = B e^{A+D}$$

3.(iii) $$e^{A} B^{n} = (B e^{D})^{n} e^{A}$$ Hint: 3(ii), induction!

3.(iv) $$e^{A} e^{B} = e^{(B e^{D})} e^{A}$$ Hint: expand e^{B}, use 3(iii).

4. Prove the identities:

(i) $$\psi_\eta(x)\, f(\{b^{+}_{q\eta}\}) = f(\{b^{+}_{q\eta} - \delta_{\eta\eta'}\alpha^{*}_{q}(x)\})\, \psi_s(x)$$

Hint: use 3(i), with $A = b^{+}_{q\eta} - \delta_{\eta\eta'}\alpha^{*}_{q}(x)$, $B = \psi_\eta(x)$

(ii) $$f(\{b^{+}_{q\eta} - \delta_{\eta\eta'}\alpha^{*}_{q}(x)\}) = e^{-i\varphi_\eta(x)} f(\{b^{+}_{q\eta}\})\, e^{i\varphi_\eta(x)}$$

Hint; use 2(iii), with $A = b^{+}_{q\eta}$, $B = i\varphi_\eta(x)$.

H4

5. Check that the bosonization identity,

$$\psi_\eta(x) = F \hat{\lambda}_\eta(x)\, e^{-i\varphi^{+}_\eta(x)}\, e^{-i\varphi_\eta(x)},$$

reproduces the anti-comm. relations $\{\psi_\eta(x), \psi_\eta(x')\} = 0$

$$\{\psi_\eta(x), \psi^{+}_\eta(x')\} = \delta_{\eta\eta'} 2\pi\, \delta(x-x')$$

6. <u>Operator Product Expansion</u>

Show that $$\psi^{+}_\eta(z+a)\, \psi_\eta(z) \xrightarrow{a\to 0} \frac{1}{a} + i\partial_z \phi_s(z) + O(\tfrac{1}{L})$$

where $z = \tau + ix$. Hint: first normal order, then take $a \to 0$.

6(i) use fermionic representation of ψ

6(ii) " bosonic " " ".

Lecture III - Odds and Ends

1

Original fermion field
(with finite bandwidth for k,
i.e. for energy of particles or holes)

$$\psi_\eta(x) = \Delta^{1/2} \sum_k e^{-|k|a} e^{-ikx} c_{k\eta} , \quad (1)$$

Boson field
(with finite bandwidth for q,
i.e. for energy of particle-excitations)

$$\varphi_\eta(x) = - \sum_{q>0} e^{-qa/2} \frac{i}{\sqrt{n_q}} e^{-iqx} b_{q\eta} \quad (2)$$

Define new fermion field
(with finite bandwidth for q, via phi):

$$\boxed{\psi^{(a)}_\eta(x) := F_\eta \underbrace{\hat\lambda_\eta(x)}_{L\,\Delta_L^{1/2}\, \underbrace{e^{-i(N_\eta - 1/2)x}}_{\to 1 \text{ for } L\to\infty}} e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)}} \quad (3)$$

Comments:

1. Eq. (3) does not require a cutoff (we may set a = 0), because exponentials on RHS are normal ordered:

$$_0\langle \vec N | \underbrace{e^{-i\varphi^\dagger_\eta(x)}}_{=1} \underbrace{e^{-i\varphi_\eta(x)}}_{=1} | \vec N \rangle_0 = 1$$

2. For $a=0$ we have an operator identity between new and old fields: $\psi^{(0)}_\eta(x) = \psi_\eta(x)$

3. For $a \neq 0$, the new field $\psi^{(a)}_\eta(x)$ is not identically equal to old field $\psi_\eta(x)$
Their long-distance behavior is the same (this is what we are interested in),
but short-distance behavior on scale of a is different (we don't care about it anyway).

