TMP-TC2: Cosmology

Solutions to Problem Set 9

27, & 29 June 2023

1. Fourth neutrino

1. When the reaction rate of the neutrino Γ is comparable to Hubble parameter H the neutrino stops interacting

$$\Gamma(T^*) \sim H(T^*) \ , \tag{1}$$

where T^* is the temperature of decoupling. The rate of the reaction can be approximated by $\Gamma \sim n_{\nu_4}\langle\sigma v\rangle$. The number density that appears in the relationship $\Gamma \sim n\langle\sigma v\rangle$ is the density of target particles. It is assumed that the most important reaction to keep the new neutrinos ν_4 at thermal equilibrium is the annihilation of neutrino with its own antiparticle. We would then put $n_{\bar{\nu}_4}$. But it is also assumed that there is no asymmetry between neutrinos and antineutrinos, so $n_{\bar{\nu}_4} = n_{\nu_4}$.
The Hubble parameter during the domination of radiation can be expressed as $H = 1.66\sqrt{g^*(T)}\frac{T^2}{M_{\rm pl}}$, so the relationship (1) becomes

$$n_{\nu_4}(T^*)\langle\sigma v\rangle = \sqrt{g^*(T)}\frac{T^{*2}}{\tilde{M}} \ , \tag{2}$$

where we defined the constant $\tilde{M} = M_{\rm pl}/1.66$.

2. The constraint to be imposed is, that the present additional energy density of the neutrino ν_4 must not exceed the density of dark matter

$$\rho_{\nu_4} = 2m_{\nu_4}n_{\nu_4}(T_0) \leq \Omega_{dm}\rho^0_{crit} \ . \tag{3}$$

The equality is satisfied if all the dark matter is given by neutrinos ν_4.
Once the neutrino interacts not anymore, the number of neutrinos per comoving volume remains constant $n_{\nu_4}R^3 = $ const. In addition, we know that the total entropy per comoving volume is conserved, i.e. $sR^3 = $ const. By combining the two conservation laws we can express the density of neutrinos today as

$$n_{\nu_4}(T_0) = n_{\nu_4}(T^*)\frac{s(T_0)}{s(T^*)} = n_{\nu_4}(T^*)\frac{g^*(T_0)}{g^*(T^*)}\left(\frac{T_0}{T^*}\right)^3 \ . \tag{4}$$

Using this relationship we can write (3) like

$$2m_{\nu_4}n_{\nu_4}(T^*)\frac{g^*(T_0)}{g^*(T^*)}\left(\frac{T_0}{T^*}\right)^3 \leq \Omega_{dm}\rho^0_{crit} \ . \tag{5}$$

The combination of relations (2) and (5) allows us to determine the temperature of decoupling and the limits on the mass of ν_4. We will be interested in two limiting cases. The neutrino is light and decouples while still relativistic, or it is very heavy and it is already non-relativistic at decoupling.

3. light neutrino ν_4

If the neutrino ν_4 is relativistic at the time of decoupling, the density at T^* is given by

$$n_{\nu_4}(T^*) = \frac{3\zeta(3)}{4\pi^2} T^{*3} \ .$$

The thermal average of the product of the cross section and the relative velocity is

$$\langle \sigma v \rangle \sim G_F^2 T^{*2} \ .$$

With these two relations, (2) gives us the decoupling temperature

$$T^* = \left(\frac{4\pi^2 \sqrt{g^*(T)}}{3\zeta(3) G_F^2 \tilde{M}} \right)^{\frac{1}{3}} \simeq 2 \left(g^*(T)\right)^{1/6} MeV \ . \quad (6)$$

We know that $g^*(T)$ is approximately 100 when all species of standard model are relativistic and about 2 when there is only photons and massless neutrinos.(cf. figure 1) For these values, $\left(g^*(T)\right)^{1/6}$ is of the order of 1. Our estimate of T^* is then

$$T^* \simeq 2MeV \ . \quad (7)$$

At this temperature, the relativistic species are γ, ν_e, ν_μ, ν_τ, ν_4, e^+, e^-. We then get

$$g^*(T^*) = 2 + \frac{7}{8} \left(4 + 6 + 2\right) \ . \quad (8)$$

At the present time, the only particles that are still relativistic are photons and massless (standard) neutrinos, so

$$g^*(T_0) = 2 + \frac{7}{8} \left(6\right) \left(\frac{T_\nu}{T_0} \right)^3 = 2 + \frac{7}{8} \left(6\right) \frac{4}{11} \ . \quad (9)$$

