TMP-TC2: Cosmology

Problem Set 6

1. Boltzmann equation

To study the evolution of the Universe we need to describe the dynamics of many interacting particles beyond thermal equilibrium. The study of these situations goes under the name of kinetic theory. In this exercise, we obtain and study the main instrument to address the problem, namely the Boltzmann equation. In kinetic theory, we approximate the state of our many-particle system by a time-dependent phase space number density function $f(\boldsymbol{x}, \boldsymbol{p}, t)$, which represents the particle number density at position $\boldsymbol{x}$ with momentum $\boldsymbol{p}$ at a particular instant in time t.

In the first part, we ignore collisions between particles.

1. First assume a non-relativistic Hamiltonian, without interactions but in an external force field $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{p}, t)$. Derive the equation of motion for f by imposing that the total derivative $\frac{\mathrm{d}f}{\mathrm{d}t}$ is zero in phase space.

2. Now we generalize the equation of motion to general relativistic conditions. Since we are in a cosmological setting, let's assume homogeneity and isotropy so that f doesn't depend on $\boldsymbol{x}$.
   Show that in a FLRW spacetime $\frac{\partial f}{\partial t}$ satisfies the equation

$$\frac{\partial f}{\partial t} - Hp\frac{\partial f}{\partial p} = 0 \tag{1}$$

   For that, impose that the total derivative of f vanishes and use the redshift of momenta.

3. Starting from $f(\boldsymbol{x}, \boldsymbol{p}, t)$, how do we obtain the space density $n(\boldsymbol{x}, t)$? In our setting when $n(\boldsymbol{x}, t) = n(t)$ by homogeneity, derive a differential equation for the time evolution $n(t)$.

4. Show that the equation you obtained is equivalent to the conservation equation

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n(t)) = 0 \tag{2}$$

Now we want to introduce interactions between our particles. If our degrees of freedom are weakly interacting, $2 \rightarrow 2$ scattering is dominant. Imagine a scattering process of the form

$$(1)_p + (2)_q \longleftrightarrow (3)_{p'} + (4)_{q'} \tag{3}$$

We imagine our interactions are fast enough with respect to the variation of f_i so that they instantaneously transfer some density from the region of momenta p, q to regions of momenta p', q' and vice versa at the same position of space $\boldsymbol{x}$.

As a first step in analyzing collisions, we imagine an idealized setting where we have a homogeneous number density in space, with particles all moving around a typical

energy. This is relevant for nearly thermal conditions like the ones of the early universe since we can imagine particles with kinetic energy peaked around $E \sim T$. Observe that this doesn't mean they're in complete thermal equilibrium, since, for example, chemical potentials may differ.

In these conditions, we estimate the variation of n_1 as caused by a collision term :

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = -\frac{\Gamma}{V}(1 + 2 \to 3 + 4) + \frac{\Gamma}{V}(3 + 4 \to 1 + 2) \tag{4}$$

where $\frac{\Gamma}{V}$ is the scattering rate per unit volume to be expressed as a function of energy T and particle densities n_i.

5. Show that $\frac{\Gamma}{V}(1 + 2 \to 3 + 4)$ can be calculated in term of the cross section as $\frac{\Gamma}{V}(1 + 2 \to 3 + 4) = \langle \sigma v \rangle_T n_1 n_2$. v denotes the relative velocity between scatterers.

6. Finally, we obtain the equation

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = \langle \sigma v \rangle_T (n_3 n_4 - n_1 n_2) \tag{5}$$

Under the previous assumption that the particles are in kinetic equilibrium (same temperature) but not necessarily in chemical equilibrium (different chemical potential) describe what happens for $H \gg \Gamma$ and $H \ll \Gamma$.