4. Advantage of $a \neq 0$: two exponentials factors can be combined (unnormal-ordered).

1. Unnormal-ordering the bosonic exponentials

2

$$\psi^{(a)}_\eta(x) \overset{(1.3)}{:=} F_\eta \hat\lambda_\eta(x) e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)} \quad (1)$$

(II.10.2iii)

$$e^A e^B = e^{A+B+\frac12[A,B]}$$

$$= F_\eta \cancel{\Delta_L^{1/2}} e^{-i\Delta_L(N_\eta - \frac12)x} e^{-i(\underbrace{\varphi^\dagger_\eta + \varphi_\eta}_{\phi_\eta(x)})} e^{-\frac12 \underbrace{[\varphi^\dagger_\eta(x), \varphi_\eta(x)]}_{(I.17.5)\ \ln(\cancel{\Delta_L} a)}} \quad (2)$$

(divergent prefactor arises because of unnormal-ordering)

Various common notations:

$$\psi^{(a)}_\eta(x) = a^{-\frac12} F_\eta(x)\, e^{-i\phi_\eta(x)}, \quad F_\eta(x) := F_\eta\, e^{-i\Delta_L(N_\eta - \frac12)x} \quad (3)$$

$$\psi^{(a)}_\eta(x) = a^{-\frac12} F_\eta\, e^{-i\Phi_\eta(x)}, \quad \Phi_\eta(x) := \phi_\eta(x) + \Delta_L(N_\eta - \tfrac12)x \quad (4)$$

"zero mode"

$$\psi^{(a)}_\eta(x) = a^{-\frac12} e^{-i\tilde\Phi_\eta(x)}, \quad \tilde\Phi_\eta(x) := \Phi_\eta(x) - \theta_\eta, \quad F_\eta := e^{-i\theta_\eta} \quad (5)$$

2. Popular (but "dangerous") notation

(I do not recommend using it, but you should know about it)

3

Introduce "phase operators" conjugate to N:

$$F^\dagger_\eta = e^{i\theta_\eta}, \qquad F_\eta = e^{-i\theta_\eta} \qquad (1)$$

$$[\hat N_\eta, i\theta_{\eta'}] \;\text{"}{:=}\text{"}\; \delta_{\eta\eta'} \qquad [\hat N_\eta, e^{\pm i\theta_{\eta'}}] = \pm\delta_{\eta\eta'} e^{\pm i\theta_{\eta'}} = (7.6, 7)! \qquad (2)$$

sloppy

correct notation

(theta's are sometime called

$$[\theta_\eta, \theta_{\eta'}] := \begin{Bmatrix} i\pi \\ 0 \\ -i\pi \end{Bmatrix} \text{ if } \eta \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \eta' \qquad (3)$$

(1), (2), (3) reproduce (II.7.4) to (II.7.7), using

$$\left.\begin{aligned} [A, e^B] &= c\, e^B \\ e^A e^B &= e^B e^A e^c \end{aligned}\right\} \text{ if } c = [A, B] = c\text{-number} \qquad (4)$$

But, (2) is sloppy notation that produces a contradiction:

$$0 = (N_\eta - N_\eta)\langle \hat N_\eta|\theta|\hat N_\eta\rangle = \langle \underbrace{N_\eta|\hat N_\eta}_{\langle N_\eta|N_\eta\; \neq} i\theta_\eta - i\theta_\eta \underbrace{\hat N_\eta|N_\eta\rangle}_{= N_\eta|N_\eta\rangle} \overset{(9)}{=} \langle \hat N_\eta|1|\hat N_\eta\rangle = 1 \qquad (5)$$

What went wrong?

Theorem: If X and Y are conjugate operators, meaning [X, iY] = 1, and the spectrum of X is the set of discrete integers, then X is hermitian only in the space of states produced by acting on a reference state by periodic functions of Y, in other words, functions of exp(iY).

But: (11) contains states not periodic in theta, namely $\hat\theta|\hat N_\eta\rangle$ so N is not hermitian!

3. Commutators of bosonized expressions

(use (1.3) with a = 0) $\quad \varphi(x) = \varphi(x + \bar n L)$

4

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} \sim \{F_\eta, F^\dagger_{\eta'}\} = 0 \quad \text{if } \eta \neq \eta' \qquad (1)$$

(suppress eta-index henceforth)