We substitute (8) and (9) in (5) and solve for m_{ν_4} to find

$$m_{\nu_4} \leq \frac{2\pi^2}{3\zeta(3)} \frac{g^*(T^*)}{g^*(T_0)} \frac{\Omega_{dm} \rho^0_{crit}}{T_0^3} \simeq 17eV \ . \quad (10)$$

4. heavy neutrino ν_4

If the new neutrino is non-relativistic when decoupling, the density is given by

$$n_{\nu_4}(T^*) = \left(\frac{m_\nu T}{2\pi} \right)^{3/2} \exp\left(-\frac{m_\nu}{T} \right) \ . \quad (11)$$

We can already see that in this case a larger mass corresponds to a smaller density. We therefore expect to find a lower limit for the new neutrino mass.

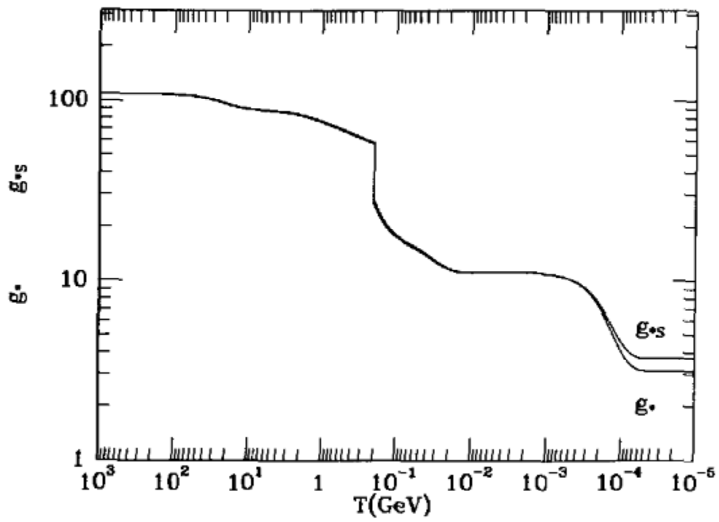

Fig. 3.5: The evolution of $g_*(T)$ as a function of temperature in the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ theory.

FIGURE 1 – $g^*(T)$

The thermal average product of the cross section and the relative velocity is now

$$\langle \sigma v \rangle \sim G_F^2 m_\nu^2 \ .$$

This result is not entirely trivial. One might expect the presence of a factor $(T/m_\nu)^n$. A detailed calculation shows that for the case of a Dirac neutrino, the dominant contribution is independent of T. The relations (2) and (5) then become

$$m_{\nu_4}^{7/2} T^{*-1/2} \exp(-m_{\nu_4}/T^*) = \frac{(2\pi)^{3/2}\sqrt{g^*(T^*)}}{G_F^2 \tilde{M}} \equiv a \qquad (12)$$

$$m_{\nu_4}^{5/2} T^{*-3/2} \exp(-m_{\nu_4}/T^*) = \frac{(2\pi)^{3/2}\Omega_{dm}\rho_{crit}^0}{2T_0^3} \frac{g^*(T^*)}{g^*(T_0)} \equiv b \ , \qquad (13)$$

where the inequality was replaced by an equality to simplify the calculations. Unfortunately we can not find an analytic solution to the above equations. We will proceed in the same way as in the previous exercise. We start by making the change of variables $x = \frac{1}{2}\frac{m_{\nu_4}}{T^*}$. The equations become

$$(2x)^{1/2} m_{\nu_4}^3 \exp(-2x) = a \qquad (14)$$

$$(2x)^{3/2} m_{\nu_4} \exp(-2x) = b \ . \qquad (15)$$

We can eliminate m_{ν_4} to find an equation for x

$$\frac{e^x}{x} = \left(\frac{16a}{b^3}\right)^{1/4} \equiv K \ . \qquad (16)$$

By taking the logarithm, we obtain

$$x - \ln x = \ln K \ . \qquad (17)$$

By assumption we have $x \gg 1$. We will try to solve the equation by iterations. As $\ln x < x$, we can neglect the logarithm to find a first approximation

$$x_0 = \ln K \ . \tag{18}$$

To find the solution, we reinsert x_0 in the equation

$$x \simeq x_1 = \ln K + \ln x_0 = \ln(K \ln K) \ . \tag{19}$$

By inserting x_1 in the second equation we obtain

$$T^* \simeq \frac{b e^{2x_1}}{(2x_1)^{5/2}} \ . \tag{20}$$

As before, we must analyze the dependence of T^* on $g^*(T^*)$. In this case the result is more sensitive to the value of g^*. But as we are only interested in orders of magnitude, we will keep it simple. You can insert different values for g^* and compare the result with the figure1. For $g^* = 20$ we get $T^* \simeq 419$MeV, which is not far from the right result. We can then find the limit on the mass m_{ν_4}

