TMP-TC2: Cosmology

Solutions to Problem Set 7

6, 7, 8 June 2023

1. Proton gas

1. First we want to derive a relation between the cross section and the mean free path. Consider the collision cylinder that is given by the product of the cross section σ and the mean free path λ

$$V_c = \sigma\lambda \tag{1}$$

Assuming that there is one particle per collision cylinder, the particle density is given by

$$n = \frac{1}{V_c} = \frac{1}{\sigma\lambda} \tag{2}$$

From a previous sheet, we know that the density of protons is given by

$$n = 2\left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left(\frac{-m_p}{T}\right) \ ,$$

The average time between collisions is given by $t = \frac{\lambda}{v} = \frac{1}{\sigma n v}$, where v is the average velocity of the protons, which can be evaluated assuming that the kinetic energy of the gas is equal to $\frac{3}{2}T$:

$$v = \sqrt{\frac{3T}{m_p}} \ .$$

At the time t_d, we compare the evolution of the universe

$$H = \frac{1}{2t} = 1.66 g_*^{1/2} \frac{T^2}{M_{Pl}}$$

where we used the time-temperature relation from a previous sheet, with the interaction rate :

$$\frac{1}{2t} = \frac{1}{m_\pi^2}\left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left(\frac{-m_p}{T}\right)\sqrt{\frac{3T}{m_p}}.$$

We assume that the decoupling takes place when these two rates are equal, i.e.

$$T_d = \frac{m_p}{\ln\left(\frac{\sqrt{3}M_{Pl}m_p}{1.66 g_*^{1/2}(2\pi)^{3/2}m_\pi^2}\right)} = \frac{m_p c^2}{k_B \ln\left(\frac{\sqrt{3}M_{Pl}m_p}{1.66 g_*^{1/2}(2\pi)^{3/2}m_\pi^2}\right)} \sim 3.6\cdot 10^{11}\ [K]$$

2. The density ratio is

$$\frac{n_p}{n_\gamma} = \frac{2\left(\frac{m_p c^2 T_d}{2\pi k_B}\right)^{3/2} \exp\left(\frac{-m_p c^2}{k_B T_d}\right)}{\frac{2\zeta(3)}{\pi^2} T_d^3} \sim 6.6 \cdot 10^{-12}$$

3. Actually $T = T_0 = 2.73[K]$, then

$$n_\gamma^0 = \frac{2\zeta(3)}{\pi^2} T_0^3 = \frac{2\zeta(3)}{\pi^2} \frac{T_0^3 k_B^3}{c^3 \hbar^3} \sim 4.2 \cdot 10^8 \; [m^{-3}]$$

   Assuming that the reactions of annihilation are negligible since decoupling, the density ratio has not changed, so $n_p^0 = 3 \cdot 10^{-3} \; [m^{-3}]$. In reality, it is about $n_p^0 = 1$. Therefore, a more complex mechanism is necessary to explain the presence of matter in the universe.

4. We suppose that $T \gg m_p$, there is an asymmetry $\frac{n_p - n_{\bar{p}}}{n_p + n_{\bar{p}}} = 10^{-10}$. We know that the entropy $S = sa^3$ is conserved during the expansion of the Universe. So the quantity $\Delta n = \frac{n_p - n_{\bar{p}}}{s}$ is also conserved. At $T \gg m_p$ (relativistic protons), we have

$$\Delta n = \frac{2 \cdot 10^{-10} n_p(T)}{s(T)}$$

   with $s(T) = \frac{2\pi^2}{45} g_* T^3$, with $g_* = 2 \times 2 \times \frac{7}{8} + 2 + 3$, since all species contribute to the effective number of (relativistic) degrees of freedom We get : $\Delta n = 0.1 \cdot 10^{-10}$.
   At $T \ll m_p$, the entropy comes from photons only ($s^0 = \frac{2\pi^2}{45} 2T^3$). On the other hand $n_\gamma^0 = \frac{\zeta(3)}{\pi^2} 2T^3$, thus

$$\frac{n_p^0}{n_\gamma^0} = \frac{s^0}{n_\gamma^0} \Delta n \sim 0.35 \cdot 10^{-10}.$$

   Note that $n_{\bar{p}}^0 = 0$ today. This number being close to reality, we see that a baryon asymmetry of the order of 10^{-10} must exist at an energy of more m_p.

