
TMP-TC2: Cosmology

Solutions to Problem Set 6

30, 31 May & 1 June 2023

1. Boltzmann Equation

1. Since particle number is conserved, we want that the variation of the phase space density at a certain point corresponds to a flux of particles to the nearby points. This amount to require that the total derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\mathbf{x}} \cdot \nabla_x + \dot{\mathbf{p}} \cdot \nabla_p \quad (1)$$

vanishes when acting on f . If we further use the equation of motion for a particle :

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m}; \quad \dot{\mathbf{p}} = \mathbf{F}(\mathbf{x}, \mathbf{p}, t) \quad (2)$$

we obtain that f satisfies :

$$\begin{aligned} \frac{\partial f(\mathbf{x}, \mathbf{p}, t)}{\partial t} &= -\dot{\mathbf{x}} \cdot \nabla_x f(\mathbf{x}, \mathbf{p}, t) - \dot{\mathbf{p}} \cdot \nabla_p f(\mathbf{x}, \mathbf{p}, t) \\ &= -\frac{\mathbf{p}}{m} \cdot \nabla_x f(\mathbf{x}, \mathbf{p}, t) - \mathbf{F}(\mathbf{x}, \mathbf{p}, t) \cdot \nabla_p f(\mathbf{x}, \mathbf{p}, t) \end{aligned} \quad (3)$$

2. To generalize the previous equation to a relativistic setting, we should replace the classical equation of motion with the general relativistic ones, namely the geodesic equation on a curved spacetime. However, in the homogeneous and isotropic setting of cosmology, we can take a shortcut. Indeed, the density doesn't depend on x , and we know that momenta evolve by redshifting ($p(t)a(t) = \text{const}$), i.e.

$$\frac{dp}{dt} = -Hp \quad (4)$$

Moreover, f can depend only on the absolute value of \mathbf{p} by isotropy. Then, we obtain

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial p} \dot{p} = Hp \frac{\partial f}{\partial p} \quad (5)$$

This equation is called collisionless Boltzmann equation.

3. To obtain an equation for the space density n , we integrate in d^3p .

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial f}{\partial t} - H \int \frac{d^3p}{(2\pi)^3} p \frac{\partial f}{\partial p} = 0 \quad (6)$$

After switching to polar coordinates, we can integrate the second term by parts. Since we know that to have a finite spatial density $f(p)$ must go to zero at infinity,

$$\int \frac{d^2\Omega}{(2\pi)^3} \int_0^\infty p^2 dp p \frac{\partial f}{\partial p} = -3 \int \frac{d^2\Omega}{(2\pi)^3} \int_0^\infty p^2 dp f(p) \quad (7)$$

The final result is

$$\frac{dn(t)}{dt} + 3Hn(t) = 0 \quad (8)$$

4. By the product rule, we recognize that the previous equation is nothing else than the familiar conservation equation for the number density :

$$\frac{1}{a^3} \frac{d}{dt} (a^3 n(t)) = 0 \quad (9)$$

5. In this point, we want to relate the scattering rate of a certain $2 \rightarrow 2$ process at finite density of scatterers n_1 and n_2 to the cross section of the same process. To this end, recall that the cross section for a general event is defined as

$$\sigma = \frac{\Gamma}{nv} \quad (10)$$

Where Γ is the scattering rate of the process, v the relative speed and n the density of the targets. We recall that in QFT Γ generally behaves as $\frac{1}{V}$ and the density for a single particle target is $\frac{1}{V}$, so that the two contributions simplify. In our case we have,

$$\frac{\Gamma}{V} = \frac{\sigma(s) v}{V} N_1 N_2 = \sigma(s) v n_1 \frac{N_2}{V} = \sigma(s) v n_1 n_2 \quad (11)$$

6. Let's call μ the chemical potential. Then, if we denote $n^{(0)}$ the density at zero chemical potential, we have that $n = e^{\frac{\mu}{T}} n^{(0)}$. then,

$$e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} = \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \quad (12)$$

holds. By kinetic equilibrium, we have for the densities without chemical potential that $n_1^{(0)} n_2^{(0)} = \int e^{(E_1+E_2)/T} = \int e^{(E_3+E_4)/T} = n_3^{(0)} n_4^{(0)}$. The equation becomes :

$$a^{-3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \quad (13)$$

We see that we have two regimes :

- If $H \gg \Gamma$, the scattering rate is negligible. Then, the Boltzmann equation reduces to the collisionless case, and the matter undergoes a free expansion.
- if $H \ll \Gamma$, we can neglect the expansion factor. Then, from the previous equation we see that we end up in chemical equilibrium (see also the next exercise).