$$m_{\nu_4} = 2x_1 T^* \geq 10\text{GeV} \tag{21}$$

2. Baryon Asymmetry of the Universe

The number of anti-protons changes due to annihilation with protons and proton-anti-proton pair creation. These processes contribute to the Boltzmann equation. As we already discussed in the previous exercise, decoupling happens when the expansion term and the reaction term in the Boltzmann equation are of same order :

$$H(T_d) \sim \Gamma(T_d) \tag{22}$$

The reaction rate is given by

$$\Gamma = \langle \sigma v \rangle n \tag{23}$$

where n is the density of the target particles. Therefore, for anti-protons the targets are the protons. From sheet 5 we know that the Hubble parameter is

$$H \approx 1.65 \sqrt{g_*} \frac{T^2}{M_{pl}} \tag{24}$$

In the following we will ignore any prefactors, because at the end we want to get just order of magnitudes. Hence, we have $H \sim \frac{T^2}{M_{pl}}$, $n = \eta n_\gamma$ and $\langle \sigma v \rangle \sim \lambda_p^2$ with λ_p the Compton wavelength of a proton.
Decoulpling happens at

$$\frac{T_d^2}{M_{pl}} \sim \eta \lambda_p^2 T_d^3 \tag{25}$$

This gives a decoupling temperature

$$T_d \sim \frac{1}{M_{pl}\eta\lambda_p^2} \sim 0.02eV \tag{26}$$

Assuming that the processes are in equilibrium at decoupling to obtain an order of magnitude for the particle density of the anti-protons :

$$n^{eq} = 2\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m-\mu}{T}} \tag{27}$$

$$\bar{n}^{eq} = 2\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m+\mu}{T}} \tag{28}$$

$$\tag{29}$$

The product is

$$n^{eq}\bar{n}^{eq} \sim (mT)^3 e^{-\frac{2m}{T}} \tag{30}$$

with $n^{eq} = \eta n_\gamma \sim \eta T^3$ we obtain for the particle density of the anti-protons

$$\bar{n}^{eq} \sim \frac{m^3}{\eta} e^{-\frac{2m}{T}} \sim e^{-10^1 1} \tag{31}$$

This number is so small that we can say that the universe contains practically no anti-protons.

3. Recombination

The Saha's equilibrium formula reads

$$n_H = n_e n_p \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{I/T} \ , \tag{32}$$

where $I = m_p + m_e - m_H = 13.6$ eV is the binding energy of the hydrogen atom.

To get this formula, we start from the equilibrium distributions for non relativistic species

$$n_i^{\rm eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right) \tag{33}$$

and we consider the ratio

$$\frac{n_H}{n_p n_e} \tag{34}$$

Since we are at equilibrium and photons have zero chemical potential $\mu_p + \mu_e = \mu_H$, and the Saha equlibrium condition follows.
For an electrically neutral plasma, $n_e = n_p$, so we obtain

$$n_H = n_p^2 \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{I/T} \ . \tag{35}$$

where we used $g_p = g_e = 2$ and $g_H = 4$ (1 triplet and 1 singlet). Denoting with $x \equiv \frac{n_p}{n_B}$, where $n_B = n_p + n_H$ is the number of baryons, we can write the above as

$$\frac{1-x}{x^2} = n_B \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{I/T} = \eta n_\gamma \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{I/T} \ , \tag{36}$$

where

$$\eta \equiv \frac{n_B}{n_\gamma} \approx 10^{-10} \ , \tag{37}$$

and

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \ . \tag{38}$$

Therefore

$$\frac{1-x}{x^2} = \frac{2\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{2T}{m_e}\right)^{3/2} e^{I/T} \ , \tag{39}$$

and as a result, the temperature at a first approximation is given by

$$T \approx \frac{I}{\log\left[\left(\frac{1-x}{x^2}\right) \frac{\sqrt{\pi}}{2\zeta(3)} \eta^{-1}\right]} \approx 0.6 \text{ eV}. \tag{40}$$

A more careful calculation shows that the temperature is even lower

$$T \approx 0.3 \text{ eV} \ ,$$

which is much smaller than the value one might expect $T \approx I$. This is due to the smallness of the η parameter, i.e. the fact that there is an over-abundance of photons for each hydrogen atom.