2. Neutrino decoupling

1. We know from the previous sheet that decoupling occurs when the scattering rate Γ is of the same order of the expansion rate H. We need to estimate both and compare them.

   — First, let's consider the scattering rate. Since we are at temperature T, the typical scattering energy is $E \sim T$. Then, by dimensional analysis the cross section is

$$\sigma \sim \frac{\alpha_W^2 T^2}{M_W^4} \tag{3}$$

where the factor $\frac{\alpha_W^2}{M_W^4}$ comes from the amplitude squared.
Since the species are relativistic, $v \sim 1$. Then, the rescattering rate is

$$\Gamma \sim \sigma n \sim \frac{\alpha_W^2 T^5}{M_W^4} \tag{4}$$

— On the other hand, we know that $H \sim GT^2$.
By comparing the two quantities, we find the decoupling temperature as

$$T \sim G^{1/3} \alpha_W^{2/3} M_W^{4/3} \tag{5}$$

2. Before the decoupling of neutrinos, γ, ν_e and $e^{\pm}$ form a plasma.
3. The entropy density is given by

$$s = \frac{2\pi^2}{45} g_* T^3$$

with $g_* = 2 + \frac{7}{8}(2 \cdot 2 + 2 \cdot 3)$. After neutrino decoupling we have

$$s = \frac{2\pi^2}{45}(2 + \frac{7}{8} \cdot 4)T^3 + \frac{2\pi^2}{45}(\frac{7}{8} \cdot 6)T_\nu^3$$

The temperature of the neutrinos and the other particles are the same until electrons and positrons annihilate. After annihilation, they don't contribute to the entropy anymore. Therefore,

$$s = \frac{2\pi^2}{45}(2)T^3 + \frac{2\pi^2}{45}(\frac{7}{8} \cdot 6)T_\nu^3$$

Using the fact that the entropy stays constant, we find at the moment of decoupling

$$\frac{2\pi^2}{45}(2 + \frac{7}{8} \cdot 4)T^3 + \frac{2\pi^2}{45}(\frac{7}{8} \cdot 6)T_\nu^3 = \frac{2\pi^2}{45}(2)T^3 + \frac{2\pi^2}{45}(\frac{7}{8} \cdot 6)T_\nu^3$$

This gives at the end

$$T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma \tag{6}$$

Therefore, the cosmic neutrino background temperature is

$$T_\nu \sim 1.94[K] \tag{7}$$

where we used $T_\gamma = 2.72[K]$

3. Decoupling and concentration

1. Before decoupling there are γ, ν's, $e^{\pm}$, $\mu^{\pm}$ and s that form the plasma.

2. In this exercise, the strategy is the same as in the case of the decoupling of neutrinos. Let's see what are the stages by which the system passes :
   — *Before decoupling* s *:* γ, ν's, $e^{\pm}$, $\mu^{\pm}$ and s forming a plasma. The entropy density is
$$s = \frac{2\pi^2}{45} g_* T^3$$
   where $g_* = 2 + 1 + \frac{7}{8}(2 \times 2 \times 2 + 6) = \frac{57}{4} + 1$
   — *After decoupling* s *:* s decouples but its temperature remains the same as the residual plasma :
$$s = \frac{2\pi^2}{45}(g_* - 1)T^3 + \frac{2\pi^2}{45} T_s^3$$
   — *Annihilation of the* μ^+ *:* The entropy of the particle s keeps the same expression, the plasma's entropy is changed by the disappearance of the muonic contribution in g_*
$$s = \frac{2\pi^2}{45}\left(\frac{57}{4} - \frac{7}{8}(2 \times 2)\right) T^3 + \frac{2\pi^2}{45} T_s^3$$
   Note that it is now more true that the plasma temperature is identical to that of the particle s. In fact, the fewer degrees of freedom lead to an increase of Ta in order to preserve entropy. One can easily find that $T_s^3/T_{\text{plasma}}^3 = 43/57$ after the annihilation of μ^+.
   — *Decoupling of neutrinos :* As before, for the particle s, neutrinos decouple but their temperature is the same as the residual plasma :
$$s = \frac{2\pi^2}{45}\left(\frac{43}{4} - \frac{7}{8} \times 6\right) T^3 + \frac{2\pi^2}{45}\left(\frac{7}{8} \times 6\right) T_\nu^3 + \frac{2\pi^2}{45} T_s^3$$
   — *Positron annihilation :* The $e^{\pm}$ disappear from the plasma and will cause a change in temperature between the plasma and neutrinos. The particle s is not affected because its entropy is conserved.
$$s = \frac{2\pi^2}{45}(2)T^3 + \frac{2\pi^2}{45}\left(\frac{7}{8} \times 6\right) T_\nu^3 + \frac{2\pi^2}{45} T_s^3$$
   If we impose the conservation of entropy between the last two points, the part for the particle s is trivial; but the "plasma-neutrino" gives us the factor $T_\nu^3/T_\gamma^3 = 4/11$ valid after the annihilation.