$e^{-i\Delta_L x}$ $\qquad$ (II.7.6) $FN = (N+1)F$ $\qquad$ $e^{i\Delta_L x'}$

$$\eta = \eta': \; = \Big(\Delta_L^{1/2} \cancel{F}\, e^{-i\Delta_L(N-\frac12)x} e^{-i\varphi^\dagger(x)} e^{-i\varphi(x)} \overset{①}{} e^{i\varphi^\dagger(x')} \overset{②}{} e^{i\varphi(x')} e^{i\Delta_L(N-\frac12)x'} \overbrace{F F^\dagger}^{1} \Delta_L^{1/2}$$

$$+ \Delta_L^{1/2} e^{i\varphi^\dagger(x')} e^{i\varphi(x')} e^{i\Delta_L(N-\frac12)x'} \underbrace{F^\dagger F}_{=1} e^{-i\Delta_L(N-\frac12)x} e^{-i\varphi^\dagger(x)} e^{-i\varphi(x)} \Delta_L^{1/2} \Big) \qquad (2)$$

$$= \Big[\Delta_L \underbrace{e^{-i\Delta_L(N-\frac12)(x-x')}}_{} \cdot \underbrace{e^{-i(\varphi^\dagger(x) - \varphi^\dagger(x'))} e^{-i(\varphi(x)-\varphi(x'))}}_{=1,\ \text{since } \varphi(x) = \varphi(x+L)} \Big] A(x,x') \qquad (3)$$

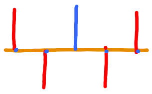
if $x - x' = \bar n L$: $\overset{(5)}{=1}$ $\quad [e^{-i\frac{2\pi}{L}\frac12 \bar n L}] = (-1)^{\bar n}$

$$A(x,x') = \Big(e^{\overset{①}{[\varphi(x), \varphi^\dagger(x')]}} e^{-i\Delta_L(x-x')} + e^{\overset{②}{[\varphi(x'), \varphi^\dagger(x)]}} \Big) \qquad (4)$$

$$= \Big(\frac{y}{1-y} + \frac{1}{1-y^{-1}} \Big) = \sum_{n=0}^{\infty} (y^{n+1} + y^{-n}) = \sum_{n\in\mathbb{Z}} y^n = L \sum_{\bar n} \delta(x - x' - \bar n L) \qquad (5)$$

$$\boxed{\begin{aligned} & e^A e^B \overset{(2.10.2ii)}{=} e^B e^A e^{[A,B]} \\ & [\varphi(x), \varphi^\dagger(x')] \overset{(I.17.4),\ a=0}{=} -\ln(1-y) \\ & y = e^{-i\Delta_L(x-x')} \end{aligned}}$$

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} \overset{(3)}{=} 2\pi \sum_{\bar n} \delta(x - x' - \bar n L)(-1)^{\bar n} \checkmark$$



antiperiodic

4. Bosonizing linearized kinetic Hamiltonian (suppress index η below) $v_F = 1$

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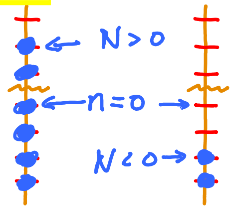
Linearized fermionic kinetic Hamiltonian:

$$\boxed{H_0 = \sum_{k=-\infty}^{\infty} k \; {}^{\times}_{\times} c^\dagger_{k\eta} c_{k\eta} {}^{\times}_{\times} \,, \qquad k = \Delta_L (n - 1/2)} \qquad (1)$$

Energy of N-particle ground state $|N\rangle_0$:

$$E_0^N = {}_0\langle N | H | N \rangle_0 = \Delta_L \left\{ \begin{array}{ll} \sum_{n=1}^{N} (n - 1/2) & \text{for } N \geq 0 \\ \sum_{n=N+1}^{0} -(n - 1/2) & \text{for } N < 0 \end{array} \right\} = \frac{\Delta_L}{2} N^2 \xrightarrow{L\to\infty} 0 \qquad (2)$$



Consider:

$$\boxed{[H_0, b_q^\dagger]} = \sum_{kk'} k \Big[\overbrace{c^\dagger_k c_k, \; c^\dagger_{k'+q}}^{\delta_{kk'}} c_{k'} \Big] \frac{i}{\sqrt{n_q}}, \quad \underbrace{c_k, \; c^\dagger_{k'+q}}_{\delta_{k,k'+q}} \quad = \sum_k k \big(c^\dagger_k c_{k-q} - \underset{k \to k+q}{c^\dagger_{k+q} c_k} \big) \frac{i}{\sqrt{n_q}} = \boxed{q \, b_q^\dagger} \qquad (3)$$

Thus, boson creation op. are energy ladder op:

$$H_0 b_q^\dagger |N\rangle_0 = (E_0^N + q) \, b_q^\dagger |N\rangle_0 \qquad (4)$$