The physical picture here is clear : as long as the scattering rate is big enough compared to the expansion rate, the particles can reequilibrate at the new temperature tracking the varying radius of the universe. If however the scattering rate is too small, the particles can't rethermalize fast enough and they'll just undergo a free expansion decoupled from the surroundings.

7. The expression is straightforward given the formula for the scattering rate per unit volume :

$$dR_{\beta\alpha} = \prod_{i=1}^2 \frac{d^3 p_i}{(2\pi)^3 2E_{\mathbf{p}_i}} \prod_{j=1}^{n'} \frac{d^3 p'_j}{(2\pi)^3 2E_{\mathbf{p}'_j}} (2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) |M_{\beta\alpha}|^2 \quad (14)$$

Let's take for example the forward reaction $1 + 2 \rightarrow 3 + 4$. To get the total scattering rate, first we need to integrate over all the possible final states. This justifies the

$$\int \frac{d^3 p'}{(2\pi)^3 2E_3(p')} \int \frac{d^3 q'}{(2\pi)^3 2E_4(q')} |\mathcal{M}|^2 \quad (15)$$

Then, we need to sum the scattering rate over the densities of incoming particles to get the total rate. This justifies

$$\int \frac{d^3 p}{(2\pi)^3 2E_2(p)} \int \frac{d^3 q}{(2\pi)^3 2E_2(q)} f_1(\mathbf{p}) f_2(\mathbf{q}) \quad (16)$$

The reasoning is analogous for the backward scattering.

2. Conservation of chemical potential

1. Denote by f_i the distribution function of i 's particle participating in the reaction $(1) + (2) \rightarrow (3) + (4)$. In equilibrium

$$f_i = \frac{1}{e^{\frac{E_i - \mu_i}{T}} \pm 1}, \quad (17)$$

where '+' sign holds for fermions, and '-' sign for bosons. The equality $I_{eq} = 0$ is equivalent to the relation

$$(1 \pm f_1)(1 \pm f_2) f_3 f_4 = (1 \pm f_3)(1 \pm f_4) f_1 f_2, \quad (18)$$

where the upper sign holds for bosons and the lower sign for fermions. The last equation can be rewritten as

$$\log \frac{f_1}{1 \pm f_1} + \log \frac{f_2}{1 \pm f_2} = \log \frac{f_3}{1 \pm f_3} + \log \frac{f_4}{1 \pm f_4}. \quad (19)$$

Therefore, $\log \frac{f}{1 \pm f}$ is an additive constant of motion. Substituting Eq.(17), we have

$$\log \frac{f_i}{1 \pm f_i} = -\frac{E_i - \mu_i}{T}. \quad (20)$$

Hence, Eq.(19) implies

$$\mu_1 + \mu_2 = \mu_3 + \mu_4. \quad (21)$$

- The vanishing of n_γ is an immediate consequence of photon number non-conservation. Indeed, there is no quantum number associated with photon which must be preserved during interactions in equilibrium systems, unlike, for example, electric charge for electrons or baryon charge of quarks. Since μ itself is preserved, it follows that $\mu_\gamma = 0$.
- Consider the following reaction



Since $\mu_\gamma = 0$, it follows that $\mu_p = -\mu_{\bar{p}}$.