   We now ask about the relationship between the temperature of photons and s after the annihilation of positrons. To do this we equate the entropy after the decoupling of s with the entropy after the annihilation of positrons. As already mentioned, the entropy of decoupled species is separately conserved for each species. Therefore, we obtain
$$\begin{cases} \frac{57}{4}(Ta)_{\text{in}}^3 = 2(Ta)_{\text{out}}^3 + \frac{7}{8} \times 6(T_\nu a)_{\text{out}}^3 \\ (T_s a)_{\text{in}}^3 = (T_s a)_{\text{out}}^3 \end{cases}$$

But, as argued above, the temperature of s after decoupling is equal to the temperature of the plasma at this time. Then $(T_s a)^3_{\text{in}} = (Ta)^3_{\text{in}}$. Therefore the first equation can be simplified by a^3_{out} and it gives

$$T_s^3 = \frac{4}{57}\left(2T^3 + \left(\frac{7}{8}\times 6\right)T_\nu^3\right)$$

where, to lighten the notation, we did not mention that this equation is valid only after the annihilation of positrons. Noting that the plasma is composed only of photons, we can identify $T = T_\gamma$. In addition we have seen above $T_\nu^3/T_\gamma^3 = 4/11$. So finally

$$\left(\frac{T^s}{T_\gamma}\right)_{\text{out}} = 0.65$$

3. After the annihilation of $e^\pm$, the universe expands without interactions. The particle s becomes non-relativistic, but is not in equilibrium, so we can not directly use a formula for its density. However, it is known that the ratio between the number density and entropy density (n_s/s) is conserved. Thus :

$$n(T_0) = \frac{n_s(T_d)}{s(T_d)}s(T_0) = \frac{g}{g_*}\frac{\zeta(3)}{\pi^2}T^3_{\gamma,0} \approx 3\times 10^7\ [m^{-3}]$$

4. The mass density induced by the particle s is :

$$\rho_s = n_s m_s = 3\times 10^9\ [eV/m^3]$$

Compare this with the critical density :

$$\rho_c = 5\times 10^9\ [eV/m^3]$$

The particle s is responsible for almost all the universe density!

4. Investigating the kinetic equation numerically

1. (a) The non-relativistic limit $T \ll m$ corresponds to $x = \frac{m}{T} \gg 1$. The distribution function takes the form

$$f_{eq}(\vec{k}) = e^{-m/T}e^{-\frac{k^2}{2mT}}, \tag{8}$$

and the integration can be performed explicitly,

$$n_{eq} = \frac{1}{2\pi^2}\int_0^\infty dk k^2 e^{-m/T}e^{-\frac{k^2}{2mT}} = \frac{1}{2\pi^2}e^{-m/T}(mT)^{3/2}\int_0^\infty dy y^2 e^{-y^2/2}. \tag{9}$$

The last integral equals $\sqrt{\pi/2}$, and we have

$$F(x) \equiv \frac{n_{eq}}{T^3} \approx (2\pi)^{-3/2}x^{3/2}e^{-x}, \quad x\to\infty. \tag{10}$$

One can see from Fig.1 that this is indeed the correct asymptotic of the function $F(x)$.