The only bosonic operator that also satisfies (2) and (3) for all q is: $[H_0, b_q^\dagger] = q \, b_q^\dagger$

$$\boxed{H_0 := \sum_{q>0} q \, b_q^\dagger b_q + \frac{\Delta_L}{2} \hat{N}^2} \qquad (5)$$

(seemingly quartic in $c^\dagger c c^\dagger c$!!)

hence: (1) = (5)

5. Imaginary-time-ordered boson correlator at T = 0

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Imaginary-time evolution:

$$\phi(\tau, x) \overset{(\text{II}.2.9)}{=} - \sum_{q>0} \frac{e^{-aq/2}}{\sqrt{n_q}} \left[e^{-q(ix+\tau)} b_q + e^{q(ix+\tau)} b_q^\dagger \right] = \phi(z) \qquad (1)$$

$$\boxed{{}_0\langle 0 | \mathcal{T} \phi(z) \phi(0) | 0 \rangle_0}$$

$\langle \phi(-z) \phi(0) \rangle$ by time translational invariance

$$= \theta(\tau) \langle \phi(z) \phi(0) \rangle + \theta(-\tau) \underset{\langle \phi(-z)\phi(0)\rangle}{\langle \phi(0) \phi(z) \rangle} \qquad (2)$$

$\sigma = \text{sign}(\tau)$

$$= \langle \phi(\sigma z) \phi(0) \rangle, \quad \text{in } \langle (b_q + b_q^\dagger)(b_{q'} + b_{q'}^\dagger) \rangle \text{ only } \delta_{qq'} b_q b_q^\dagger \text{ contributes} \qquad (3)$$

$$= \sum_{q>0} \frac{e^{-aq}}{n} e^{-qz\sigma} \overbrace{\langle b_q b_q^\dagger \rangle}^{=1} \qquad \text{with } q = \Delta_L n \,, \quad y = e^{-\Delta_L (z\sigma + a)} \qquad (4)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} y^n = -\ln(1 - y) = -\ln\left(1 - e^{-\Delta_L (z\sigma + a)}\right) \xrightarrow{L\to\infty} \boxed{-\ln\left(\Delta_L (\sigma z + a)\right)} \qquad (5)$$

Time evolution of Klein factor:

$$F_\eta(\tau, x) = e^{H\tau} F_\eta \, e^{-i\Delta_L (N_\eta - \frac{1}{2}) x} e^{-H\tau} = F_\eta \, e^{\mp \Delta_L (N_\eta - \frac{1}{2}) z} \qquad (6)$$

$$F_\eta^\dagger \qquad F_\eta^\dagger \qquad F_\eta^\dagger$$

6. Using bosonization to calculate fermion correlators

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Theorem: for free boson Hamiltonian $\hat H = \sum_q \omega_q b_q^\dagger b_q$ and $\hat B = \sum_q (\lambda_q b_q + \bar\lambda_q b_q^\dagger)$

ground state or thermal expectation values of exponentials of bosons satisfy:

$$\langle e^{\hat B}\rangle = e^{\frac12\langle \hat B^2\rangle}, \quad \text{where } \langle \hat O\rangle = \mathrm{Tr}\left(e^{-\beta\hat H}\hat O\right)/\mathrm{Tr}\, e^{-\beta \hat H} \qquad (1)$$

(1) and (II.10.2iii) imply:

$$\langle e^{\hat B_1} e^{\hat B_2}\rangle = e^{\langle \hat B_1\hat B_2 + \frac12 \hat B_1^2 + \frac12 \hat B_2^2\rangle} \qquad (2)$$

Bosonize fermion correlator:

$$\boxed{\langle 0|T\psi(z)\,\psi^\dagger(0)|0\rangle} = \frac1a \langle 0|T \underbrace{F F^\dagger}_{=1} e^{-\Delta_L(\underset{=1}{N}-1/2)z}\, e^{-i\phi(z)}\, e^{i\phi(0)}\, \cancel{F^\dagger}|0\rangle \qquad (3)$$

$$\overset{(2)}{=} \frac{\sigma}{a}\, e^{-\Delta_L z/2}\, e^{\langle 0|T\phi(z)\phi(0) - \frac12\phi(z)\phi(z) - \frac12 \phi(0)\phi(0)|0\rangle} \qquad (4)$$

same

$$\xrightarrow{L\to\infty} \frac{\sigma}{a}\, e^{-\left(\ln \cancel{\Delta_L}(\sigma z+a) - 2\frac12 \ln \cancel{\Delta_L} a\right)}$$

$$\langle T\phi(z)\phi(0)\rangle \overset{(6.5)}{=} -\ln\Delta_L(\sigma z+a)$$

$$= \frac{\cancel{\sigma}}{\cancel{a}}\,\frac{\cancel{a}}{\cancel{\sigma} z + a\sigma} = \boxed{\frac{1}{z+\sigma a}} = (I.11.5)$$