If we want to understand the Boltzmann equation in a more general setting, we can consider collisions for arbitrary phase space density distributions f_i. Then, the process causes a variation of $n_1(t)$ (we suppressed $\boldsymbol{x}$) given by

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = \int \frac{d^3 p}{(2\pi)^3 2E_2(p)} \int \frac{d^3 q}{(2\pi)^3 2E_2(q)} \int \frac{d^3 p'}{(2\pi)^3 2E_3(p')} \int \frac{d^3 q'}{(2\pi)^3 2E_4(q')} |\mathcal{M}|^2$$
$$\times (2\pi)^4 \delta_{\mathrm{D}}^{(3)}[\boldsymbol{p} + \boldsymbol{q} - \boldsymbol{p}' - \boldsymbol{q}'] \delta_{\mathrm{D}}^{(1)}[E_1(p) + E_2(q) - E_3(p') - E_4(q')]$$
$$\times \{f_3(\boldsymbol{p}') f_4(\boldsymbol{q}') - f_1(\boldsymbol{p}) f_2(\boldsymbol{q})\}. \tag{6}$$

The right-hand side takes the name of the collision integral.

7. Justify the above expression.

   Indication: Recall the expression of the relativistic scattering rate per unit volume from QFT :

$$\mathrm{d}R_{\beta\alpha} = \prod_{i=1}^{2} \frac{d^3 p_i}{(2\pi)^3 2E_{\boldsymbol{p_i}}} \prod_{j=1}^{n'} \frac{\mathrm{d}^3 p'_j}{(2\pi)^3 2E_{\boldsymbol{p_j}}} (2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) |M_{\beta\alpha}|^2 \tag{7}$$

We add without proving that the boson/fermion statistics change the factor $f_3(\boldsymbol{p}') f_4(\boldsymbol{q}') - f_1(\boldsymbol{p}) f_2(\boldsymbol{q})$ to

$$f_3(\boldsymbol{p}') f_4(\boldsymbol{q}') [1 \pm f_1(\boldsymbol{p})] [1 \pm f_2(\boldsymbol{q})]$$
$$- f_1(\boldsymbol{p}) f_2(\boldsymbol{q}) [1 \pm f_3(\boldsymbol{p}')] [1 \pm f_4(\boldsymbol{q}')] \tag{8}$$

These factors can be interpreted as an enhancement coming from Bose condensation for bosons or a suppression by the Pauli exclusion principle for fermions.

If we approximate the previous integral assuming kinetic equilibrium, i.e. same temperature but possibly different chemical potential, we can then average the cross-section over the thermal distribution and reduce ourselves back to the easier previous form.

2. Conservation of chemical potential

In this exercise we expand on the last point of the previous exercise. We saw that the Boltzmann equation has the structure of the balance equation, whose right-hand side (the collision integral I) measures the difference between gain and loss processes in a given cell of the phase space. In equilibrium, these processes must balance each other, hence $I_{eq} = 0$.

1. Prove that in equilibrium the chemical potential μ is preserved by reactions of the form $(1) + (2) \to (3) + (4)$, that is, $\mu_1 + \mu_2 = \mu_3 + \mu_4$.

2. Bearing in mind the conservation of chemical potential in equilibrium systems, show that $\mu_\gamma = 0$.

3. Show that the chemical potentials of particle (p) and antiparticle $(\bar{p})$ are related by $\mu_p = -\mu_{\bar{p}}$.

3. Abundances evolution

The density of decaying particles $X(X \to qq)$, obeys the equation

$$\frac{dn_X}{dt} + 3Hn_X = -\Gamma_X(n_X - n_{eq}) \ ,$$

where the term $\Gamma_X n_{eq}$ comes from inverse process $(qq \to X)$, and n_{eq} is the equilibrium density of X.
Assuming that $\Gamma_X = \alpha m$ with m the mass of the particle, what value of α should be chosen in order to get non-equilibrium behaviour ?
To answer this question, effectuate the following steps :

1. To get rid of the friction term $3Hn_X$, we introduce the quantity $Y = n_X/n_\gamma$, where the photon density n_γ obeys a collisionless Boltzmann equation . In addition, it can sometimes be useful to introduce the dimensionless variable $x = m/T$.

2. Deduce from the equation for n_X, a differential equation for $Y(x)$.

3. Solve the problem numerically with the initial condition $Y(T \to \infty) = Y_{eq}(T)$. *Indication:* Distinguish the cases $T \gg m$ and $T \ll m$.