(b) In the ultra-relativistic limit $T \gg m$, $x \ll 1$, $k \approx E$, and the number density equals

$$n_{eq} = \frac{1}{2\pi^2} \int_0^\infty \frac{dE E^2}{e^{E/T} + 1} = \frac{3\zeta(3)}{4\pi^2} T^3. \qquad (11)$$

Hence, $F(x)$ must approach the constant as $x \to 0$, which is indeed the case.

2. Recall that

$$n_{eq} = \int \frac{d^3\vec{k}}{(2\pi)^3} f_{eq}(\vec{k}), \qquad (12)$$

where the fermion distribution function

$$f_{eq}(\vec{k}) = \frac{1}{e^{E/T} + 1}, \quad E = \sqrt{k^2 + m^2}. \qquad (13)$$

Introduce the dimensionless variable $x = m/T$ and rewrite n_{eq} as follows,

$$n_{eq} = \frac{T^3}{2\pi^2} \int_0^\infty dy \frac{y^2}{e^{\sqrt{x^2+y^2}} + 1} = T^3 F(x). \qquad (14)$$

The function $F(x)$ is presented in Fig.1. Let us check the limits of small and large temperatures.

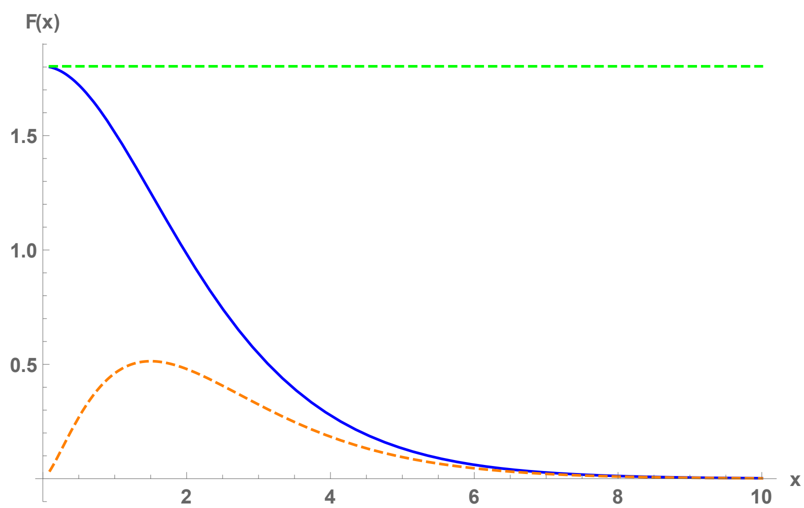

FIGURE 1 – The function F(x) and its asymptotics.

3. To find Γ it is convenient to introduce dimensionless parameters $x = m/T$ and $y = E/T$. The cross section and velocity are written as

$$v(x, y) = \sqrt{1 - \frac{x^2}{y^2}}, \qquad (15)$$

$$\sigma(x, y) = \frac{\pi e^4}{m^2} \frac{1 - v^2}{4v} \left(\frac{3 - v^4}{v} \log \frac{1 + v}{1 - v} - 2(2 - v^2) \right). \qquad (16)$$

Then,

$$\Gamma = \frac{e^4 T^3}{2m^2\pi} \int_x^\infty dyy\sqrt{y^2 - x^2}(e^y + 1)^{-1} v(x,y)\tilde{\sigma}(x,y) = \frac{e^4 T^3}{2m^2\pi} G(x). \quad (17)$$

where $\sigma = \frac{\pi e^4}{m^2}\tilde{\sigma}$, The function $G(x)$ is presented in Fig.2. One can again assure that the limits $x \ll 1$ and $x \gg 1$ give the asymptotic behaviour

$$\Gamma \approx \frac{me^4}{2\sqrt{2\pi}} \frac{e^{-x}}{x}, \quad x \to \infty, \quad (18)$$

$$\Gamma \approx \frac{me^4 \zeta(3)}{6\pi} \frac{1}{x}, \quad x \to 0, \quad (19)$$

which coincides with the analytic estimations.