3. Abundances Evolution

- We want to cast the Boltzmann equation in a form without the friction term $3Hn_X$. We start by writing the equations that the densities n_X and n_γ obey

$$\begin{cases} \frac{dn_X}{dt} + 3Hn_X = -\Gamma(n_X - n_X^{\text{eq}}), \\ \frac{dn_\gamma}{dt} + 3Hn_\gamma = 0. \end{cases}$$

Notice that

$$H = \frac{1}{2t} = \frac{T^2}{M_P} = \frac{T^2}{m^2} \frac{m^2}{M_P} = x^{-2} H(m) \rightarrow x = \sqrt{2tH(m)},$$

thus

$$\frac{d}{dt} = \frac{dx}{dt} \frac{d}{dx} = \frac{H(m)}{x} \frac{d}{dx}$$

The Boltzmann equations are then written as

$$\begin{cases} \frac{dn_X}{dx} + 3\frac{n_X}{x} = -\frac{\Gamma x}{H(m)}(n_X - n_X^{\text{eq}}) \\ \frac{dn_\gamma}{dx} + 3\frac{n_\gamma}{x} = 0 \end{cases}$$

- Combining the above, we find

$$\frac{dY}{dx} = -\frac{\Gamma x}{H(m)}(Y - Y^{\text{eq}}),$$

where $Y \equiv n_X/n_\gamma$ and $Y^{\text{eq}} \equiv n_X^{\text{eq}}/n_\gamma$.

- To solve numerically this exercise, we will use Mathematica. The previous equation with $\Gamma_X = \alpha m$ gives

$$\frac{dY}{dx} = -\alpha x \frac{M_P}{m}(Y - Y^{\text{eq}})$$

When we are in the relativistic case we have $T \gg m$, so $x \ll 1$. The behavior of Y^{eq} in this area is constant. Indeed, the two densities are proportional to

T^3 . When the temperature drops, the particle X becomes non-relativistic, while the photon remains relativistic. Their ratio is

$$Y^{\text{eq}} = \frac{n_X^{\text{eq}}}{n_\gamma} \sim \frac{(mT)^{3/2} e^{-m/T}}{T^3} = x^{3/2} e^{-x}$$

The strategy was therefore to cut the resolution of the differential equation into two parts : relativistic regime and the non-relativistic. In the relativistic case, as expected, the abundance remains constant. In the case where the particle decouples while non-relativistic (see next point for a condition on α) we obtain a non-equilibrium behavior as can be seen in the figure 1.

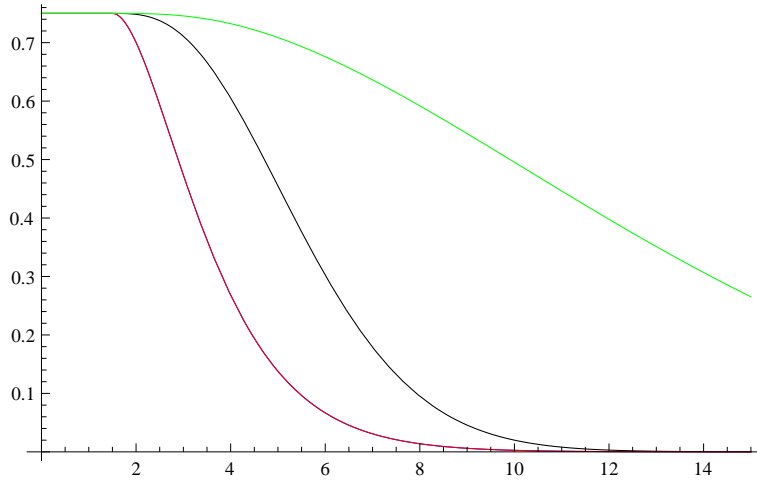


FIGURE 1 – Evolution of the abundance in the case $\alpha = 10^{-15}, 10^{-19}, 10^{-20}$

To produce a non-equilibrium behavior requires that the decoupling takes place at a temperature where the particle in question is non-relativistic. The decoupling temperature is as usual found by considering that the rate of interaction is the same as the rate of the expansion of the Universe

$$\Gamma(T^*) \simeq H(T^*) \quad \rightarrow \quad T_*^2 = \alpha m M_P .$$

For non-relativistic decoupling, we need $m > T_*$, i.e.

$$m > \sqrt{\alpha m M_P} \quad \leftrightarrow \quad 1 > \frac{\alpha M_P}{m} \quad \leftrightarrow \quad \alpha < 10^{-18} .$$