[if L is kept finite, one recovers (I.11.6)]

7. Vertex operators: general exponentials of boson fields

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Definition of "vertex operator": with charge $\lambda\in\mathbb{R}$

see (2.2)

$$V_\lambda^{(\eta)}(z) := \Delta_L^{\lambda^2/2}\; {}^{\times}_{\times} e^{i\lambda\phi_\eta(z)} {}^{\times}_{\times} = a^{-\lambda^2/2}\, e^{i\lambda\phi_\eta(z)} \qquad (1)$$

$${}^{\times}_{\times} e^{i\lambda\phi_\eta(z)} {}^{\times}_{\times} := e^{i\lambda\varphi_\eta^\dagger(z)}\, e^{i\lambda\varphi_\eta(z)}$$

unnormal-order, producing a factor $e^{-\lambda^2[\varphi^\dagger(z),\varphi(z)]} = e^{-\lambda^2\ln\Delta_L a}$

Ground state expectation value:

$${}_0\langle\vec 0|\, V_\lambda^{(\eta)}(z)\,|\vec 0\rangle_0 = \Delta_L^{\lambda^2/2} \xrightarrow[\Delta_L\to 0]{L\to\infty} \delta_{\lambda 0} \qquad (2)$$

Two-point correlator:

$${}_0\langle\vec 0|\,T\, V_\lambda^{(\eta)}(z)\, V_{\lambda'}^{(\eta')}(z')\,|\vec 0\rangle_0 = \delta_{\eta\eta'}\, \cancel{a^{-(\lambda^2+\lambda'^2)/2}}\; e^{-\lambda\lambda'\underbrace{\langle T\phi(z)\phi(0)\rangle}_{-\ln\Delta_L(\sigma z+a)} - \frac12(\lambda^2+\lambda'^2)\underbrace{\langle\phi(0)\phi(0)\rangle}_{-\ln\Delta_L \cancel{a}}}$$

$$= \delta_{\eta\eta'}\,\frac{\Delta_L^{\frac12(\lambda+\lambda')^2}}{(\sigma z+a)^{-\lambda\lambda'}} \xrightarrow[\Delta_L\to 0]{L\to\infty} \delta_{\eta\eta'}\,\frac{\delta_{\lambda,-\lambda'}}{(\sigma z+a)^{\lambda^2}} \qquad (3)$$

Similarly for n-point correlator:

$${}_0\langle\vec 0|\,T\, V_{\lambda_1}^{(\eta)}(z_1)\dots V_{\lambda_n}^{(\eta)}(z_n)\,|\vec 0\rangle_0 = \Delta_L^{\frac12\left(\sum_{j=1}^n\lambda_j\right)^2}\prod_{i<j}(z_{ij}\sigma_{ij}+a)^{\lambda_i\lambda_j} \qquad (4)$$

"charge neutrality" $\xrightarrow[\Delta_L\to 0]{L\to\infty} = 0$ unless $\sum_{j=1}^n\lambda_j = 0$

$$\left[\begin{array}{l} z_{ij} = z_i - z_j, \\ \sigma_{ij} = \mathrm{sign}(\tau_i-\tau_j) \end{array}\right]$$

$\boxed{9}$

8. Kinetic energy in position space (spinless electrons, L / R)

fermionic description: $\psi^{(t+x)}_{L/R}(x) \overset{(\text{I}.8.1)}{=} \Delta_L^{1/2}\sum_k e^{-ikx}\, c_{k\,L/R} =: \tilde{\psi}_{L/R}^{\,t\pm x\,(\pm x)}$ (1)

(mathematical L-movers) (mathematical L / R -movers)