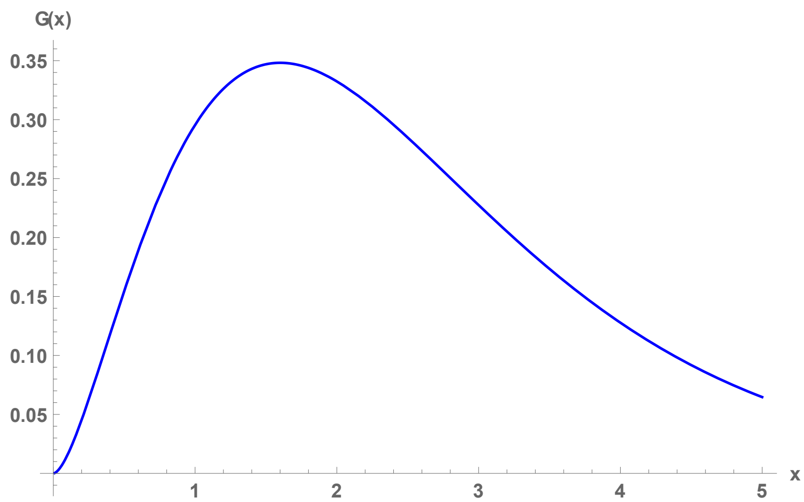

FIGURE 2 – The function G(x).

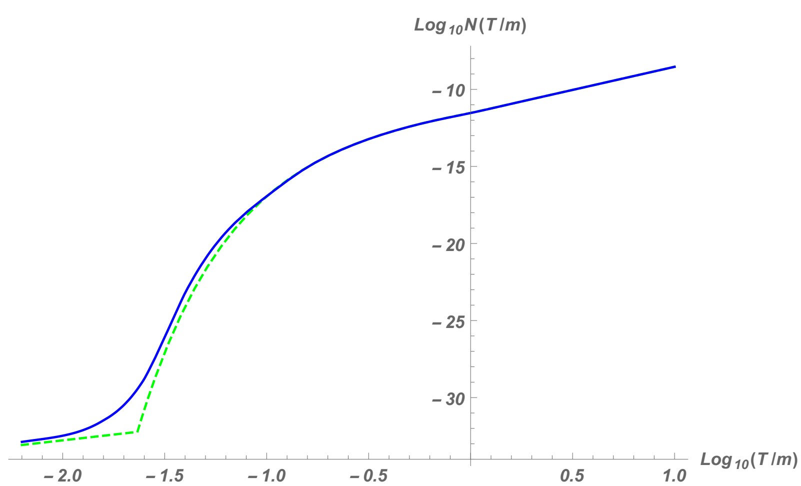

FIGURE 3 – The function $n(T)$ in conventional units. Dashed lines represent different asymptotics found analytically.

4. Consider the equation

$$\frac{\partial n}{\partial t} + 3Hn = -\Gamma(n - n_{eq}). \tag{20}$$

To solve it numerically, one should first rewrite it in dimensionless variables. In the radiation domination stage of the Universe

$$dt \approx -\frac{M_P}{T^3}dT, \quad H \approx \frac{T^2}{M_P} \tag{21}$$

(we drop the difficulties emerging from counting degrees of freedom). eq.(20) is then rewritten as

$$-\frac{\partial n}{\partial T}\frac{T^3}{M_P} + 3n\frac{T^2}{M_P} = -\frac{e^4}{2m^2\pi}T^3 G(m/T)\left(n - T^3 F(m/T)\right). \tag{22}$$

Next, we change the variable $n = T^3\tilde{n}$. The equation takes the form

$$\frac{\partial \tilde{n}}{\partial T}\frac{1}{M_P} = \frac{e^4}{2m^2\pi}G(m/T)\left(\tilde{n} - F(m/T)\right). \tag{23}$$

Finally, we turn to the variable $y = \dfrac{m}{T}\dfrac{2\pi m}{M_P e^4}$:

$$-y^2\tilde{n}'(y) = G(y)\left(\tilde{n}(y) - F(y)\right). \tag{24}$$

Eq.(24) must be supplemented by the initial condition. We have

$$\tilde{n}(0) = \lim_{T\to\infty}\frac{n(y)}{T^3} = \frac{3\zeta(3)}{4\pi^2}, \tag{25}$$

according to Eq.(11). The solution to this equation is presented on Fig.3.