Kinetic energy: $H_0 \overset{(5.1)}{=} \sum_{\eta=L,R}\sum_{k=-\infty}^{\infty} k\, :c^\dagger_{k\eta}c_{k\eta}:$ (2)

$$= \int_{-L/2}^{L/2}\frac{dx}{2\pi}\, :\Big[\tilde{\psi}_L^\dagger(x)\underbrace{(i\partial_x)}_{\hookrightarrow k}\tilde{\psi}_L(x) + \tilde{\psi}_R^\dagger(x)\underbrace{(-i\partial_x)}_{\hookrightarrow k}\tilde{\psi}_R(x)\Big]: \qquad (3)$$

bosonic description: $\phi_{L/R}(x) \overset{(\text{II}.2.9)}{=} -\sum_{q>0}\frac{e^{-aq/2}}{\sqrt{n_q}}\Big[e^{-iqx}b_{q\,L/R} + e^{iqx}b^\dagger_{q\,L/R}\Big] =: \tilde{\phi}_{L/R}(\pm x)$ (4)

(mathematical L-movers) (mathematical L / R -movers)

Kinetic energy: $H_0 \overset{(5.5)}{=} \sum_{\eta=L,R}\Big[\sum_{q>0} q\, b^\dagger_{q\eta}b_{q\eta} + \frac{\Delta_L}{2}\hat{N}_\eta^2\Big]$ (5)

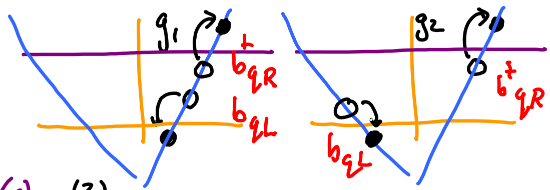
$$= \sum_{\eta=L,R}\Big[\int_{-L/2}^{L/2}\frac{dx}{2\pi}\, :\frac{1}{2}(\partial_x\tilde{\phi}_\eta)^2: + \frac{\Delta_L}{2}\hat{N}_\eta^2\Big] \qquad (6)$$

$\hookrightarrow \left(\sqrt{\tfrac{1}{n_q}}\,n_q\right)^2 \sim q$

$\boxed{10}$

9. Electron-electron interactions

$$\hat{V}_{ee} = \frac{1}{2L}\sum_{kk'q}^{\text{all}} V_{ee}(q)\, c^\dagger_{k-q}\, c^\dagger_{k'+q}\, c_{k'}\, c_k \qquad (1)$$



$$= \frac{1}{2}\int dx\,dx'\; \psi^\dagger(x)\,\psi^\dagger(x')\,\underbrace{V_{ee}(x-x')}_{\delta(x-x')}\,\psi(x')\,\psi(x) \qquad (2)$$

$$\psi_{phys} = e^{-ik_Fx}\tilde{\psi}_L(x) + e^{ik_Fx}\tilde{\psi}_R(x) \qquad (3)$$

$$= \frac{1}{2}\int\frac{dx}{2\pi}\frac{dx'}{2\pi}\,\delta(x-x')\Big\{\frac{g_4}{2}\Big[\tilde{\psi}^\dagger_R(x)\,\tilde{\psi}^\dagger_R(x')\,\tilde{\psi}_R(x')\,\tilde{\psi}_R(x)\Big] + R \leftrightarrow L$$

$$g_2\,[R^\dagger \quad L^\dagger \quad L \quad R] \qquad "$$

$$e^{-2ik_Fx}\,[R^\dagger \quad R \quad R^\dagger \quad L] \qquad "$$

$$e^{-4ik_Fx}\,[{}^\dagger \quad R^\dagger \quad L \quad L] \qquad " \qquad (4)$$

Umklapp $q \sim 2k_F$

double-Umklapp $q \sim 4k_F$

drop: high-energy!

Tomonaga-Luttinger model:

$$\boxed{H_{int} = \int_{-L/2}^{L/2}\frac{dx}{2\pi}\, :\Big[g_2\,\tilde{\rho}_L(x)\,\tilde{\rho}_R(x) + \frac{1}{2}g_4\big[\tilde{\rho}_L^2(x) + \tilde{\rho}_R^2(x)\big]\Big]:} \qquad (5)$$

this is already in bosonized form, since

$$\tilde{\rho}_{L/R}(x) \overset{(\text{II}.2.10)}{=} \pm\,\partial_x\tilde{\phi}_{L/R}(x) + \Delta_L\hat{N}_{L/R} \qquad (6)$$